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REVERSE MATHEMATICS, COMPUTABILITY, AND PARTITIONS OF TREES

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Abstract. We examine the reverse mathematics and computability theory of a form of Ramsey's theorem in which the linear n-tuples of a binary tree are colored.

Let $2^{<\mathbb{N}}$ denote the full binary tree of height ω. We identify nodes of the tree with finite sequences of zeros and ones, and refer to any subset of the nodes as a subtree. For positive integers n, let $[2^{<\mathbb{N}}]^n$ denote the set of all linearly ordered n-tuples of nodes in $2^{<\mathbb{N}}$. We say that a subtree S of $2^{<\mathbb{N}}$ is isomorphic to $2^{<\mathbb{N}}$ if every node of S has exactly two immediate successors. More formally, $S \subseteq 2^{<\mathbb{N}}$ is isomorphic to $2^{<\mathbb{N}}$ if there is a bijective function $f : 2^{<\mathbb{N}} \to S$ such that for all $\sigma, \tau \in 2^{<\mathbb{N}}$, we have $\sigma \subseteq \tau$ if and only if $f(\sigma) \subseteq f(\tau)$. This weak form of isomorphism does not preserve minima. Using this terminology, we can formulate the following version of Ramsey's theorem for trees.

TT^n_k: Suppose that $[2^{<\mathbb{N}}]^n$ is colored with k colors. Then there is a subtree S isomorphic to $2^{<\mathbb{N}}$ such that $[S]^n$ is monochromatic.

Although we have not found this principle stated verbatim in the literature, TT^1_k is an immediate consequence of Theorem 1.3 of [8] and also of Theorem 2.1 of [2]. In his dissertation [7], McNicholl used TT^1_k to find the combinatorial conditions that are necessary and sufficient to carry out a kind of priority construction. Iterated applications of Theorem 2.3 of [2] could be used to prove TT^n_k for all finite n and k. All these cited results use a stronger notion of isomorphism than is used in our formulation.

The goal of this paper is to examine the reverse mathematics and computability theory of this form of Ramsey's theorem. Section 1 gives the reverse mathematical analysis of TT^n_k and consequently includes proofs of TT^n_k. Section 2 proves upper and lower bounds on the complexity of the monochromatic sets, paralleling the similar bounds proved by Jockusch in [5] for Ramsey's theorem on the integers. We conclude the paper by noting extensions of these results to other infinite trees and listing some questions.

§1. Reverse mathematics. In this section, we analyze TT^n_k using the hierarchy of subsystems of second order arithmetic detailed in Simpson's book [10]. We need

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only three systems. The base system RCA_0 includes induction restricted to Σ^0_1 sets and a form of set comprehension for computable sets. Our proof of TT^1_k uses RCA_0 with induction for Σ^0_2 formulas appended. Proofs for higher exponents require the use of ACA_0, which adds comprehension for arithmetically definable sets to RCA_0. We begin by carrying out a proof of TT^1_k in a weak system.

LEMMA 1.1. (RCA_0) *Let* $f : 2^{<\mathbb{N}} \to \{\mathsf{red}, \mathsf{blue}\}$ *be a two coloring of the nodes of the full binary tree. For any node* σ *of the tree either* (1) *above* σ *there is a subtree isomorphic to* $2^{<\mathbb{N}}$ *in which every nonempty node is colored* red, *or* (2) σ *can be extended to a node* τ *such that every node properly extending* τ *is colored* blue.

PROOF. Suppose f and σ are as hypothesized. Enumerate the pairs of nodes of $2^{<\mathbb{N}}$. Construct the red tree as follows. Let $p_{\langle\rangle} = \sigma$. Given p_α, let $p_{\alpha^\frown 0}$ and $p_{\alpha^\frown 1}$ be the first pair of incomparable red nodes extending p_α. If this process never fails, then $\{p_\alpha \mid \alpha \in 2^{<\mathbb{N}}\}$ is the desired red tree. If the process fails, then there is a least node $\beta \supset \sigma$ such that all red extensions of β are comparable. If no extension of β is red, let $\tau = \beta$. Every proper extension of τ is blue. If there is a node $\alpha \supset \beta$ such that α is red, pick the least such node and write $\alpha = \beta^\frown\gamma^\frown\varepsilon$, where $\varepsilon \in \{0, 1\}$. Then let $\tau = \beta^\frown\gamma^\frown(1-\varepsilon)$ and note that every proper extension of τ is blue. ⊣

In the following, we will refer to the red tree in the proof of Lemma 1.1 as the *standard red subtree* (of $2^{<\mathbb{N}}$ for σ using f). We will refer to the blue subtree as the *full subtree* (of $2^{<\mathbb{N}}$ for σ). The proof of the next result uses induction on Σ^0_2 formulas. Consequently, the statement of the theorem refers to $\mathsf{RCA}_0 + \Sigma^0_2$-$\mathsf{IND}$.

THEOREM 1.2. $(\mathsf{RCA}_0 + \Sigma^0_2\text{-}\mathsf{IND})$ *For all* k, TT^1_k. *That is, for any finite coloring of* $2^{<\mathbb{N}}$, *there is a monochromatic subtree isomorphic to* $2^{<\mathbb{N}}$.

PROOF. Suppose $f : 2^{<\mathbb{N}} \to k$ is a finite coloring of the nodes of $2^{<\mathbb{N}}$. Consider the set $C = \{j < k \mid \exists\sigma\forall\tau(\tau \supseteq \sigma \to j \le f(\tau))\}$. By bounded Σ^0_2 comprehension, which is provable from Σ^0_2-IND in RCA_0 (see [10], page 72), the set C exists. Now $0 \in C$, so C is nonempty and finite. Find the largest element of C and call it j. Since $j \in C$, there must be a witness $\sigma \in 2^{<\mathbb{N}}$ such that $\forall\tau(\tau \supseteq \sigma \to j \le f(\tau))$. Consider the two coloring on extensions of σ defined by $g(\tau) = \mathsf{red}$ if $f(\tau) = j$ and $g(\tau) = \mathsf{blue}$ otherwise. The existence of a full blue subtree for g contradicts the choice of j as maximal. Consequently, Lemma 1.1 shows there is a standard red subtree for g above σ, which is an isomorphic copy of $2^{<\mathbb{N}}$ on which f constantly takes the value j. ⊣

It is easy to deduce the infinite pigeonhole principle from TT^1_k; simply color nodes according to their level. The infinite pigeonhole principle is equivalent to the bounding principle $\mathsf{B}\Pi^0_1$ (see [3] or [1]) which is strictly weaker than Σ^0_2-IND. Thus, the exact strength of TT^1_k is at least $\mathsf{B}\Pi^0_1$ and at most Σ^0_2-IND. To carry out a proof of TT^n_k, we need to prove TT^2_k as a base case.

THEOREM 1.3. (ACA_0) *For all* k, TT^2_k. *That is, for any finite coloring of pairs of comparable nodes of* $2^{<\mathbb{N}}$, *there is a monochromatic subtree isomorphic to* $2^{<\mathbb{N}}$.

PROOF FOR TWO COLORS ONLY. Suppose $f : [2^{<\mathbb{N}}]^2 \to \{\mathsf{red}, \mathsf{blue}\}$ is a two coloring of pairs of comparable nodes of the full binary tree. Given any $\sigma \in 2^{<\mathbb{N}}$, we define an induced map on single nodes $f_\sigma : \{\tau \in 2^{<\mathbb{N}} \mid \tau \supset \sigma\} \to 2$ by setting $f_\sigma(\tau) = f(\sigma, \tau)$.

