Combinatorial principles equivalent to weak induction

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Abstract

We consider two combinatorial principles, ERT and ECT. Both are easily proved in RCA_0 plus Σ^0_2 induction. We give two proofs of ERT in RCA_0, using different methods to eliminate the use of Σ^0_2 induction. Working in the weakened base system RCA_0^*, we prove that ERT is equivalent to Σ^0_1 induction and ECT is equivalent to Σ^0_2 induction. We conclude with a Weihrauch analysis of the principles, showing $\mathsf{ERT} \equiv_{\mathrm{W}} \mathsf{LPO}^* <_{\mathrm{W}} \mathsf{TC}^*_{\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{ECT}$.

In their logical analysis of vertex colorings of hypergraphs, Davis, Hirst, Pardo, and Ransom [5] isolate the combinatorial principle ERT, and relate it to the nonexistence of finite conflict-free colorings for a particular hypergraph. The principle asserts that for any finite coloring of the natural numbers $\mathbb{N}$ there is a tail of the coloring such that every color appearing in the tail appears at least twice in the tail. ERT stands for *e*ventually *r*epeating *t*ail, and can be formulated as follows.

ERT. If $f : \mathbb{N} \to k$ for some $k \in \mathbb{N}$, then there is a $b \in \mathbb{N}$ such that for all $x \geq b$, there is a $y \geq b$ such that $x \neq y$ and $f(x) = f(y)$.

The principle ERT is an immediate consequence of the principle ECT introduced by Hirst [6]. ECT stands for *e*ventually *c*onstant palette *t*ail, and asserts that for any finite coloring of $\mathbb{N}$ there is a tail of the coloring such that the colors appearing in any final segment of the tail are exactly those appearing in the entire tail. A more formal version follows.

ECT. If $f : \mathbb{N} \to k$ for some $k \in \mathbb{N}$, then there is a $b \in \mathbb{N}$ such that for all $x \geq b$, there is a $y > x$ such that $f(x) = f(y)$.

Both Davis et al. [5] and Hirst [6] work in the usual framework of reverse mathematics. In particular, they prove equivalences over the subsystem of second order arithmetic RCA_0. This axiom system includes basic natural number arithmetic axioms, an induction scheme restricted to Σ^0_1 formulas (denoted $\mathsf{I}\Sigma^0_1$), and a recursive comprehension axiom that essentially asserts that computable sets of natural numbers exist. See Simpson's book [11] for more about RCA_0 and reverse mathematics. Theorem 6 of Hirst [6] shows that over RCA_0, ECT is equivalent to $\mathsf{I}\Sigma^0_2$, an induction scheme for Σ^0_2 formulas. RCA_0 can prove that ECT implies ERT, so RCA_0 proves that $\mathsf{I}\Sigma^0_2$ implies ERT. As we will see in the next section, $\mathsf{I}\Sigma^0_2$ is not necessary in this proof.

1. RCA_0 proves ERT

Davis et al. [5] show that $\mathsf{I}\Sigma^0_2$ is not needed in the proof of ERT by deriving ERT from a restricted form of Ramsey's theorem and applying a result of Chong, Slaman, and Yang [3]. There, Ramsey's theorem is restricted to stable colorings of pairs, that is to functions $f : [\mathbb{N}]^2 \to k$ such that for all x, $\lim_{y\to\infty} f(x, y)$ exists. Stable Ramsey's theorem for pairs and two colors is denoted by SRT^2_2 and can be formalized as follows.

SRT^2_2. If $f : [\mathbb{N}]^2 \to 2$ is stable, then there is an infinite set $H \subset \mathbb{N}$ and a color $c \in \{0, 1\}$ such that for all $(x, y) \in [H]^2$, $f(x, y) = c$.

The next result appears as Theorem 11 in Davis et al. [5]. The RCA_0 in parentheses indicates that the proof can be carried out in the formal system RCA_0. For completeness, we give a minimal sketch of the proof.

Lemma 1. (RCA_0) SRT^2_2 *implies* ERT.

Proof. Working in RCA_0, let $f : \mathbb{N} \to k$ be a coloring of $\mathbb{N}$ as in the statement of ERT. Define a coloring of pairs, $g : [\mathbb{N}]^2 \to 2$ by $g(a, b) = 1$ if and only if for some x in the half open interval of natural numbers $[a, b)$, $f(x)$ appears exactly once in the range of f restricted to $[a, b)$. Because g is stable, by SRT^2_2 there is an infinite homogeneous set H. An argument based on the first $3 \cdot 2^{k-1}$ elements of H shows that g is identically equal to 0. Consequently, the minimum element of H satisfies the requirements of the bound b in the statement of ERT. (For a more complete proof, see Davis et al. [5]). □

By Corollary 2.6 of Chong, Slaman, and Yang [3], SRT^2_2 cannot prove $\mathsf{I}\Sigma^0_2$, so neither can ERT. Thus although $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2$ proves ERT, the full strength of $\mathsf{I}\Sigma^0_2$ is not necessary. Using a recent conservation result of Patey and Yokoyama [9], together with an alternative formalization of ERT, we can show that RCA_0 proves ERT, completely eliminating the use of $\mathsf{I}\Sigma^0_2$.

Lemma 2. (RCA_0) *The following are equivalent.*

(1) ERT.

(2) ERT' : *If $f : \mathbb{N} \to k$ for some $k \in \mathbb{N}$, then there is a number $b \in \mathbb{N}$, a set $I \subset [0, k)$ consisting of the range of f on $[b, \infty)$, and a witness set $\{(x_i, y_i) \mid i \in I\}$ such that for every $z \geq b$, we have $f(z) \in I$, $b \leq x_{f(z)} < y_{f(z)}$, and $f(z) = f(x_{f(z)}) = f(y_{f(z)})$.*

Proof. We will work in RCA_0. Note that for any f, the number b provided by ERT' also satisfies the statement of ERT. Thus ERT follows immediately from ERT'.

