Partitions of trees and ACA_0'

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Abstract

We show that a version of Ramsey's theorem for trees for arbitrary exponents is equivalent to the subsystem ACA_0' of reverse mathematics.

In [1], a version of Ramsey's theorem for trees is analyzed using techniques from computability theory and reverse mathematics. In particular, it is shown that for each standard integer $n \geq 3$, the usual Ramsey's theorem for n-tuples is equivalent to the tree version for n-tuples. The main result of this note shows that the universally quantified versions of these forms of Ramsey's theorem are also equivalent. Because there are so few examples of proofs involving ACA_0' in the literature, we have included a somewhat detailed exposition of the proof.

The main subsystems of second order arithmetic used in this paper are RCA_0, which includes a comprehension axiom for computable sets, and ACA_0, which appends a comprehension axiom for sets definable by arithmetical formulas. For details on the axiomatization of these subsystems, see [4]. More about the subsystem ACA_0' appears below.

If $2^{<\mathbb{N}}$ is the full binary tree of height ω, we may identify each node with a finite sequence of zeros and ones. We refer to any subset of the nodes as a subtree, and say that a subtree S is isomorphic to $2^{<\mathbb{N}}$ if every node of S has exactly two immediate successors in S. Formally, $S \subseteq 2^{<\mathbb{N}}$ is isomorphic to $2^{<\mathbb{N}}$ if and only if there is a bijection $b : 2^{<\mathbb{N}} \to S$ such that for all $\sigma, \tau \in 2^{<\mathbb{N}}$, we have $\sigma \subseteq \tau$ if and only if $b(\sigma) \subseteq b(\tau)$. (For sequences, $\sigma \subseteq \tau$ means σ is an initial segment of τ, and $\sigma \subset \tau$ means σ is a proper initial segment of τ.)

For any subtree S, we write $[S]^n$ for the set of linearly ordered n-tuples of nodes in S. All the nodes in any such n-tuple are pairwise comparable in the tree ordering. In [1], the following version of Ramsey's theorem is presented.

> $\mathsf{TT}(n)$: Fix $k \in \mathbb{N}$. Suppose that $[2^{<\mathbb{N}}]^n$ is colored with k colors. Then there is a subtree S isomorphic to $2^{<\mathbb{N}}$ such that $[S]^n$ is monochromatic.

In applying $\mathsf{TT}(n)$, we often think of the coloring as a function $f : [2^{<\mathbb{N}}]^n \to k$, in which case S is monochromatic precisely when f is constant on $[S]^n$.

Let $\Phi^X_{e,t}(m)\downarrow$ denote a fixed formalization of the assertion that the Turing machine with code number e, using an oracle for the set X, halts on input m with the entire computation bounded by t. We will assume that t is a bound on all aspects of the computation, including codes for inputs from the oracle. This formalization can be based on Kleene's T-predicate or any similar arithmetization of computation. In RCA_0, we use the notation $Y \leq_T X$ to denote the existence of two codes e and e' such that

$$\forall m(m \in Y \leftrightarrow \exists t \Phi^X_{e,t}(m)\downarrow)$$

and

$$\forall m(m \notin Y \leftrightarrow \exists t \Phi^X_{e',t}(m)\downarrow).$$

The preceding formalizes the notion that Y is Turing reducible to X if and only if both Y and its complement are computably enumerable in X.

As in [2], we can also use this notation to formalize ACA_0'. Given any set X, let $Y = X'$ denote the statement

$$\forall \langle m, e\rangle(\langle m, e\rangle \in Y \leftrightarrow \exists t \Phi^X_{e,t}(m)\downarrow),$$

where $\langle m, e\rangle$ denotes an integer code for the ordered pair (m, e). To formalize the nth jump for $n \geq 1$, we write $Y = X^{(n)}$ if there is a finite sequence $X_0, \ldots, X_n$ such that $X_0 = X$, $X_n = Y$, and for every $i < n$, $X_{i+1} = X_i'$. In this notation, $Y = X'$ if and only if $Y = X^{(1)}$, and we will often write X'' for $X^{(2)}$. The subsystem ACA_0' consists of ACA_0 plus the assertion that for every X and every n, there is a set Y such that $Y = X^{(n)}$.