Define p_σ, T_σ, and c_σ as follows. Set $p_{\langle\rangle} = \langle\rangle$ and $T_{\langle\rangle} = 2^{<\mathbb{N}}$. Suppose p_σ and T_σ have been defined. If there is a full blue subtree of T_σ for p_σ using f_{p_σ}, then make the following assignments:

- $c_\sigma = \mathsf{blue}$.
- Let $p_{\sigma^\frown 0}$ and $p_{\sigma^\frown 1}$ be the first two nonempty nodes of the full blue subtree of T_σ for p_σ using f_{p_σ}.
- For $\varepsilon \in \{0, 1\}$, set $T_{\sigma^\frown\varepsilon} = \{\tau \in T_\sigma \mid \tau \supseteq p_{\sigma^\frown\varepsilon}\}$.

If there is no full blue subtree of T_σ for p_σ using f_{p_σ}, then make the following assignments:

- $c_\sigma = \mathsf{red}$.
- Let $p_{\sigma^\frown 0}$ and $p_{\sigma^\frown 1}$ be the first two nonempty nodes of the standard red tree of T_σ for p_σ using f_{p_σ}.
- For $\varepsilon \in \{0, 1\}$, $T_{\sigma^\frown\varepsilon}$ consists of those nodes of the standard red tree of T_σ for p_σ using f_{p_σ} which extend $p_{\sigma^\frown\varepsilon}$.

Let $S = \{p_\sigma \mid \sigma \in 2^{<\mathbb{N}}\}$. Since p_σ is arithmetically definable from the values of p_τ and c_τ for $\tau \subset \sigma$, ACA_0 proves the existence of S. By the construction, whenever $\sigma \subset \tau \in 2^{<\mathbb{N}}$, we have $f(p_\sigma, p_\tau) = c_\sigma$. The map $p_\sigma \mapsto c_\sigma$ is a 2 coloring of S whose existence is provable in ACA_0. Since ACA_0 implies Σ^0_2-IND, an application of Theorem 1.2 yields a color c and a subtree T of S such that $p_\sigma \in T$ implies $c_\sigma = c$. Consequently, if $p_\sigma \subset p_\tau$ are elements of T, then $f(p_\sigma, p_\tau) = c$, completing the proof. ⊣

Proof of Theorem 1.3 for k colors. Suppose $f : [2^{<\mathbb{N}}]^2 \to k$ is a finite coloring of pairs of comparable nodes of $2^{<\mathbb{N}}$. Given any $\sigma \in 2^{<\mathbb{N}}$, we define an induced map on single nodes $f_\sigma : \{\tau \in 2^{<\mathbb{N}} \mid \tau \supset \sigma\} \to k$ by setting $f_\sigma(\tau) = f(\sigma, \tau)$.

Define p_σ, T_σ, and c_σ as follows. Set $p_{\langle\rangle} = \langle\rangle$ and $T_{\langle\rangle} = 2^{<\mathbb{N}}$. Given p_σ and T_σ, define c_σ as follows. Let j be the least integer such that there is no $p \supset p_\sigma$ in T_σ such that $\forall \tau \in T_\sigma(\tau \supset p \to j < f_{p_\sigma}(\tau))$. Since j is the least such integer, there is a $p \supset p_\sigma$ in T_σ such that $\forall \tau \in T_\sigma(\tau \supset p \to j \leq f_{p_\sigma}(\tau))$. Fix this p, and note that by the definition of j, there is no $q \supset p$ in T_σ such that $\forall \tau \in T_\sigma(\tau \supset q \to j < f_{p_\sigma}(\tau))$. If we treat the color j as red and the colors greater than j as blue, by Lemma 1.1, the standard j-colored subtree of T_σ for p using f_{p_σ} exists and is isomorphic to $2^{<\mathbb{N}}$. Call this tree T. Let $c_\sigma = j$. Let $p_{\sigma^\frown 0}$ and $p_{\sigma^\frown 1}$ be the two level one elements of T. For $\varepsilon \in \{0, 1\}$, let $T_{\sigma^\frown\varepsilon}$ be the subtree of T with root $p_{\sigma^\frown\varepsilon}$.

Let $S = \{p_\sigma \mid \sigma \in 2^{<\mathbb{N}}\}$. Since p_σ is arithmetically definable from the values of p_τ and c_τ for $\tau \subset \sigma$, ACA_0 proves the existence of S. By the construction, whenever $\sigma \subset \tau \in 2^{<\mathbb{N}}$, we have $f(p_\sigma, p_\tau) = c_\sigma$. The map $p_\sigma \mapsto c_\sigma$ is a finite coloring of S whose existence is provable in ACA_0. Since ACA_0 implies Σ^0_2-IND, an application of Theorem 1.2 yields a color c and a subtree T of S such that $p_\sigma \in T$ implies $c_\sigma = c$. Consequently, if $p_\sigma \subset p_\tau$ are elements of T, then $f(p_\sigma, p_\tau) = c$, completing the proof. ⊣

We complete the proof of TT^n_k in ACA_0 using the following inductive step. We abbreviate $\forall k(\mathsf{TT}^n_k)$ by TT^n.

Theorem 1.4. (ACA_0) *For all* $n \geq 1$, TT^n *implies* TT^{n+1}.

Proof. We will generalize the proof of Theorem 1.3 to handle higher exponents by constructing a subtree S such that the color of any $n+1$-tuple is determined by its first n elements, and applying TT^n to S to obtain the desired monochromatic tree. Suppose $f : [2^{<\mathbb{N}}]^{n+1} \to k$ is a finite coloring of the $n+1$-tuples of comparable nodes of $2^{<\mathbb{N}}$. If $P = \{p_\tau \mid \tau \subseteq \sigma\}$ is a sequence of comparable nodes terminating in p_σ, we define an induced coloring of single nodes $\tau \supset p_\sigma$ by setting

$$f_{p_\sigma}(\tau) = \prod_{\vec{m} \in [P]^n} \mathsf{pr}(\vec{m})^{f(\vec{m},\tau)}.$$

where if r is the integer code for the sequence $\vec{m}$, then $\mathsf{pr}(\vec{m})$ is the r^{th} prime. Note that f_{p_σ} uses no more than $k^{|P|^n}$ colors.

Define p_σ, T_σ, and c_σ as follows. Set $p_{\langle\rangle} = \langle\rangle$ and $T_{\langle\rangle} = 2^{<\mathbb{N}}$. Given p_σ and T_σ, let c_σ be the greatest integer in the range of f_{p_σ} such that there is a $p \supset p_\sigma$ in T_σ such that

$$\forall \tau \in T_\sigma \; (\tau \supset p \to c_\sigma \leq f_{p_\sigma}(\tau)).$$

Fix the least such p, and note that the standard c_σ-colored subtree of T_σ for p using f_{p_σ} exists and is isomorphic to $2^{<\mathbb{N}}$. Call this tree T. Let $p_{\sigma^\frown 0}$ and $p_{\sigma^\frown 1}$ be the two level one elements of T and let $T_{\sigma^\frown \varepsilon}$ be the subtree of T with root $p_{\sigma^\frown \varepsilon}$ for $\varepsilon \in \{0, 1\}$.

Since p_σ is arithmetically definable from the values of p_τ and c_τ for $\tau \subset \sigma$, ACA_0 proves the existence of the tree $S = \{p_\sigma \mid \sigma \in 2^{<\mathbb{N}}\}$. By the construction of S, given any increasing sequence of elements of S of the form

$$p_1 \subset p_2 \subset \cdots \subset p_n \subset p_{n+1} \subset p_{n+2},$$

we have $f_{p_n}(p_{n+1}) = f_{p_n}(p_{n+2})$, and so $f(p_1, \ldots, p_n, p_{n+1}) = f(p_1, \ldots, p_n, p_{n+2})$. Consequently, the function $g : [S]^n \to k$ defined for $p_{\sigma_1} \subset \cdots \subset p_{\sigma_n}$ by

$$g(p_{\sigma_1}, \ldots, p_{\sigma_n}) = f(p_{\sigma_1}, \ldots, p_{\sigma_n}, p_{\sigma_n^\frown 0})$$

indicates the color of any $n+1$-tuple extending $(p_{\sigma_1}, \ldots, p_{\sigma_n})$. By TT^n there is a subtree of S which is isomorphic to $2^{<\mathbb{N}}$, monochromatic for g, and so also monochromatic for f. ⊣

The use of ACA_0 in the preceding results is necessary, as shown by the reversal included in the following theorem.