To prove the converse, let $f : \mathbb{N} \to k$ and apply ERT to obtain b. The set $I = \{j < k \mid \exists t(t \geq b \wedge f(t) = j)\}$ exists by bounded Σ^0_1 comprehension, a consequence of RCA_0 [11, Theorem II.3.9]. For each $i \in I$, there are at least two distinct values $x_i \geq b$ and $y_i \geq b$ such that $f(x_i) = f(y_i) = i$. Picking the least such witness pair for each i, recursive comprehension proves the existence of the witness set $\{(x_i, y_i) \mid i \in I\}$. Routine arguments verify that b and this witness set satisfy the requirements of ERT'. □

Applying the two lemmas and using a result of Patey and Yokoyama [9], we can easily prove ERT in RCA_0, answering a question of Davis et al. [5]. An alternative direct proof of Theorem 3 is included in the next section in the proof of Theorem 6.

Theorem 3. RCA_0 *proves* ERT.

Proof. (RCA_0) By Lemma 1, $\mathsf{RCA}_0 + \mathsf{SRT}^2_2$ proves ERT. Thus, by Lemma 2, $\mathsf{RCA}_0 + \mathsf{SRT}^2_2$ proves ERT'. By Theorem 7.4 of Patey and Yokoyama [9], $\mathsf{RCA}_0 + \mathsf{SRT}^2_2$ is a conservative extension of RCA_0 for formulas of the form $\forall X \varphi(X)$, where φ is Π^0_3. ERT' has this form, so RCA_0 proves ERT'. By Lemma 2, RCA_0 proves ERT. □

The conservation result of Patey and Yokoyama is a powerful tool for eliminating the use of Σ^0_2 induction in the proofs of combinatorial principles.

Their result actually holds for Ramsey's theorem for pairs and two colors, so it is not necessary to limit ourselves to stable colorings. The principle ERT' can be formalized in the form $\forall X \varphi(X)$ where φ is Σ^0_2. Clearly, we have made use of less than the full strength of this technique in our example. On the other hand, if ERT is directly formalized in the form $\forall X \theta(X)$, the formula θ is Σ^0_3, so Patey and Yokoyama's result does not apply. Lemma 2 is a necessary step in the argument.

2. Reverse mathematics: ERT is $\mathsf{I}\Sigma^0_1$ and ECT is $\mathsf{I}\Sigma^0_2$

In this section, we prove that our combinatorial principles are equivalent to induction schemes over the weakened base system RCA^*_0. The axioms of RCA^*_0 are those of RCA_0 less the Σ^0_1 induction scheme, with the addition of a Σ^0_0 induction scheme and function symbols and axioms for integer exponentiation. The subsystem is described in Chapter X of Simpson's book [11]. The following lemma incorporates results from an early work of Simpson and Smith [12]. Note the change in the base system at the beginning of the statement of the lemma.

Lemma 4. (RCA^*_0) *The following are equivalent.*

(1) $\mathsf{I}\Sigma^0_1$, *the* Σ^0_1 *induction scheme.*

(2) *The universe of total functions is closed under primitive recursion.*

(3) *Bounded* Σ^0_1 *comprehension.*

(4) *Bounded* Π^0_1 *comprehension.*

Proof. The equivalence of items (1), (2), and (3) are included in Lemma 2.5 of the article of Simpson and Smith [12]. Recursive comprehension proves the existence of complements of sets, so items (3) and (4) are also equivalent. □

For our arguments, it is useful to formalize the concept of a partial function. Working in RCA^*_0, we can define a code for a finite partial function as a set of ordered pairs $f \subset [0, k) \times \mathbb{N}$ such that for all i, n, and m, if $(i, n) \in f$ and $(i, m) \in f$, then $n = m$. Using this notion, we can state another equivalent form of $\mathsf{I}\Sigma^0_1$.

Lemma 5. (RCA^*_0) *The following are equivalent:*

(1) $\mathsf{I}\Sigma^0_1$.

(2) *Finite partial functions have bounded ranges. That is, if $f \subset k \times \mathbb{N}$ is a finite partial function, then*

$$\exists b\, \forall i < k\, \forall n((i,n) \in f \to n \leq b).$$

Proof. To prove (1) implies (2), working in RCA^*_0, assume $\mathsf{I}\Sigma^0_1$ and let f be a finite partial function contained in $k \times \mathbb{N}$. By Lemma 4, we may apply bounded Σ^0_1 comprehension and find the set $D = \{x < k \mid \exists y (x,y) \in f\}$. By recursive comprehension, there is a total function f' satisfying

$$f'(n) = \begin{cases} \min\{m \mid (n,m) \in f\} & \text{if } n \in D \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 4, we may apply primitive recursion to find the summation function $g(n) = \sum_{i=0}^{n} f'(i)$. The integer $g(k-1)$ is a suitable bound for the range of f.

To prove the converse, we will use (2) to prove bounded Σ^0_1 comprehension. Let $\theta(m,n)$ be a Σ^0_0 formula and fix a bound k. We will prove that the set $\{m < k \mid \exists n \theta(m,n)\}$ exists. Using recursive comprehension, we can find the set of pairs

$$f = \{(m,n) \mid \theta(m,n) \wedge \forall y < n\, \neg\theta(m,y)\}.$$

Note that f is a partial function from k into $\mathbb{N}$. By (2), there is a bound b for the range of f. Thus, for all $m < k$, $\exists n \theta(m,n)$ if and only if $\exists n \leq b\, \theta(m,n)$. So $\{m < k \mid \exists n \theta(m,n)\}$ is identical to $\{m < k \mid \exists n \leq b\, \theta(m,n)\}$, and its existence follows from recursive comprehension. □

We can now show that ERT is equivalent to $\mathsf{I}\Sigma^0_1$ over RCA^*_0. Because RCA^*_0 plus $\mathsf{I}\Sigma^0_1$ is RCA_0, this provides a direct proof of ERT in RCA_0, without the use of conservation results. Following the proof of the theorem, we will comment on this as a technique for eliminating $\mathsf{I}\Sigma^0_2$ in proofs of combinatorial results.

Theorem 6. (RCA^*_0) *The following are equivalent.*

(1) $\mathsf{I}\Sigma^0_1$.

(2) ERT.