Using all this terminology, we can prove a formalized version of the implication from $\mathsf{TT}(n)$ to $\mathsf{TT}(n+1)$, including a formalized computability theoretic upper bound.

Lemma 1. (RCA_0) *Suppose R is a tree isomorphic to $2^{<\mathbb{N}}$, $f : [R]^{n+1} \to k$ is a finite coloring of the $(n+1)$-tuples of comparable nodes of R, and both*

$R \leq_T A$ and $f \leq_T A$. Suppose that A'' exists. Then we can find a tree S and a coloring $g : [S]^n \to k$ such that $S \leq_T A''$, $g \leq_T A''$, S is a subtree of R isomorphic to $2^{<\mathbb{N}}$, and every monochromatic subtree of S for g is also monochromatic for f.

Proof. Working in RCA_0, suppose R, f, and A are as in the statement of the lemma. We will essentially carry out the proof of Theorem 1.4 of [1], replacing uses of arithmetical comprehension by recursive comprehension relative to A''. Toward this end, given a sequence $P = \{\rho_\tau \mid \tau \subseteq \sigma\}$ of comparable nodes of R such that the sequence terminates in ρ_σ, define an induced coloring of single nodes $\tau \supset \rho_\sigma$ by setting

$$f_{\rho_\sigma}(\tau) = \langle \{(\vec{m}, f(\vec{m}, \tau)) \mid \vec{m} \in [P]^n\} \rangle$$

where the angle brackets denote an integer code for the finite set. Since $f \leq_T A$, for any finite set P we have $f_{\rho_\sigma} \leq_T A$.

For each $\sigma \in 2^{<\mathbb{N}}$, define p_σ, T_σ, and c_σ as follows. Let $\rho_{\langle\rangle}$ be the root of R and $T_{\langle\rangle} = R$. Given ρ_σ and T_σ computable from A, use A'' to compute a c_σ which is the greatest integer in the range of f_{ρ_σ} such that

$$\exists \rho \in T_\sigma(\rho \supset \rho_\sigma \wedge \forall \tau \in T_\sigma(\tau \supset \rho \to c_\sigma \leq f_{\rho_\sigma}(\tau))).$$

Using A'', compute the least such ρ. Let T denote the subtree of T_σ isomorphic to $2^{<\mathbb{N}}$ defined by taking ρ as the root and letting the immediate successors of each node be the least pair of incomparable extensions in T_σ that are assigned c_σ by f_{ρ_σ}. Because of the choice of c_σ, T is isomorphic to $2^{<\mathbb{N}}$, and its nodes can be located in an effective manner. (In [1], this T is called the standard c_σ-colored subtree of T_σ for ρ using f_{ρ_σ}.) Let $\rho_{\sigma^\frown 0}$ and $\rho_{\sigma^\frown 1}$ be the two level one elements of T and let $T_{\sigma^\frown \varepsilon}$ be the subtree of T with root $\rho_{\sigma^\frown \varepsilon}$ for each $\varepsilon \in \{0, 1\}$. Note that given any finite chain of elements and colors $\{\langle \rho_\tau, c_\tau \rangle \mid \tau \subseteq \sigma\}$, sufficiently large initial segments of each T_τ can be computed to determine $\rho_{\sigma^\frown 0}$, $\rho_{\sigma^\frown 1}$, $c_{\sigma^\frown 0}$, and $c_{\sigma^\frown 1}$, using only A''. Consequently, the subtree $S = \{\rho_\sigma \mid \sigma \in 2^{<\mathbb{N}}\}$ is computable from A''.