THEOREM 1.5. *For $n \geq 3$ and $k \geq 2$, RCA_0 proves that the following are equivalent:*

(1) ACA_0.
(2) TT^n.
(3) TT^n_k.

PROOF. Theorems 1.3 and 1.4 show that (1) implies (2). Since (3) is a special case of (2), it remains only to show that (3) implies (1). Note that (3) implies Ramsey's theorem for n-tuples and two colors by the following argument. If $g : [\mathbb{N}]^n \to 2$, then we may define $f : [2^{<\mathbb{N}}]^n \to 2$ by setting $f(\sigma_1, \ldots, \sigma_n) = g(\mathrm{lh}(\sigma_1), \ldots, \mathrm{lh}(\sigma_n))$. From any monochromatic subtree for f, we can construct an infinite monochromatic set for g. Whenever $n \geq 3$, Ramsey's theorem for n-tuples and two colors implies ACA_0 (see Lemma III.7.5 of [10]), completing the proof. ⊣

We have shown that TT^3_2 implies ACA_0, but the exact strength of TT^2 and TT^2_2 remain open. Using the level coloring argument of Theorem 1.5, it easy to show that TT^2 implies Ramsey's theorem for pairs, but whether or not the converse is provable in RCA_0 remains open. See Section 3 for more comments on this.

§2. Computability theory. This section turns to computability theory, presenting bounds on the complexity of the monochromatic sets paralleling those of [5]. Since every coloring of $[\mathbb{N}]^n$ induces a coloring of $[2^{<\mathbb{N}}]^n$ via the level coloring technique in the proof of Theorem 1.5, Theorem 5.1 of [5] yields the following theorem.

THEOREM 2.1. *If $n \geq 2$ then there is a computable 2-coloring of $[2^{<\mathbb{N}}]^n$ with no Σ^0_n monochromatic subtree.*

A Π^0_2 bound. The proofs of the corresponding upper bound results are significantly more involved. Consider the following result, which is analogous to Theorem 4.2 of [5].

THEOREM 2.2. *Every computable finite coloring of pairs of comparable nodes of $2^{<\mathbb{N}}$ has a Π^0_2 monochromatic subtree that is isomorphic to $2^{<\mathbb{N}}$.*

PROOF. We will carry out the proof for two colors, and then indicate how to extend the result to an arbitrary finite number of colors. Suppose $f : [2^{<\mathbb{N}}]^2 \to \{\mathsf{red}, \mathsf{blue}\}$ is a computable two coloring of the pairs of comparable nodes of $2^{<\mathbb{N}}$. Any computable monochromatic subtree would be Π^0_2 definable, so for the remainder of the proof we may assume no computable monochromatic subtree exists.

Emulating the proof of Jockusch's Theorem 4.2 in [5], we will show that the complement of the desired monochromatic subtree is computably enumerable in $0'$ and then apply the strong hierarchy theorem [9]. Initially, we will need to enumerate the complement of an analog of the tree S in the proof of Theorem 1.3. This enumeration, also computable from $0'$, will be built using moving markers.

Intuitively, the markers used in this proof eventually settle on nodes corresponding to the p_σ nodes of S in Theorem 1.3. By making initial guesses at the associated colors and allowing for later revisions, we can execute the construction using only a $0'$ oracle. In this respect, this proof closely follows Jockusch's proof. However, arranging for the monochromatic tree to be isomorphic to $2^{<\mathbb{N}}$ complicates the selection of the nodes, especially when the color blue is assigned to a node.

As in the proof of Theorem 1.3, for each $\alpha \in 2^{<\mathbb{N}}$ we will have a marker p_α. We write p^s_α to indicate the location of p_α at stage s. Similarly, we will use the colors $c^s_\alpha \in \{\mathsf{red}, \mathsf{blue}\}$. At each stage, if $\alpha \subset \beta$ and p^s_α and p^s_β are in use, we require $p^s_\alpha \subset p^s_\beta$ and $f(p^s_\alpha, p^s_\beta) = c^s_\alpha$. As before, we will frequently write $f_{p_\alpha}(p_\beta)$ for $f(p_\alpha, p_\beta)$.

At each stage s, we will also have a finite set M^s consisting of all $\alpha \in 2^{<\mathbb{N}}$ such that p_α is in use, and a finite set E^s which is a correct initial segment of the complement of the analog of S. (M is for *map* and E is for *ejected*.) In selecting locations for newly introduced markers, we will be careful to avoid elements of E^s.

For each $\alpha \in 2^{<\mathbb{N}}$ and stage s, we will also have a tree T^s_α of possible extensions of p^s_α. Just as we intend for p^s_α to converge to a node in the analog of S, we intend for T^s_α to converge to a tree isomorphic to $2^{<\mathbb{N}}$. However, T^s_α may be a finite tree at some stages, due to erroneous selections. Regardless of the size of T^s_α, it is completely described by E^s together with a finite sequence of pairs called a *descriptor*. Descriptors are defined inductively as follows. The sequence of no pairs, $\langle\rangle$, is a descriptor for $2^{<\mathbb{N}}$. Writing $d(T^s_\alpha)$ for the descriptor of T^s_α, if $p \in T^s_\alpha$ then $d(T^s_\alpha)^\frown(p, \mathsf{red})$ is the descriptor for the tree obtained by following the algorithm for constructing the standard red subtree of T^s_α for p using f_p, avoiding all nodes in E^s. In executing the algorithm to find the standard red subtree, we will assume that all elements at level k are determined before any elements at level $k+1$. Similarly, $d(T^s_\alpha)^\frown(p, \mathsf{full})$ is the descriptor for the tree of all elements of T^s_α extending the least extension of p which lies above all elements of E^s. Call this least extension the root. Because of the way we will construct E^s, the root of the tree with descriptor $d(T^s_\alpha)^\frown(p, \mathsf{full})$ is always a proper extension of p. Because of the way descriptors are defined, for any s and α, T^s_α is either isomorphic to $2^{<\mathbb{N}}$ or finite. Since descriptors are always finite, they can be encoded by an integer and tagged onto markers.

In the following construction, the behavior of each marker is very limited. Initially, we place p_α and guess that red is the appropriate color for c_α. As long as no difficulties arise in locating extensions of p_α in standard red subtrees, p_α and c_α remain unchanged. If the search for extensions fails, then (some) p_α must have a full blue subtree for f_{p_α}. In this case, we change c_α to blue and attempt to move p_α to a successor of its current location. This move is a necessary complication, allowing us to decode a monochromatic subtree from the analog of S. If c_α is blue at stage s, then p_α will not be moved unless the descriptor $d(T^s_\alpha)$ is shortened or for some $\beta \subset \alpha$, p_β is modified.

Stage 0: Let $p^0_{\langle\rangle} = \langle\rangle$, $c^0_{\langle\rangle} = \mathsf{red}$, $d(T^0_{\langle\rangle}) = (\langle\rangle, \mathsf{red})$, $E^0 = \emptyset$, and $M^0 = \{\langle\rangle\}$. Thus, we have assigned the empty node the color red, will search for successors of this marker in the standard red subtree of $2^{<\mathbb{N}}$ for $\langle\rangle$ using $f_{\langle\rangle}$, have determined no elements in the complement of the analog of S, and have placed exactly one marker, corresponding to the location of $\langle\rangle$ in $2^{<\mathbb{N}}$. All other markers are unassigned.