(3) $\forall j \mathsf{ERT}(j)$. *Here* $\mathsf{ERT}(j)$ *generalizes* ERT, *requiring that at or after the bound* b, *any value of* f *that appears must appear at least* j *times.*

Proof. To show that (1) implies (2), we could simply cite Theorem 3. We present a direct proof using sequential applications of bounded comprehension that will be adapted to prove Theorem 8 below. Working in RCA_0^*, assume $\mathsf{I}\Sigma_1^0$. By Lemma 4, we may apply bounded Σ_1^0 comprehension. We will prove ERT for $f : \mathbb{N} \to k$. By bounded Σ_1^0 comprehension, we can find the set of (codes for) non-repeating finite sequences of values less than k such that the colors appear in this order somewhere in the range of f. More formally, bounded Σ_1^0 comprehension proves the existence of a set S of (codes for) sequences such that $\sigma \in S$ if and only if

- $\text{length}(\sigma) < k$,
- $\forall i < \text{length}(\sigma)\, (\sigma(i) < k)$,
- $\forall i < \text{length}(\sigma)\, \forall j < \text{length}(\sigma)\, (\sigma(i) = \sigma(j) \to i = j)$,

and there is a finite witness sequence τ such that

- $\text{length}(\sigma) = \text{length}(\tau)$,
- $\forall i < \text{length}(\tau)\, \forall j < \text{length}(\tau)\, (i < j \to \tau(i) < \tau(j))$,
- $\forall i < \text{length}(\tau)(f(\tau(i)) = \sigma(i))$.

By Lemma 4, we may also use bounded Π_1^0 comprehension. Using S as a parameter and applying bounded Π_1^0 comprehension, we can find a subset T of S consisting of the empty sequence and all those sequences σ such that the first time the colors appear in the specified order, the last color never reappears. When selecting the first witness sequence, we assume that for sequences differing in a single entry, the sequence with the smaller entry appears first. Thus, σ is in T if and only if σ is empty, or $\sigma \in S$ and for the first witness sequence τ for σ and any $j > \text{length}(\tau) - 1$, $f(j) \neq \sigma(\text{length}(\tau) - 1)$. T is a subset of the finite set of non-repeating sequences of numbers less than k, so RCA_0^* can answer questions about whether or not sequences are in T. In particular, we can define a subset $T_0 \subset T$ of sequences σ such that no extension of σ is in T and every initial segment of σ is in T. Suppose $\sigma_0 \in T_0$. If σ_0 is empty, then every color in the range of f appears at least twice, and $b = 0$ is the desired bound for ERT. If σ_0 is nonempty, let

τ_0 be the first witness sequence for σ_0, and define $b = \tau_0(\text{length}(\sigma_0) - 1) + 1$. Because σ_0 is in T and none of its extensions are, every color appearing at or after b must appear at least twice. Summarizing, the bound b satisfies the requirements of ERT.

Next, we will show that (2) implies (1) by proving the contrapositive. Suppose RCA_0^* holds and $\mathsf{I}\Sigma^0_1$ fails. By Lemma 5, there is a finite partial function $g \subset k \times \mathbb{N}$ with an unbounded range. Define the function $f : \mathbb{N} \to k + 1$ by

$$f(n) = \begin{cases} j & \text{if } j < k \wedge (j, n) \in g \\ k & \text{otherwise.} \end{cases}$$

The function f exists by recursive comprehension, and for any b there is an $n > b$ such that $f(n) < k$ and the value of $f(n)$ appears only once in the range of f. Thus no b can be a bound for ERT applied to f, and ERT fails.

Because (2) is a special case of (3), to complete the proof of the theorem, it suffices to show in RCA_0^* that $\forall j \mathsf{ERT}(j)$ follows from ERT. By our previous work, $\mathsf{I}\Sigma^0_1$ follows from ERT, so we may work in RCA_0. Fix j and suppose $f : \mathbb{N} \to k$. Our goal is to find a b such that every color appearing at or after b appears at least j times in the range of f at or after b. Define $g : \mathbb{N} \to k \times j$ by setting

$$g(n) = (f(n), \text{mod}_j |\{i < n \mid f(i) = f(n)\}|).$$

Intuitively, if f takes the value i at locations $x_0, x_1, \ldots x_j$ (and nowhere before or in between), then $g(x_0) = (i, 0)$, $g(x_1) = (i, 1)$, $\ldots$, $g(x_{j-1}) = (i, j - 1)$, and $g(x_j) = (i, 0)$. Using a bijection between $k \times j$ and the natural numbers less than $k \cdot j$, we can view g as a function from $\mathbb{N}$ into $k \cdot j$. Let b be a bound for ERT applied to g. Suppose color i appears at or after b in the range of f. Let x_0 be the first such location. Then for some $m < j$, $g(x_0) = (i, m)$. Note that x_0 is the first location at or after b where g takes this value. By ERT for g, there is an $x_1 > x_0$ such that $g(x_1) = g(x_0)$. By the definition of g, there are at least j places in $[x_0, x_1)$ where f takes the value i. Thus b is a bound for $\mathsf{ERT}(j)$ for f. This completes the proof of (3) from (2) and the proof of the theorem. □

For use in the proof of Theorem 11, note that the set T defined in the preceding proof can be used to compute the minimum bound satisfying ERT. Because we are making a computability theoretic argument, we are not restricted to RCA_0^*. If every color in the range of f appears at least twice, then no sequences of length one appear in T_0, so σ_0 is the empty sequence and

$b = 0$ is the minimum bound. Otherwise, define finite sequences σ and τ as follows. Let $\sigma(0)$ be the last appearing among colors that appear exactly once, and let $\tau(0)$ be the location where $\sigma(0)$ appears. Let $\sigma(i+1)$ be the last appearing among colors that appear exactly once after $\tau(i)$ if such a color exists, and let $\tau(i+1)$ be the last location where $\sigma(i)$ appears. If no such color exists, terminate the sequences. Routine verifications show that $\sigma \in T_0$ and that τ is the first witness for σ, so that $b = \tau(\text{length}(\sigma) - 1) + 1$ is a bound for ERT. From the construction, if b' is any bound for ERT, then $b' > \tau(0)$, and for $i < \text{length}(\sigma)$, if $b' > \tau(i)$ then $b' > \tau(i+1)$. Thus b is minimal. Consequently, the minimum bound can be calculated by listing T_0, calculating the value b for each sequence in T_0, and then selecting the minimum bound.