Define $g : [S]^n \to k$ by $g(p_{\sigma_1}, \ldots, \rho_{\sigma_n}) = f(\rho_{\sigma_1}, \ldots, \rho_{\sigma_n}, \rho_{\sigma_n^\frown 0})$. Since $S \leq_T A''$, we also have $g \leq_T A''$. By the construction of S, given any increasing sequence of elements of S of the form $\rho_1 \subset \rho_2 \subset \cdots \subset \rho_n$, and extensions $\rho_n \subset \rho_{n+1}$ and $\rho_n \subset \rho_{n+2}$, we have $f_{\rho_n}(\rho_{n+1}) = f_{\rho_n}(\rho_{n+2})$, so $f(\rho_1, \ldots, \rho_n, \rho_{n+1}) = f(\rho_1, \ldots, \rho_n, \rho_{n+2})$. Thus any monochromatic subtree for g is also monochromatic for f, and the proof is complete. □

Extracting the computability theoretic content of the previous argument, given a computable coloring of n-tuples we can find a monochromatic set computable from $0^{(2n-2)}$. This is not an optimal bound, since applying the Strong Hierarchy Theorem to Theorem 2.7 of [1] yields a monochromatic set computable from $0^{(n)}$. However, the preceding result does enable us to complete the proof of the next theorem, and avoids formalization of the long proof of Theorem 2.7 of [1].

Theorem 2. (RCA_0) *The following are equivalent:*

(1) ACA_0'

(2) $\forall n \mathsf{TT}(n)$

Proof. To prove that (1) implies (2), assume ACA_0' and let $f : [2^{<\mathbb{N}}]^n \to k$ be a coloring. By ACA_0', the jump $f^{(2n-2)}$ exists, so by discarding the odd jumps we can find a sequence of sets $X_0, X_1, \ldots, X_{n-1}$ such that $X_0 = f$ and for each i, $X_{i+1} = X_i''$. Note that $f \leq_T X_0$ and $2^{<\mathbb{N}} \leq_T X_0$. By Lemma 1, for any X_i, given indices witnessing that a subtree isomorphic to $2^{<\mathbb{N}}$ and a coloring of the $(n-i)$-tuples of that subtree are each computable from X_i, we can find indices for computing an infinite subtree and a coloring of $(n-i-1)$-tuples from X_{i+1} satisfying the conclusion of Lemma 1. Thus, by induction on arithmetical formulas (which is a consequence of ACA_0'), we can prove the existence of a sequence of indices, the last of which can be used to compute a subtree T_{n-1} and a function $f_{n-1} : [T_{n-1}]^1 \to k$ such that T_{n-1} is isomorphic to $2^{<\mathbb{N}}$ and any monochromatic subtree for f_{n-1} is also monochromatic for f. Since ACA_0' includes RCA_0 plus induction for Σ^0_2 formulas, by Theorem 1.2 of [1], T_{n-1} contains a subtree which is monochromatic for f_{n-1} and isomorphic to $2^{<\mathbb{N}}$. This subtree is also monochromatic for f, so $\mathsf{TT}(n)$ holds for f.

To prove that (2) implies (1), assume RCA_0 and (2). Given any coloring of n-tuples of integers, $f : [\mathbb{N}]^n \to k$, we may define a coloring $g : [2^{<\mathbb{N}}]^n \to k$ on n-tuples of elements of $2^{<\mathbb{N}}$ by

$$g(\sigma_1, \ldots, \sigma_n) = f(\mathrm{lh}(\sigma_1), \ldots \mathrm{lh}(\sigma_n))$$

where $\mathrm{lh}(\sigma)$ denotes the length of the sequence σ. Any monochromatic tree for g contains an infinite path which encodes an infinite monochromatic set for f. Thus, as noted in the proof of Theorem 1.5 of [1], $\forall n \mathsf{TT}(n)$ implies the usual full Ramsey's theorem, denoted by $\forall n \mathsf{RT}(n)$. ACA_0' can be deduced from $\forall n \mathsf{RT}(n)$ by Theorem 8.4 of [3], or by applying Proposition 4.4 of [2]. □

A typical proof of $\forall n \mathsf{TT}(n)$ would proceed by induction on n and require the use of induction on Π^1_2 formulas. In the preceding argument, the existence of the nth jump is used to push the application of induction down to arithmetical formulas. The proof of Theorem 2 together with Proposition 4.4 of [2] provide a detailed exposition of a proof and reversal in ACA_0' and show that the full versions of the usual Ramsey's theorem, the polarized version of Ramsey's theorem, and Ramsey's theorem for trees are all equivalent to ACA_0' over RCA_0.

Bibliography

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