Stage $s+1$: We will use two cases to describe the action at this stage.

Case 1: For each leaf $\beta \in M^s$, we can locate incomparable proper extensions $p_{\beta,0}$ and $p_{\beta,1}$ of p^s_β in T^s_β. Note that we can determine whether or not this case holds on the basis of finitely many queries to $0'$. When this case holds, do the following:

- For each leaf $\beta \in M^s$ and each $\varepsilon \in \{0,1\}$:
  - Set $p^{s+1}_{\beta^\frown\varepsilon} = p_{\beta,\varepsilon}$ and $c^{s+1}_{\beta^\frown\varepsilon} = \mathsf{red}$;
  - add all elements of M^s and $\beta^\frown\varepsilon$ to M^{s+1};
  - Let the descriptor for $T^{s+1}_{\beta^\frown\varepsilon}$ be $d(T^s_\beta)^\frown(p_{\beta,\varepsilon}, \mathsf{red})$.
- For all other α, set $p^{s+1}_\alpha = p^s_\alpha$, $c^{s+1}_\alpha = c^s_\alpha$, and $d(T^{s+1}_\alpha) = d(T^s_\alpha)$.
- Let $L = \max\{lh(p^{s+1}_\alpha) \mid \alpha \in M^{s+1}\}$, and set

$$E^{s+1} = E^s \cup \{\tau \in 2^{<\mathbb{N}} \mid lh(\tau) < L \wedge \forall \alpha \in M^{s+1}(\tau \neq p^{s+1}_\alpha)\}.$$

Case 2: Case 1 fails, so there is a leaf $\beta \in M^s$ with no incomparable proper extensions of p^s_β in T^s_β. Using $0'$ we can fix such a leaf β.

Intuitively, whenever this situation arises, we need to create a blue marker. For example, if c^s_β is red and we can find no such extensions, then we should change c^s_β to blue. Though not as obvious, blue markers with no extensions arise from erroneous red nodes in descriptors. To complicate matters, simply changing the color of a marker creates problems with extracting the final monochromatic tree from our analog of S. Consequently, in this case we will move some marker and color it blue.

We will search each tree B in a list for a pair of nodes $p_0 \subset p_1$ such that $\forall \tau \in B(p_1 \subset \tau \to f_{p_0}(\tau) = \mathsf{blue})$. The trees fall into two categories. If $\alpha \subseteq \beta$ and $c^s_\alpha = \mathsf{red}$, then the descriptor $d(T^s_\alpha)$ is of the form $d^\frown(p^s_\alpha, \mathsf{red})$. For each (possibly empty) sequence $p_{\alpha,0} \subset p_{\alpha,1} \subset \cdots \subset p_{\alpha,k}$ of nodes in the tree with descriptor $d^\frown(p^s_\alpha, \mathsf{full})$, add the tree with descriptor

$$d^\frown(p^s_\alpha, \mathsf{full})^\frown(p_{\alpha,0}, \mathsf{red})^\frown \dots ^\frown (p_{\alpha,k}, \mathsf{red})$$

to the list. If $\alpha \subseteq \beta$ and $c^s_\alpha = \mathsf{blue}$, then the descriptor $d(T^s_\alpha)$ may be of the form $d^\frown(p_{\alpha,0}, \mathsf{red})^\frown \dots ^\frown (p_{\alpha,k}, \mathsf{red})^\frown(p_{\alpha,k+1}, \mathsf{full})^\frown(p_{\alpha,k+2}, \mathsf{full})$ where $k \geq 0$. If so, then for each $j \leq k$ add the tree with the descriptor

$$d^\frown(p_{\alpha,0}, \mathsf{red})^\frown \dots ^\frown (p_{\alpha,j-1}, \mathsf{red})^\frown(p_{\alpha,j}, \mathsf{full})$$

to the list. Search all trees in the list until p_0 and p_1 as described at the beginning of this paragraph are found. (We allow p_0 to be the root of a tree; in particular, if the descriptor of B terminates in $(p_{\alpha,j}, \mathsf{full})$, we may let p_0 be the root of B, which is the least extension of $p_{\alpha,j}$ lying above all elements of E^s.) A proof that this search always terminates is given in Claim 1 below. Remember the descriptor of the tree for which the search succeeded, including the value of α.

Suppose we have found p_0, p_1, and $\alpha \subseteq \beta$ as specified in the preceding paragraph. Do the following:

- Let $M^{s+1} = \{\gamma \in M^s \mid \gamma \not\supset \alpha\}$.
- For $\gamma \in M^{s+1} - \{\alpha\}$, let $p^{s+1}_\gamma = p^s_\gamma$, $c^{s+1}_\gamma = c^s_\gamma$, and $d(T^{s+1}_\gamma) = d(T^s_\gamma)$.
- Let $E^{s+1} = E^s$.

Denote the descriptor of the tree for which the search succeeded by d_0 and do the following:

- Set $c^{s+1}_\alpha = \mathsf{blue}$ and $p^{s+1}_\alpha = p_0$.
- Let $d(T^{s+1}_\alpha) = d_0^\frown(p_0, \mathsf{full})^\frown(p_1, \mathsf{full})$.

This completes the construction. The next four claims show that the construction yields the desired enumeration of the complement of the analog of S.

Claim 1: The search described in Case 2 of Stage $s + 1$ always terminates.

Proof of Claim 1: Suppose there is a leaf $\beta \in M^s$ with no proper extensions of p^s_β in T^s_β. The absence of extensions indicates that T^s_β is not isomorphic to $2^{<\mathbb{N}}$, so T^s_β must be finite. Each initial segment of the descriptor $d(T^s_\beta)$ is a descriptor for some tree. Since the empty sequence is the descriptor for $2^{<\mathbb{N}}$, there must be a first pair (p, c) such that the initial segment of $d(T^s_\beta)$ terminating in (p, c) describes a finite tree.

If d is the descriptor for a tree isomorphic to $2^{<\mathbb{N}}$ containing p, then $d^\frown(p, \mathsf{full})$ is also isomorphic to $2^{<\mathbb{N}}$. Thus the pair (p, c) in the preceding paragraph must be of the form (p, red). The node p must either be a p_α for some $\alpha \subseteq \beta$, or a node on a path leading to some p_α for which $c^s_\alpha = \mathsf{blue}$. We will consider these situations in order.

First suppose (p, c) is of the form (p_α, red) for some $\alpha \subseteq \beta$ where $c^s_\alpha = \mathsf{red}$. Then $d(T^s_\beta)$ is of the form $d^\frown(p^s_\alpha, \mathsf{red})^\frown\hat{d}$. (Note that the following holds when $\hat{d} = \emptyset$.) The tree with descriptor $d^\frown(p^s_\alpha, \mathsf{full})$ is isomorphic to $2^{<\mathbb{N}}$. Suppose by way of contradiction that the search fails. That is, given any (possibly empty) sequence $p_{\alpha,0} \subset p_{\alpha,1} \subset \cdots \subset p_{\alpha,k}$ of nodes in the tree with descriptor $d^\frown(p^s_\alpha, \mathsf{full})$, if we let B be the tree with descriptor

$$d^\frown(p^s_\alpha, \mathsf{full})^\frown(p_{\alpha,0}, \mathsf{red})^\frown \dots ^\frown (p_{\alpha,k}, \mathsf{red}),$$

then there is no pair $p_0 \subset p_1$ in B such that $\forall \tau \in B\ (p_1 \subset \tau \to f_{p_0}(\tau) = \mathsf{blue})$. Consequently, for any such B, p_0, and p_1, there is a $\tau \supset p_1$ in B such that $f_{p_0}(\tau) = \mathsf{red}$. We can use this feature to construct a computable monochromatic red tree as follows.