The existence of the set T in the proof that (1) implies (2) above used an application of bounded Σ^0_1 comprehension followed by an application of bounded Π^0_1 comprehension. Naïvely concatenating the associated formulas to construct T with a single application results in a use of bounded Σ^0_2 comprehension, a principle equivalent to $\mathsf{I}\Sigma^0_2$ [11, Exercise II.3.13]. Conversely, it may be possible to eliminate unnecessary uses of $\mathsf{I}\Sigma^0_2$ in proofs, particularly in the guise of bounded Σ^0_2 or bounded Π^0_2 comprehension, by using a sequence of applications of bounded Σ^0_1 or bounded Π^0_1 comprehension. In the case of the preceding proof, the sequential applications can be combined into a single application, as in the second part of the proof of Theorem 8 below.

We complete this section with a proof of the equivalence of $\mathsf{I}\Sigma^0_2$ and ECT, showing that ERT is strictly weaker than ECT over RCA^*_0. This result differs from those in the article of Hirst [6] in the use of the weaker base system RCA^*_0. The arguments here sidestep the tree colorings used for [6, Theorem 6] and in the alternative argument following [6, Theorem 7], which is based on the conservation result of Corduan, Groszek, and Mileti [4].

Theorem 7. (RCA^*_0) *The following are equivalent.*

(1) $\mathsf{I}\Sigma^0_2$.

(2) ECT.

Proof. To prove that (1) implies (2), assume $\mathsf{I}\Sigma^0_2$ and fix $f : \mathbb{N} \to k$. Because $\mathsf{I}\Sigma^0_2$ implies $\mathsf{I}\Sigma^0_1$, we may work in RCA_0. By bounded Π^0_2 comprehension, a consequence of $\mathsf{I}\Sigma^0_2$ ([11, Exercise II.3.13], plus complementation via recursive comprehension), the set

$$T = \{j < k \mid \forall n \exists x (x > n \wedge f(x) = j)\}$$

exists. If $j \notin T$, then after some point j ceases to appear in the range of f. Formally,

$$\forall j < k\, \exists s \forall x((j \notin T \wedge x > s) \to f(x) \neq j).$$

By the Π^0_1 bounding principle, a consequence of $\mathsf{I}\Sigma^0_2$ [11, Exercise II.3.15], there is a b such that

$$\forall j < k\, \forall x((j \notin T \wedge x > b) \to f(x) \neq j).$$

In particular, if $j \notin T$ then for all $x \geq b$ we have $f(x) \neq j$. Summarizing, the range of f at or after b is exactly T, and every value of T appears infinitely often in the the range. Thus b satisfies the requirements of ECT.

We will prove that (2) implies (1), by a two stage bootstrapping argument. For the first step, working in RCA^*_0, note that ECT implies ERT. By Theorem 6, we may deduce $\mathsf{I}\Sigma^0_1$, so from now on we can work in RCA_0.

For the second step, we will use ECT to prove bounded Π^0_2 comprehension, and then deduce $\mathsf{I}\Sigma^0_2$. Fix k and consider $T = \{j < k \mid \forall x \exists y \theta(j, x, y)\}$ where θ is a Σ^0_0 formula. Our goal is to prove the existence of T. Suppose (j, x, y) is the n^{th} triple in a bijective enumeration of $k \times \mathbb{N} \times \mathbb{N}$. Define $f(n) = j$ if y is the first witness that $\forall s < x\, \exists t \leq y\, \theta(j, s, t)$, and let $f(n) = k$ otherwise. The function f exists by recursive comprehension. For any $j < k$, j appears infinitely often in the range of f if and only if $\forall x \exists y \theta(j, x, y)$. Apply ECT to f and obtain a bound b. Then

$$T = \{j < k \mid \exists x(x \geq b \wedge f(x) = j)\}.$$

By bounded Σ^0_1 comprehension, a consequence of RCA_0 [11, Theorem II.3.9], the set T exists, proving bounded Π^0_2 comprehension. To complete the proof, recall that by the first step above, we may work in RCA_0. By complementation, bounded Π^0_2 comprehension implies bounded Σ^0_2 comprehension. Applying Exercise II.3.13 of Simpson [11], Σ^0_2 induction follows from RCA_0 and bounded Σ^0_2 comprehension. □

3. Weihrauch analysis

The goal of this section is to analyze ERT and ECT using Weihrauch reductions. Because ERT and ECT have number outputs rather than set outputs, Weihrauch reducibility yields meaningful results where other forms of computability-theoretic reducibility would not. We will consider Weihrauch problems defined by subsets of $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}$. Each problem P can be viewed as a multifunction mapping instances $I \in \text{domain}(P)$ into solutions S with $(I, S) \in P$. A problem P is Weihrauch reducible to a problem Q, written $P \leq_W Q$, if instances of P can be uniformly computably transformed into instances of Q whose solutions can be uniformly computably transformed into solutions of the problem P. This last transformation may make use of the original instance of P. More formally, $P \leq_W Q$ if there are computable functionals Φ and Ψ such that for all $I \in \text{domain}(P)$, $\Phi(I) \in \text{domain}(Q)$, and for all S such that $(\Phi(I), S) \in Q$, we have $(I, \Psi(I, S)) \in P$. We write $P \equiv_W Q$ if $P \leq_W Q$ and $Q \leq_W P$, and write $P <_W Q$ if $P \leq_W Q$ and $Q \not\leq_W P$.

The relation $\equiv_W$ is an equivalence relation on the Weihrauch problems. The equivalence classes are called Weihrauch degrees, and many have well-known representing problems. For example, many Weihrauch problems are known to be equivalent to the Weihrauch problem LPO (*L*imited *P*rinciple of *O*mniscience). This problem takes as an instance any $f \in \mathbb{N}^{\mathbb{N}}$, and outputs 0 if $\exists n f(n) = 0$ and 1 otherwise. For an introduction to Weihrauch reducibility and many Weihrauch degrees, see the article of Brattka and Gherardi [1] and the survey of Brattka, Gherardi, and Pauly [2].

Many operators on Weihrauch problems preserve reducibility. For example, for a problem P, the problem P^n is the result of n parallel applications of P. The problem P^* is the result of an arbitrary finite number of parallel applications of P. Thus, for each n, we have $P^n \leq_W P^*$. Pauly introduces the concept of P^* in [10] and in Theorem 6.5 shows that $P \leq_W Q$ implies $P^* \leq_W Q^*$. Thus $\cdot^*$ can be viewed as an operator that preserves Weihrauch reducibility.

We may view ERT as a Weihrauch problem, where the input is a number k and a function $f : \mathbb{N} \to k$, and the solution is a value b as provided by ERT, that is,

$$\forall n \geq b \, \exists m \geq b \, (m \neq n \wedge f(m) = f(n)).$$

In a similar fashion, ECT can be viewed as a Weihrauch problem. Our goal is to find a known Weihrauch problems equivalent to ERT and to ECT, and to separate ERT and ECT in the Weihrauch setting. As a first step, we can state the following theorem.

Theorem 8. *$\mathsf{ERT} \equiv_W \mathsf{LPO}^*$.*

Proof. First we show that $\mathsf{LPO}^* \leq_W \mathsf{ERT}$. Given k LPO instances $f_0, \dots f_{k-1}$ we define a coloring $g : \mathbb{N} \to k + 1$ as follows. For $i < k$, let $g(nk + i) = i$

if and only if $f_i(n) = 0$ and $\forall m < n(f_i(m) \neq 0)$. Else, set $g(nk + i) = k$. Note that by construction, all colors but k appear at most once in the range of g. Thus any solution to ERT for g must be an upper bound for the first occurrence of 0 in the range of any f_i, which allows us to solve LPO for each f_i.

For the converse reduction, we can adapt the first part of the proof of Theorem 6, substituting LPO^* for the uses of bounded comprehension. Given $f : \mathbb{N} \to k$ we can use finitely many parallel applications of LPO to find the non-repeating sequences of colors in the set S. Simultaneously, we can use finitely many parallel applications of LPO to find those sequences that appear and whose last color reappears. Call the set of these sequences T'. A sequence is in the set T defined in the proof of Theorem 6 if and only if it is in S and is not in T'. Given the set T, we can find the bound b satisfying ERT for f by the construction in the proof of Theorem 6. This shows that $\mathsf{ERT} \leq_{\mathrm{W}} \mathsf{LPO}^*$. Summarizing, $\mathsf{ERT} \equiv_{\mathrm{W}} \mathsf{LPO}^*$. □

Next, we turn to the Weihrauch analysis of ECT. The principle Discrete Choice, denoted $\mathsf{C}_{\mathbb{N}}$, takes as an input an enumeration of the complement of a nonempty set A and outputs an element of A. The article of Neumann and Pauly [8] introduces and studies $\mathsf{TC}_{\mathbb{N}}$, the total continuation of $\mathsf{C}_{\mathbb{N}}$. $\mathsf{TC}_{\mathbb{N}}$ accepts the enumeration of the complement of any set A, empty or not, and outputs a number, which will be an element of A if A is nonempty. Clearly, $\mathsf{C}_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}}$, and consequently $\mathsf{C}_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}}^*$. Lemma 5 of Neumann and Pauly [8] includes $\mathsf{LPO}^* <_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}$. Concatenating the relations, $\mathsf{LPO}^* <_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}}^*$. The next theorem links $\mathsf{TC}_{\mathbb{N}}^*$ and ECT.

Theorem 9. $\mathsf{ECT} \equiv_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}}^*$.

Proof. To see that $\mathsf{ECT} \leq_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}}^*$, suppose the coloring $f : \mathbb{N} \to k$ is an instance of ECT. Our goal is to use finitely many applications of $\mathsf{TC}_{\mathbb{N}}$ to find a value b such that every color appearing at or after b appears infinitely often in the range of f. For each $i < k$ construct an enumeration of the complement of the set

$$A_i = \{n \mid \forall m \geq n(f(m) \neq i)\}.$$

Apply $\mathsf{TC}_{\mathbb{N}}$ to each of these sets to obtain numbers b_i such that if the color i appears only finitely often, then it no longer appears after b_i. The number $b = 1 + \max\{b_i \mid i < k\}$ is a solution to the ECT instance.

For the converse direction, suppose A_i for $1 \leq i < k$ is a finite list of $\mathsf{TC}_{\mathbb{N}}$ instances, where for each i, e_i enumerates the complement of A_i. Fix a bijective pairing function $(\cdot,\cdot) : \mathbb{N} \times k \to \mathbb{N}$, and define the coloring $c : \mathbb{N} \to k$ by

$$c((s,i)) = \begin{cases} i & \text{if } i \neq 0 \wedge e_i(s) = \min\{n \mid \forall t < s (e_i(t) \neq n)\} \\ 0 & \text{otherwise.} \end{cases}$$

Apply ECT to c to find a bound b. If some color $i \neq 0$ appears infinitely often in the range of c, then $A_i = \emptyset$. Otherwise, if i never appears after b and s is sufficiently large that $(s,i) \geq b$, then $\min\{n \mid \forall t < s(e_i(t) \neq n)\}$ is in A_i. In either case, $\min\{n \mid \forall t < s(e_i(t) \neq n)\}$ is a valid output for $\mathsf{TC}_{\mathbb{N}}$ applied to the input A_i. □

Summarizing, we have shown that $\mathsf{ERT} \equiv_{\mathrm{W}} \mathsf{LPO}^*$, $\mathsf{LPO}^* <_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}}^*$, and $\mathsf{TC}_{\mathbb{N}}^* \equiv_{\mathrm{W}} \mathsf{ECT}$, so $\mathsf{ERT} <_{\mathrm{W}} \mathsf{ECT}$. We have captured the strength of ERT and ECT in terms of known Weihrauch degrees, and shown that ERT is strictly weaker than ECT in the Weihrauch degrees.

Both Theorem 8 and Theorem 9 fail for strong Weihrauch reducibility. In strong reducibility, the solution to the input problem must be computed from any solution of the transformed problem without further reference to the original input. Using the notation from the first paragraph of this section, $P \leq_{\mathrm{sW}} Q$ if there are computable functionals Φ and Ψ such that for all $I \in \mathrm{domain}(P)$, $\Phi(I) \in \mathrm{domain}(Q)$, and for all S such that $(\Phi(I), S) \in Q$, we have $(I, \Psi(S)) \in P$.