Let B denote the tree with descriptor $d^\frown(p^s_\alpha, \mathsf{full})$. Let $q_{\langle\rangle}$ denote the root of this tree; that is, $q_{\langle\rangle}$ is the least extension of p^s_α lying above all elements of E^s. By the preceding paragraph, there is no $p_1 \supset q_{\langle\rangle}$ such that $\forall \tau \in B\ (p_1 \subset \tau \to f_{q_{\langle\rangle}}(\tau) = \mathsf{blue})$. Let $B_{\langle\rangle}$ be the tree with descriptor $d^\frown(q_{\langle\rangle}, \mathsf{red})$. By Lemma 1.1, $B_{\langle\rangle}$ is isomorphic to $2^{<\mathbb{N}}$. Suppose q_α and B_α are defined and B_α is isomorphic to $2^{<\mathbb{N}}$. Let $q_{\alpha^\frown 0}$ and $q_{\alpha^\frown 1}$ be the first pair of incomparable elements of B_α. For each $\varepsilon \in \{0, 1\}$, treating $q_{\alpha^\frown\varepsilon}$ as p_0, by the preceding paragraph, the tree with descriptor

$$d^\frown(q_{\langle\rangle}, \mathsf{red})^\frown \ldots (q_\alpha, \mathsf{red})^\frown(q_{\alpha^\frown\varepsilon}, \mathsf{red})$$

(which will be $B_{\alpha^\frown\varepsilon}$) is isomorphic to $2^{<\mathbb{N}}$. Note that if $\alpha \subset \beta$, then $q_\beta \in B_\alpha$, so $f_{q_\alpha}(q_\beta) = \mathsf{red}$. Thus $\{q_\alpha \mid \alpha \in 2^{<\mathbb{N}}\}$ is a computable monochromatic tree for f.

The existence of a computable monochromatic tree for f contradicts the first paragraph of the proof of this theorem. Consequently, when c^s_α is red, the search must terminate, completing the proof for this situation.

Now suppose (p, c) is of the form $(p_{\alpha,j}, \mathsf{red})$ for some $\alpha \subseteq \beta$ where $c^s_\alpha = \mathsf{blue}$. Then $d(T^s_\beta)$ is of the form

$$d^\frown(p_{\alpha,j}, \mathsf{red})^\frown \ldots^\frown (p_{\alpha,k}, \mathsf{red})^\frown(p_{\alpha,k+1}, \mathsf{full})^\frown(p_{\alpha,k+2}, \mathsf{full})^\frown \hat{d}.$$

(Note that the following holds if $\hat{d} = \emptyset$ and also if $j = k$.) Since $(p_{\alpha,j}, \mathsf{red})$ was the first pair yielding a finite tree, the tree with descriptor $d^\frown(p_{\alpha,j}, \mathsf{full})$ is isomorphic to $2^{<\mathbb{N}}$. As in the preceding paragraphs, if we cannot find p_0 and p_1 satisfying the search, then we can construct a computable monochromatic tree, yielding a contradiction. This completes the proof of the claim that the search always succeeds.

Claim 2: For every $\alpha \in 2^{<\mathbb{N}}$, the following limits exist: $\lim_s p^s_\alpha = p_\alpha$, $\lim_s c^s_\alpha = c_\alpha$, and $\lim_s d(T^s_\alpha) = d_\alpha$.

Proof of Claim 2: Consider the possible behaviors for $p^s_{\langle\rangle}$. If $p^s_{\langle\rangle}$ is never moved, then $p_{\langle\rangle} = \langle\rangle$, $c_{\langle\rangle} = \mathsf{red}$, and $T_{\langle\rangle}$ is the tree with descriptor $(\langle\rangle, \mathsf{red})$. At some stage s, $p^s_{\langle\rangle}$ may be moved, in which case $c^s_{\langle\rangle} = \mathsf{blue}$ and $T^s_{\langle\rangle}$ has a new descriptor of some length n. At any successive stage t, $c^t_{\langle\rangle} = \mathsf{blue}$ and if $p^t_{\langle\rangle}$ moves, then the descriptor of $T^t_{\langle\rangle}$ is shortened. Consequently, the process must eventually converge to a limiting p_α and d_α.

If p_α, c_α and d_α have achieved their limits at stage s, then the only allowable changes in $p^t_{\alpha^\frown\varepsilon}$, $c^t_{\alpha^\frown\varepsilon}$, and $d(T^t_{\alpha^\frown\varepsilon})$, for $\varepsilon \in \{0, 1\}$ and $t > s$ are exactly those in the preceding paragraph. Thus, all the markers must achieve their limits.

Furthermore, each time Case 2 of Stage $s + 1$ is executed, either M^s is decreased in size, or for some $\alpha \in M^s$, either c^s_α is changed from red to blue or the descriptor of T^s_α is shortened. Since M^s and all descriptors are finite, Case 1 of Stage $s + 1$ must occur infinitely often. Consequently, once p_α achieves its limit, $p^s_{\alpha^\frown\varepsilon}$ must eventually be introduced, and will also achieve its limit. Thus, for every $\alpha \in 2^{<\mathbb{N}}$, p_α and c_α are assigned, and T_α is a nonempty tree.

Claim 3: For every $\alpha \in 2^{<\mathbb{N}}$, if $\alpha \subset \beta$ then $p_\alpha \subset p_\beta$, $d_\alpha \subset d_\beta$, and $f_{p_\alpha}(p_\beta) = c_\alpha$.

Proof of Claim 3: Detailed examination of the construction shows that for each s, if $\alpha \subset \beta \in M^s$, then $p^s_\alpha \subset p^s_\beta$, $p^s_\beta \in T^s_\alpha$, and $d(T^s_\beta)$ extends $d(T^s_\alpha)$. Consequently, $f_{p^s_\alpha}(p^s_\beta) = c^s_\alpha$ and $T^s_\beta \subset T^s_\alpha$. Since these relationships are preserved at each stage, they must hold in the limit.

Claim 4: $\overline{\{p_\alpha \mid \alpha \in 2^{<\mathbb{N}}\}} = \bigcup_s E^s$.

Proof of Claim 4: As shown in Claims 2 and 3, for each $\alpha \in 2^{<\mathbb{N}}$, p_α exists, and if $\beta \supset \alpha$, then $p_\beta \supset p_\alpha$. Thus, the length of p^s_α can be forced to exceed any fixed bound in $\mathbb{N}$ by picking suitably large values of s and α. By virtue of the definition of E^{s+1} in Case 1 of Stage $s+1$ (which occurs infinitely often), $\bigcup_s E_s \supseteq \overline{\{p_\alpha \mid \alpha \in 2^{<\mathbb{N}}\}}$. Since each T^s_α is defined so as to avoid elements of E^s, no p_α can be an element of $\bigcup_s E_s$.

Summarizing the proof to this point, we have a subtree $\{p_\alpha \mid \alpha \in 2^{<\mathbb{N}}\}$ which is isomorphic to $2^{<\mathbb{N}}$, and satisfies $f_{p_\alpha}(p_\beta) = c_\alpha$ whenever $\alpha \subset \beta$. Furthermore, the complement of this set is the union of finite sets each of which can be computed with the aid of $0'$. Consequently, the complement is computably enumerable in $0'$. Thus, we have found an analog of S whose complement is c.e. in $0'$. It remains to extract a monochromatic subtree and describe an enumeration for its complement.

First, suppose there is an α such that for all $\beta \supseteq \alpha$, $c_\beta = \mathsf{red}$. Then the subtree $T = \{p_\beta \mid \beta \supseteq \alpha\}$ is the desired monochromatic red tree. To enumerate the complement of T, repeat the construction, adding p^s_γ to E^s whenever $p^s_\gamma \not\supset p_\alpha$. Since the complement of T is computably enumerable in $0'$, by the strong hierarchy theorem T is Π^0_2 definable.

Finally, suppose that for every α there is a $\beta \supset \alpha$ such that $c_\beta = \mathsf{blue}$. We repeat the construction, adding new markers $\{t_\alpha \mid \alpha \in 2^{<\mathbb{N}}\}$, new finite subsets of the complement of the monochromatic tree $\{F^s \mid s \in \mathbb{N}\}$, and new maps $\{N^s \mid s \in \mathbb{N}\}$ where N^s contains those α for which t_α is attached at stage s.

Run the construction until the first c^s_α is set to blue. Let $t^s_{\langle\rangle} = p^s_\alpha$, $N^s = \{\langle\rangle\}$, and $F^s = \{p^s_\beta \mid \beta \in M^s \wedge \beta \neq \alpha\}$. Note that if $p^s_\beta \in F^s$, then $c^s_\beta = \mathsf{red}$.

At stage $s+1$, execute the process for constructing S, and then consider three cases.

Case 1: For each leaf $\beta \in N^s$, given that $t^s_\beta = p^s_\gamma$, suppose we can locate extensions $p^s_{\delta_0} \supset p^s_{\gamma^\frown 0}$ and $p^s_{\delta_1} \supset p^s_{\gamma^\frown 1}$ such that $c^s_{\delta_0} = \mathsf{blue}$ and $c^s_{\delta_1} = \mathsf{blue}$. In this case, do the following:

- For each leaf $\beta \in N^s$ and each $\varepsilon \in \{0, 1\}$,
  - set $t^{s+1}_{\beta^\frown\varepsilon} = p^s_{\delta_\varepsilon}$, and
  - add all elements of N^s and $\beta^\frown\varepsilon$ to N^{s+1}.
- For all other $\alpha \in N^s$, set $t^{s+1}_\alpha = t^s_\alpha$.
- Define F^{s+1} by the equation

$$F^{s+1} = F^s \cup \{\tau \in 2^{<\mathbb{N}} \mid \exists \alpha \in M^{s+1}(\tau = p^{s+1}_\alpha) \wedge \forall \alpha \in N^{s+1}(\tau \neq t^{s+1}_\alpha)\}.$$

Case 2: For some $\beta \in N^s$, a predecessor of t^s_β is moved. Let p^s_δ be this predecessor node. Because of the way nodes are added in Case 1, there is a unique least $\alpha \subseteq \beta$ such that $t^s_\alpha \supseteq p^s_\delta$. Find this α, and do the following:

- Let $N^{s+1} = \{\gamma \in N^s \mid \gamma \not\supset \alpha\}$.

- For $\gamma \in N^{s+1} - \{\alpha\}$, let $t_\gamma^{s+1} = t_\gamma^s$.
- Let $F^{s+1} = F^s$.
- Let $t_\alpha^{s+1} = p_\delta^{s+1}$.

This last step is possible because p_δ^s was moved to a previously unassigned location, guaranteeing that $p_\delta^s \notin F^s$. Furthermore, since p_δ^s moved, $c_\delta^{s+1} = \mathsf{blue}$.

Case 3: If neither Case 1 nor Case 2 holds, let $N^{s+1} = N^s$, $F^{s+1} = F^s$, and $t_\alpha^{s+1} = t_\alpha^s$ for all $\alpha \in N^{s+1}$.

It is not difficult to show that for each α, the limit $t_\alpha = \lim_s t_\alpha^s$ exists, and that it marks some p_γ such that $c_\gamma = \mathsf{blue}$. Also, $\overline{\{t_\alpha \mid \alpha \in 2^{<\mathbb{N}}\}} = \bigcup_s E_s \cup \bigcup_s F_s$, so $\{t_\alpha \mid \alpha \in 2^{<\mathbb{N}}\}$ is the complement of a set which is computably enumerable in $0'$. Thus, in this situation, $\{t_\alpha \mid \alpha \in 2^{<\mathbb{N}}\}$ is a blue monochromatic tree which is Π_2^0 definable.

We have completed the proof for two colors. To extend the result to an arbitrary finite number of colors, we modify the construction, assigning colors 0 through k in order. Initially c_α^s is assigned 0. In Case 2 of Stage $s+1$, if c_α^s is assigned $j < k$ then we search for $p_0 \subset p_1$ such that $\forall \tau \in B(p_1 \subset \tau \to j < f_{p_0}(\tau))$ and set $c_\alpha^{s+1} = j+1$. The color k behaves like blue in the original construction.

The claims are proved as before, yielding an analog of S with each c_α in $\{0, \dots, k\}$. Pick the least j such that there is an α such that for all $\beta \supseteq \alpha$, $c_\beta \leq j$. If $j = 0$, then $T = \{p_\beta \mid \beta \supseteq \alpha\}$ is the desired monochromatic subtree. Otherwise, rerun the construction using new markers to extract a j-colored subtree. The Π_2^0 bounds follow as before. ⊣

The Π_n^0 bound. In the proof of Theorem 2.7, the preceding theorem acts as a base case for deducing the bounds for colorings of $n+1$-tuples. In the argument, our goal is to produce a subtree with a controlled level of complexity such that the coloring of any $n+1$-tuple depends only on the first n elements. The color blocks defined below aid in controlling the complexity of the desired tree. In all of the following, let $f : 2^{<\mathbb{N}} \to k$. Also, for each $\alpha \in 2^{<\mathbb{N}}$, we let T_α denote the full subtree of $2^{<\mathbb{N}}$ extending α, that is $T_\alpha = \{\tau \in 2^{<\mathbb{N}} \mid \alpha \subseteq \tau\}$.

Definition 1. A *color block for f* is a set of $k+1$ chains with the following properties:

1. Each chain consists of k nodes, exactly one of each color.
2. Any two nodes chosen from distinct chains are incomparable.

Definition 2. For $c < k$, we say f has a *full c-avoiding tree* if there is some node τ such that for all $\sigma \supset \tau$, $f(\sigma) \neq c$.

Lemma 2.3. *Either there is a c such that f has a full c-avoiding tree or there is a color block for f.*

Proof. We search $2^{<\mathbb{N}}$ for a color block for f. If the search fails, it is because we have discovered a full c-avoiding tree for some c.

Begin by selecting $k+1$ pairwise incomparable nodes in $2^{<\mathbb{N}}$, the least $k+1$ such nodes will do. For each node σ in this collection, do the following:

> Let $\sigma_0 = \sigma$. For $0 \leq i \leq k-2$, given σ_i, let σ_{i+1} be the least node extending σ_i with $f(\sigma_{i+1}) \neq f(\sigma_j)$ for $j \leq i$, if such a node exists.

(Note that establishing the existence of such a σ_{i+1} requires a query to $0'$ when f is computable.)

If this search fails for some i, it is because all nodes τ extending σ_i have $f(\tau) = f(\sigma_j)$ for some $j \leq i$, thus T_{σ_i} is c-avoiding for any $c \notin \{f(\sigma_j) | j \leq i\}$.

If the search does not fail, we have successfully completed the construction of $k+1$ non-intersecting chains, each consisting of k distinctly colored nodes. Further, when f is computable, this construction may be carried out with only finitely many queries to $0'$. $\dashv$

Definition 3. If f is a k-coloring and S is a subset of the colors, an *S color block for f* is a collection of $|S| + 1$ chains, each of which is composed of exactly one node from each color in S, satisfying the incomparability requirement in the definition of a color block.

Given a coloring of $[2^{<\mathbb{N}}]^{n+1}$ and an $\alpha \in 2^{<\mathbb{N}}$, we can color nodes of T_α by fixing n-tuples at or below α and assigning colors to each node above α. We present some definitions and lemmas about collections of colorings which could be induced in this fashion. Intuitively, the $\langle f_\alpha \rangle$-forest defined below consists of finite approximations to the tree used for reducing $n+1$-tuples to n-tuples.