As an example using familiar problems, we will show that $\mathsf{LPO}^* <_{\mathrm{sW}} \mathsf{TC}_{\mathbb{N}}^*$. To see that $\mathsf{LPO} \leq_{\mathrm{sW}} \mathsf{TC}_{\mathbb{N}}$, given an instance f of LPO, construct an instance g of $\mathsf{TC}_{\mathbb{N}}$ by setting $g(n) = n+1$ if $f(n) \neq 0$ and $g(n) = 0$ otherwise. If the solution for g is positive, then the solution for f is 0. If the solution for g is 0, then the solution for f is 1. Similarly, sequences of LPO problems can be transformed to sequences of $\mathsf{TC}_{\mathbb{N}}$ problems, so $\mathsf{LPO}^* \leq_{\mathrm{sW}} \mathsf{TC}_{\mathbb{N}}^*$. We know $\mathsf{TC}_{\mathbb{N}}^* \not\leq_{\mathrm{W}} \mathsf{LPO}^*$, so $\mathsf{TC}_{\mathbb{N}}^* \not\leq_{\mathrm{sW}} \mathsf{LPO}^*$, and thus $\mathsf{LPO}^* <_{\mathrm{sW}} \mathsf{TC}_{\mathbb{N}}^*$. The next theorem summarizes strong reducibility relations for ERT and ECT.

Theorem 10. $\mathsf{ERT} <_{\mathrm{sW}} \mathsf{ECT} <_{\mathrm{sW}} \mathsf{TC}_{\mathbb{N}}^*$, $\mathsf{LPO} \not\leq_{\mathrm{sW}} \mathsf{ECT}$, *and* $\mathsf{ERT} \not\leq_{\mathrm{sW}} \mathsf{LPO}^*$.

Proof. Identity functionals witness $\mathsf{ERT} \leq_{\mathrm{sW}} \mathsf{ECT}$. We know $\mathsf{ECT} \not\leq_{\mathrm{W}} \mathsf{ERT}$, so $\mathsf{ECT} \not\leq_{\mathrm{sW}} \mathsf{ERT}$ and thus $\mathsf{ERT} <_{\mathrm{sW}} \mathsf{ECT}$.

The first paragraph of the proof of Theorem 9 shows that $\mathsf{ECT} \leq_{\mathrm{sW}} \mathsf{TC}_{\mathbb{N}}^*$. The failure of the converse relation and $\mathsf{ECT} <_{\mathrm{sW}} \mathsf{TC}_{\mathbb{N}}^*$ both follow from $\mathsf{LPO} \not\leq_{\mathrm{sW}} \mathsf{ECT}$, which we prove next.

To see that $\mathsf{LPO} \not\leq_{\mathrm{sW}} \mathsf{ECT}$, suppose by contradiction that Φ and Ψ witness $\mathsf{LPO} \leq_{\mathrm{sW}} \mathsf{ECT}$. Suppose f_1 and f_2 are LPO problems with distinct solutions. Let $\Phi(f_1) = g_1$ and $\Phi(f_2) = g_2$ be the associated ECT problems. Let b_1 be a solution for g_1 and b_2 be a solution for g_2. Then $b = \max\{b_1, b_2\}$ is a solution for both g_1 and g_2. Then $\Psi(b)$ is a solution for both f_1 and f_2, yielding a contradiction. Thus $\mathsf{LPO} \not\leq_{\mathrm{sW}} \mathsf{ECT}$. This argument is an example of the general principle that no multifunction of the form $f : X \rightrightarrows \mathbb{N}$ where all $f(x)$ are upwards closed can compute a non-trivial multifunction $g : X \rightrightarrows k$ for finite k.

To see that $\mathsf{ERT} \not\leq_{\mathrm{sW}} \mathsf{LPO}^*$, we again argue by contradiction, supposing that Φ and Ψ witness $\mathsf{ERT} \leq_{\mathrm{sW}} \mathsf{LPO}^*$. Let f_1 be the instance of ECT consisting of a two-coloring that is constantly zero. Suppose $\Phi(f_1) = (g_1, \ldots, g_n)$ is a sequence of n instances of LPO. The computation of $\Phi(f_1)$ uses only a finite initial segment of f_1. Denote the length of this segment by k. The LPO problems $g_1, \ldots, g_n$ have solutions $s_1, \ldots, s_n$. Thus $\Psi(s_1, \ldots, s_n)$ computes a bound m satisfying ERT for f_1. Now consider the ERT problem f_2, consisting of a two-coloring that contains $k + m$ zeros, followed by a single one, followed by an infinite string of zeros. Because f_2 and f_1 agree on the first k values, $\Phi(f_2) = \Phi(f_1) = (g_1, \ldots, g_n)$. These LPO problems are the same as before, and so still have the solutions $s_1, \ldots, s_n$. Thus $\Psi(s_1, \ldots, s_n) = m$ should be a bound satisfying ERT for f_2. However, by the construction of f_2, any bound for f_2 must be at least $k + m + 1$, which is strictly larger than m. Thus Φ and Ψ cannot be witnesses of $\mathsf{ERT} \leq_{\mathrm{sW}} \mathsf{LPO}^*$, and we have shown that $\mathsf{ERT} \not\leq_{\mathrm{sW}} \mathsf{LPO}^*$. □

Minor alterations in the formulations of ERT and ECT can result in interesting variations in their Weihrauch strengths. For example, let minERT denote the principle that outputs the minimum bound satisfying ERT. Define minECT similarly.

Theorem 11. $\mathsf{ERT} \equiv_{\mathrm{W}} \mathsf{minERT}$ *and* RCA_0^* *proves* $\mathsf{ERT} \leftrightarrow \mathsf{minERT}$.

Proof. Every solution of minERT is a solution of ERT, so $\mathsf{ERT} \leq_{\mathrm{W}} \mathsf{minERT}$. For the converse, apply the second paragraph of the proof of Theorem 8, using LPO^* to find the set T. By the note following the proof of Theorem 6,

T can be used to calculate the minimum bound. Thus $\mathsf{minERT} \leq_W \mathsf{LPO}^*$. By Theorem 8, $\mathsf{LPO}^* \leq_W \mathsf{ERT}$, so $\mathsf{minERT} \leq_W \mathsf{ERT}$.

For the reverse mathematics result, RCA_0^* proves minERT implies ERT trivially. To prove the converse, assume ERT and let $f : \mathbb{N} \to k$. By ERT, we can find a bound b. By Theorem 6, ERT implies Σ_1^0 induction, so by Lemma 4 we can use bounded Σ_1^0 comprehension to find $Y = \{c < k \mid \exists x(x \geq b \wedge f(x) = c\}$, the range of f on $[b, \infty)$. By the Σ_0^0 least element principle, there is a least $n \leq b$ such that for all $t \in [n, b]$, either $f(t) \in Y$ or $f(t)$ appears at least twice in $[n, b]$. This least n satisfies minERT. □

In contrast to Theorem 11, we will prove below that $\mathsf{ECT} <_W \mathsf{minECT}$. Our proof uses the following characterization of minECT in terms of $\mathsf{TC}_\mathbb{N}$ and $\mathsf{isInfinite}$. The principle $\mathsf{isInfinite}$ takes an infinite binary string as input, outputs 1 if it has infinitely many ones, and outputs 0 otherwise. The notation $P \times Q$ denotes the principle corresponding to solving P and Q in parallel.

Theorem 12. $\mathsf{minECT} \equiv_W \mathsf{TC}_\mathbb{N}^* \times \mathsf{isInfinite}^*$.

Proof. To see that $\mathsf{minECT} \leq_W \mathsf{TC}_\mathbb{N}^* \times \mathsf{isInfinite}^*$, let $f : \mathbb{N} \to k$ be an instance of minECT. For each $j < k$, we can use one instance of $\mathsf{isInfinite}$ to determine if j appears infinitely many times in the range of f, and one instance of $\mathsf{TC}_\mathbb{N}$ to find the last occurrence of j in the case that j appears only finitely many times. Adding one to the maximum of the positions for the values that do not appear infinitely many times yields the desired output for minECT.

The converse relation takes a few steps. By Theorem 9, $\mathsf{TC}_\mathbb{N}^* \equiv_W \mathsf{ECT}$. Trivially, $\mathsf{ECT} \leq_W \mathsf{minECT}$, so $\mathsf{TC}_\mathbb{N}^* \leq_W \mathsf{minECT}$.

To see that $\mathsf{isInfinite} \leq_W \mathsf{minECT}$, let p denote an infinite binary sequence. Let r be the sequence consisting of a 1 followed by the result of alternating 0 with digits from p. Then $\mathsf{minECT}(r)$ is 0 if and only if 1 appears infinitely many times in p.

Next, we show that minECT is idempotent, or to be more precise, that $\mathsf{minECT} \times \mathsf{minECT} \leq_W \mathsf{minECT}$. Let $\langle \cdot, \cdot \rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a bijective map such that if $m_0 \leq m_1$ and $n_0 \leq n_1$, then $\langle m_0, n_0 \rangle \leq \langle m_1, n_1 \rangle$. Let p and q be instances of minECT. Replace $p(0)$ with a color not appearing in the range of p. This increases the value of minECT by one only in the case that every color appears infinitely often in the original sequence. We can now assume that at least one color appears only finitely many times in p. Make the same adjustment and assumption for q. Define the coloring r by $r(\langle n, m \rangle) = \langle p(n), q(m) \rangle$. If n_0 is the last time some color c_0 appears in p, and n_1 is the last time that some color c_1 appears in q, then $\langle n_0, n_1\rangle$ is the last time that $\langle c_0, c_1\rangle$ appears in r. Conversely, if $\langle c_0, c_1\rangle$ appears for the last time at position $\langle n_0, n_1\rangle$, then c_0 must appear last in p at n_0, and c_1 must appear last in q at n_1. Thus, solutions for p and q can be extracted from the solution for r.

Iterated applications of the idempotence of minECT (or an application of Proposition 4.4 of [2]) show that $\mathsf{minECT}^* \leq_{\mathrm{W}} \mathsf{minECT}$. Because $\mathsf{isInfinite} \leq_{\mathrm{W}} \mathsf{minECT}$, we have $\mathsf{isInfinite}^* \leq_{\mathrm{W}} \mathsf{minECT}^* \leq_{\mathrm{W}} \mathsf{minECT}$. We have already shown that $\mathsf{TC}_{\mathbb{N}}^* \leq_{\mathrm{W}} \mathsf{minECT}$, so $\mathsf{TC}_{\mathbb{N}}^* \times \mathsf{isInfinite}^* \leq_{\mathrm{W}} \mathsf{minECT} \times \mathsf{minECT} \leq_{\mathrm{W}} \mathsf{minECT}$, completing the proof of the theorem. □

The next result assists in separating ECT and minECT.

Theorem 13. $\mathsf{isInfinite} \not\leq_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}}^*$.

Proof. Suppose by way of contradiction that Φ and Ψ witness $\mathsf{isInfinite} \leq_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}}^*$. The function mapping sequences p to the number of instances of $\mathsf{TC}_{\mathbb{N}}$ in $\Phi(p)$ is computable and therefore continuous. Let σ_0 and n be such that $\Phi(p)$ consists of n instances of $\mathsf{TC}_{\mathbb{N}}$ for all $p \succcurlyeq \sigma_0$, that is for all p extending σ_0. Denote the ranges of these instances by $\Phi(p)_1, \ldots, \Phi(p)_n$. For $i \leq n$ and $m \in \mathbb{N}$, define $C_{m,i} = \{\sigma \succcurlyeq \sigma_0 \mid m \in \Phi(\sigma)_i\}$. Let σ_1 be an extension of σ_0 such that for each $i \in [1, n]$, either every $C_{m,i}$ is dense below σ_1, or there is an m_i such that $C_{m_i,i}$ contains no extension of σ_1. Let F be the set of all i such that m_i is defined.