For what follows, assume that for each $\alpha \in 2^{<\mathbb{N}}$, the function $f_\alpha : T_\alpha \to k_\alpha$ is a finite coloring of T_α. For a node α of length n, denote the initial segments of α by $\langle\rangle = \alpha_0 \subset \alpha_1 \subset \alpha_2 \subset \cdots \subset \alpha_n = \alpha$. Using this notation, define $k^*_\alpha = \{(j_0, \dots, j_n) \mid \forall i \; j_i < k_{\alpha_i}\}$, and the functions $f^*_\alpha : T_\alpha \to k^*_\alpha$ by $f^*_\alpha(\tau) = (f_{\alpha_0}(\tau), f_{\alpha_1}(\tau), \dots, f_{\alpha_n}(\tau))$.

Definition 4. An $\langle f_\alpha \rangle$-forest is defined in terms of a sequence of levels $\langle L_i \rangle_{i \in \mathbb{N}}$. The levels are defined as follows: Let $L_0 = \{\langle\rangle\}$. Note that $f^*_{\langle\rangle} = f_{\langle\rangle}$ and that the range of $f_{\langle\rangle}$ is $\{0, 1, \dots, k_{\langle\rangle} - 1\}$. Attach the tag $(\{0, 1, \dots, k_{\langle\rangle} - 1\}, \langle\rangle)$ to $\langle\rangle$.

Suppose that L_n is defined. If some $\sigma \in L_n$ has a tag, then do the following:

1. If the tag on σ is (S, τ) and $|S| > 1$, then check for an S color block for f^*_σ above τ using the algorithm from the proof of Lemma 2.3.
   (a) If such a color block is located, add all the nodes in the color block to L_{n+1}. Whenever μ is the supremum of a chain in the color block, define $S_\mu = \{(v_0, v_1, \dots, v_{|\sigma|}, \dots v_{|\mu|}) \in k^*_\mu \mid (v_0, v_1, \dots, v_{|\sigma|}) \in S\}$ and attach the tag (S_μ, μ) to μ. Remove the tag from σ.
   (b) If no such color block is found, then for some $c \in S$ and some β above τ there is a c-avoiding tree for f^*_σ above β. Change the tag on σ to $(S - \{c\}, \beta)$.
2. If the tag on σ is (S, τ) and $|S| = 1$, then the tree above τ is monochromatic for f^*_σ. Add $\tau^\frown 0$ and $\tau^\frown 1$ to L_{n+1}. For each $\varepsilon \in \{0, 1\}$, define $S_{\tau^\frown\varepsilon} = \{(v_0, \dots, v_{|\sigma|}, \dots, v_{|\tau^\frown\varepsilon|}) \in k^*_{\tau^\frown\varepsilon} \mid (v_0, \dots, v_\sigma) \in S\}$ and attach the tag $(S_{\tau^\frown\varepsilon}, \tau^\frown\varepsilon)$ to $\tau^\frown\varepsilon$. Remove the tag from σ.

If no element of L_n has a tag, then the calculation of L_{n+1} is complete, and L_{n+1} is defined. Note that this process always terminates, and that for each n, L_n is finite.

The $\langle f_\alpha \rangle$-forest consists of all finite binary subtrees T such that:

(1) the k^{th} level of T is empty or contains exactly 2^k elements from L_k, and
(2) if σ, τ_1, and τ_2 are nodes of T and τ_1 and τ_2 both extend σ, then $f^*_\sigma(\tau_1) = f^*_\sigma(\tau_2)$.

Observe that by requiring (2), we ensure that when T is a tree in an $\langle f_\alpha \rangle$-forest and τ is in T, then f_τ is monochromatic on the nodes of T above τ. Lemmas 2.4, 2.5, and 2.6 complete the construction of the tree needed for the proof of Theorem 2.7. Let $[T]$ denote some canonical integer code for a finite tree T.

LEMMA 2.4. *Suppose that for each $\alpha \in 2^{<\mathbb{N}}$, $f_\alpha : T_\alpha \to k_\alpha$ is a finite coloring of T_α. If $\langle f_\alpha \rangle_{\alpha \in 2^{<\mathbb{N}}}$ is a computable collection of computable finite colorings, then the $\langle f_\alpha \rangle$-forest is computable from $0'$. Furthermore, there is a function g such that $g \le 0'$ and for all n, if T is a height n element of the $\langle f_\alpha \rangle$-forest, then $[T] \le g(n)$.*

PROOF. Let T be a finite tree of height n, and $\mathscr{F}$ the $\langle f_\alpha \rangle$-forest. To determine whether $T \in \mathscr{F}$, we first (computably) check that it satisfies the monochromaticity requirement in (2) in the definition above and is isomorphic to $2^{\le n}$. Then check for each $k \le n$ that

$$(\forall \sigma \in T)[\sigma \text{ has } k \text{ predecessors in } T \Longrightarrow \sigma \in L_k].$$

Thus membership in $\mathscr{F}$ reduces to finitely many questions about membership in the sets L_k for $k \le n$. By the proof of Lemma 2.3, the construction of the (finite) set L_k can be carried out with the assistance of $0'$.

Since each set L_k is finite, there are only finitely many trees of a given height n belonging to $\mathscr{F}$. With the aid of $0'$, we may find these and set $g(n)$ to exceed the largest of their canonical indices. ⊣

LEMMA 2.5. *If $\langle f_\alpha \rangle$ is as in Lemma 2.4, then there is a subtree T such that the following hold:*

1. *T is isomorphic to $2^{<\mathbb{N}}$.*
2. *For each $\sigma \in T$, f_σ is constant on $\{\tau \in T \mid \tau \supset \sigma\}$.*
3. *$T' \le 0''$.*

PROOF. Let $\mathscr{F}$ denote the $\langle f_\alpha \rangle$-forest and order its elements by inclusion. By Lemma 2.4, $\mathscr{F}$ has the structure of a finitely branching tree bounded by a function g, and both $\mathscr{F}$ and g can be computed from $0'$.

Once we show below that $\mathscr{F}$ is infinite, we can apply the relativized Low Basis Theorem [6] to obtain from $\mathscr{F}$ a path P such that $P' \le 0''$. The desired tree T is the union of the elements in this path. A node of length n on the path is a finite tree isomorphic to $2^{\le n}$ and they form a nested (increasing) sequence, so their union T is indeed isomorphic to $2^{<\mathbb{N}}$. Since $T \le P$ and $P' \le 0''$, we have $T' \le 0''$.

Note that the definition of an $\langle f_\alpha \rangle$-forest ensures that for all $\sigma \in T$, f^*_σ is constant on $\{\tau \in T \mid \tau \supset \sigma\}$ and so f_σ is also constant on this set.

To see that $\mathscr{F}$ is infinite suppose by way of contradiction that there is an upper bound on the height of elements of $\mathscr{F}$. In this case, we may find a sequence of colorings $\langle h_\alpha \rangle$ such that the maximum height of a tree in the $\langle h_\alpha \rangle$-forest is minimal among all choices of colorings. Let H be a tree from the $\langle h_\alpha \rangle$-forest that has this maximal height, which we will denote by n. It follows from Lemma 2.3 and the definition of $\langle f_\alpha \rangle$-forest that $n \ge 1$. Let L_1 be the first level of the $\langle h_\alpha \rangle$-forest. We consider two cases.

First suppose L_1 consists of nodes taken from a monochromatic subtree for $h_{\langle\rangle}$; denote these by τ_0 and τ_1. For each nonempty $\alpha \in 2^{<\mathbb{N}}$, define $h^{\tau_0}_\alpha$ by $h^{\tau_0}_\alpha(\beta) = h_{\tau_0^\frown \alpha}(\tau_0^\frown \beta)$ and also define $h^{\tau_0}_{\langle\rangle}(\beta) = h^*_{\tau_0}(\tau_0^\frown \beta)$. Define $h^{\tau_1}_\alpha$ similarly, and note that for $i \in \{0, 1\}$ the trees of the $\langle h^{\tau_i}_\alpha \rangle$-forest are the extensions of τ_i in the $\langle h_\alpha \rangle$-forest.

By our choice of n, the $\langle h_\alpha^{\tau_0}\rangle$-forest and the $\langle h_\alpha^{\tau_1}\rangle$-forest each contain a tree of height n; call them T_{τ_0} and T_{τ_1}. Then $\{\langle\rangle\} \cup T_{\tau_0} \cup T_{\tau_1}$ is a tree in the $\langle h_\alpha\rangle$-forest of height $n+1$, contradicting the choice of n.

Now suppose L_1 consists of nodes in a color block for $h_{\langle\rangle}$ and let $\mu_0, \dots, \mu_j$ be the maximal elements of the $j+1$ chains in L_1. Note here that the cardinality of the range of $h_{\langle\rangle}$ on nodes in and above the chains is j. As in the previous paragraph, construct the induced sequences of colorings for each μ_i, and a monochromatic tree M_{μ_i} of height n for each μ_i. Two of these, say M_{μ_i} and M_{μ_j}, must agree in the first component of their coloring. Pick σ_i in the chain below μ_i so that $\{\sigma_i\} \cup (M_{\mu_i} - \{\mu_i\})$ is monochromatic for $h_{\langle\rangle}$. Choose σ_j for M_{μ_j} similarly, and note that $\{\langle\rangle, \sigma_i, \sigma_j\} \cup (M_{\mu_i} - \{\mu_i\}) \cup (M_{\mu_j} - \{\mu_j\})$ is a tree of height at least $n+1$ in the $\langle h_\alpha\rangle$-forest, contradicting the choice of n and completing the proof that $\mathscr{F}$ is infinite. ⊣

Lemma 2.6. *Suppose $n > 1$ and $f : [2^{<\mathbb{N}}]^{n+1} \to k$ is computable. There is a tree T which is isomorphic to $2^{<\mathbb{N}}$ such that the following hold*:

1. $T' \le 0''$.
2. *If $\sigma_1, \dots, \sigma_n$ is a sequence of n comparable elements of T and τ_1 and τ_2 are extensions of σ_n, then $f(\sigma_1, \dots, \sigma_n, \tau_1) = f(\sigma_1, \dots, \sigma_n, \tau_2)$.*

Proof. Define a computable family of colorings $\langle f_\alpha\rangle_{\alpha \in 2^{<\mathbb{N}}}$ as follows. If $\mathrm{lh}(\alpha) < n$, let $f_\alpha(\tau) = 0$ for all τ. If $\mathrm{lh}(\alpha) \ge n$, let $\vec{\sigma}_1, \dots, \vec{\sigma}_m$ be an enumeration of the n-tuples of nodes at or below α. For $\tau \supset \alpha$, let $f_\alpha(\tau) = \prod_{j \le m} \mathsf{pr}(\vec{\sigma}_j)^{f(\vec{\sigma}_j, \tau)}$, where $\mathsf{pr}(\vec{\sigma})$ denotes the rth prime for some canonical code r for $\vec{\sigma}$.

Apply Lemma 2.5 to $\langle f_\alpha\rangle$ to obtain a tree T isomorphic to $2^{<\mathbb{N}}$ with $T' \le 0''$. Let $\vec{\gamma}$ denote an increasing n-tuple $\gamma_1, \dots, \gamma_n$ of comparable elements of T. By Lemma 2.5, f_{γ_n} is constant on $\{\tau \in T \mid \tau \supset \gamma_n\}$. Let $\tau_1, \tau_2 \in T$ extend γ_n. Then $f_{\gamma_n}(\tau_1) = f_{\gamma_n}(\tau_2)$. From the definition of f_{γ_n}, the prime power factors corresponding to the n-tuple $\vec{\gamma}$ must agree, yielding:

$$\mathsf{pr}(\vec{\gamma})^{f(\vec{\gamma}, \tau_1)} = \mathsf{pr}(\vec{\gamma})^{f(\vec{\gamma}, \tau_2)}.$$

The exponents in the preceding equation must match, so $f(\vec{\gamma}, \tau_1) = f(\vec{\gamma}, \tau_2)$ as desired. ⊣

Finally, we have assembled all the machinery to prove the analog of Theorem 5.5 in [5].

Theorem 2.7. *If $f : [2^{<\mathbb{N}}]^n \to k$ is computable, then there is a Π^0_n monochromatic subtree isomorphic to $2^{<\mathbb{N}}$.*

Proof. Essentially quoting the proof of Theorem 5.5 of [5], we use induction on n. The case $n = 1$ follows from Theorem 1.2 and $n = 2$ follows from Theorem 2.2. Suppose the theorem holds for some $n \ge 2$, we will prove it for $n+1$. Let $f : [2^{<\mathbb{N}}]^{n+1} \to k$ be computable. Find T as in Lemma 2.6. Given any sequence $\sigma_0, \dots, \sigma_{n-1}$ of comparable elements of T, let σ_n be the least extension of σ_{n-1} in T and define $\hat{f}(\sigma_0, \dots, \sigma_{n-1}) = f(\sigma_0, \dots, \sigma_{n-1}, \sigma_n)$. Note that $\hat{f}$ is computable from T. By the induction hypothesis, there is a monochromatic tree $\hat{T}$ for $\hat{f}$ which is Π^0_n in T. Since $\hat{T}$ is monochromatic for f, it remains only to show that $\hat{T}$ is Π^0_{n+1}. Since $\hat{T}$ is Π^0_n in T, there is a T-computable $(n+1)$-place predicate R such that for all τ

$$\tau \in \hat{T} \leftrightarrow \forall x_1 \dots Qx_n\ R(\tau, x_1, \dots, x_n)$$

where $Qx_n R$ is one of $\exists x_n$ and $\forall x_n$. The predicate $Qx_n R$ is computable in T' and hence in $0''$. Applying Post's hierarchy theorem (e.g., Theorem VIII(b) in §14.5 of [9]), we may replace $Qx_n R$ by either a Σ^0_3 or a Π^0_3 predicate, depending on whether $Qx_n R$ is $\exists x_n$ or $\forall x_n$. The resulting predicate is the required Π^0_{n+1} definition of $\hat{T}$. ⊣

§**3. Extensions and questions.** We first note that all the preceding results can be extended to much broader classes of trees. For example, one could replace all the binary trees in this paper with ternary trees. The extensions are based on the fact that any subtree of $\mathbb{N}^{<\mathbb{N}}$ can be computably embedded into a copy of $2^{<\mathbb{N}}$. Consequently, TT^n_k can be extended as follows.

ET^n_k: If T contains a subtree isomorphic to $2^{<\mathbb{N}}$ and R is any subtree of $\mathbb{N}^{<\mathbb{N}}$ containing an infinite path, then every k-coloring of $[T]^n$ contains a subtree S isomorphic to R such that $[S]^n$ is monochromatic.

All the reverse mathematics and computability results presented previously for TT^n_k hold identically for ET^n_k. The requirement that R contains an infinite path insures that the reversals and computability theoretic lower bounds hold.

We conclude with a short list of questions.

1. Is TT^1 stronger than $\mathsf{B}\Pi^0_1$? Is it weaker than induction for Σ^0_2 formulas?
2. Does TT^2 imply ACA_0? Can Seetapun's result (as presented in [4]) be adapted to show that this is not the case?
3. What is the relative strength of TT^2 and TT^2_2? Can the work in [1] be adapted to address this?
4. To what degree can trees be replaced with other partial orders? Is there a Ramsey theorem on some class of partial orders where the theorem for pairs is equivalent to ACA_0?

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