Let p consist of σ_1 followed by an infinite sequence of zeros. The sequence p has finitely many ones. There is a solution $(a_1, \ldots, a_n)$ of $\Phi(p)$ such that $a_i = m_i$ for all $i \in F$. Then $\Psi(a_1, \ldots, a_n, p)$ returns 0, with a computation that depends only on $a_1, \ldots, a_n$ and a finite initial segment σ_2 of p. Let $g \succ \sigma_2$ be 1-generic. If $i \notin F$, then for every m, $C_{m,i}$ is dense below σ_2, so $\Phi(g)_i = \mathbb{N}$. Thus, $(a_1, \ldots, a_n)$ is a solution of $\Phi(g)$. But $\Psi(a_1, \ldots, a_n, g) = \Psi(a_1, \ldots, a_n, p) = 0$ and g has infinitely many ones, yielding the desired contradiction. □

Theorem 14. $\mathsf{ECT} <_{\mathrm{W}} \mathsf{minECT}$ *and* RCA_0^* *proves* $\mathsf{ECT} \leftrightarrow \mathsf{minECT}$.

Proof. Trivially, $\mathsf{ECT} \leq_{\mathrm{W}} \mathsf{minECT}$. To prove the strict inequality, suppose by contradiction that $\mathsf{minECT} \leq_{\mathrm{W}} \mathsf{ECT}$. By Theorem 12, $\mathsf{isInfinite} \leq_{\mathrm{W}} \mathsf{minECT}$, so by Theorem 9, $\mathsf{isInfinite} \leq_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}}^*$, contradicting Theorem 13.

Shifting focus to reverse mathematics, trivially RCA_0^* proves that minECT implies ECT. For the converse, assuming ECT, by Theorem 7, we may use Σ_2^0

induction. By the Π^0_1 least element principle (a consequence of Σ^0_1 induction), a minimal bound can be found in the first part of the proof of Theorem 7. Thus, over RCA_0^*, ECT is equivalent to minECT. □

Theorem 14 demonstrates the ability of Weihrauch reductions to make finer distinctions in this setting.

Our final result links minECT to principles considered by Hirst and Mummert [7]. The principle $\mathsf{C}^{\#}_{\max}$ takes as inputs a size n and the enumeration of the complement of a collection of finite subsets of $\mathbb{N}$, each of size at most n, and outputs an element of the collection of maximum cardinality.

Theorem 15. $\mathsf{minECT} \equiv_{\mathrm{W}} \mathsf{C}^{\#}_{\max}$.

Proof. From Theorem 12 we know $\mathsf{minECT} \equiv_{\mathrm{W}} \mathsf{TC}^*_{\mathbb{N}} \times \mathsf{isInfinite}^*$, so it suffices to show that

$$\mathsf{minECT} \leq_{\mathrm{W}} \mathsf{C}^{\#}_{\max} \leq_{\mathrm{W}} \mathsf{TC}^*_{\mathbb{N}} \times \mathsf{isInfinite}^*.$$

For the first reduction, suppose $f : \mathbb{N} \to k$ is an instance of minECT. Consider the set A of all finite sets $F \subset k \times \mathbb{N}$ such that for each $j < k$, if $(j, n) \in F$ then n is the maximum natural number such that $f(j) = n$. Here we are identifying pairs with their integer codes, so F can be viewed as a subset of $\mathbb{N}$. The set A is Π^0_1 definable using f as a parameter, and its complement can be enumerated by a function uniformly computable from f. Use this enumeration and the size k as the input for $\mathsf{C}^{\#}_{\max}$, and let F_0 be the resulting maximal output set. Adding 1 to the maximum of the second coordinates of the elements of F_0 yields the desired bound for minECT.

To prove the final reduction, it is useful to note that $\mathsf{TC}_{\mathbb{N}}$ can be used to count the numbers of ones in a binary string. Using a bijective pairing function, given a sequence $p : \mathbb{N} \to 2$, we can define an enumeration q of the (codes for) pairs that omits at most one pair, so that the first coordinate of that omitted pair corresponds to the number of ones in the range of p, provided that number is finite. Calculation of q can be viewed as a moving marker process. Place a marker on $(0, 0)$ and then alternate enumerating unmarked pairs and calculating values of p until a 1 appears in the range of p. Move the marker to the first non-enumerated pair with an initial coordinate of 1, enumerate $(0, 0)$, and continue enumerating unmarked pairs and calculating values of p until the next 1 appears in the range of p. Iterate. If there are infinitely many ones in the range of p, then q will enumerate all pairs. If only finitely many ones appear, $\mathsf{TC}_{\mathbb{N}}$ applied to q will find a pair with the desired first coordinate.

To prove that $\mathsf{C}^{\#}_{\max} \leq_{\mathrm{W}} \mathsf{TC}^*_{\mathbb{N}} \times \mathsf{isInfinite}^*$, let $f : \mathbb{N} \to \mathbb{N}^{<\mathbb{N}}$ enumerate the complement of a set A of finite subsets of $\mathbb{N}$, each of size less than k. For each positive $i < k$, let e_i be an enumeration of all the subsets of $\mathbb{N}$ of size exactly i. For each positive $i < k$, define the instance p_i of $\mathsf{isInfinite}$ as follows. Set $p_i(n) = 1$ if there is a $t \leq n$ such that $f(t) = e_i(c_t)$ where $c_t = |\{j < n \mid p_i(j) = 1\}|$, and set $p_i(n) = 0$ otherwise. Thus f enumerates all sets of size i if and only if the range of p_i contains infinitely many ones, and the range of p_i contains a total of n ones if and only if $e_i(n)$ is the first set enumerated by e_i that is in A. For each p_i, let q_i be the associated instance of $\mathsf{TC}_{\mathbb{N}}$ that counts the ones in the range of p_i. Given the solutions to $\mathsf{isInfinite}$ for each p_i and to $\mathsf{TC}_{\mathbb{N}}$ for each q_i for all $i < k$, we can find the maximum j such that $\mathsf{isInfinite}$ fails for p_j. If n is the output from $\mathsf{TC}_{\mathbb{N}}$ for q_j, then $e_j(n)$ is a maximal element of A, solving the instance $\mathsf{C}^{\#}_{\max}$ corresponding to f. $\square$

Hirst and Mummert [7] showed that $\mathsf{C}^{\#}_{\max}$ is Weihrauch equivalent and provably equivalent over RCA_0 to several principles formalizing calculation of bases for bounded dimension matroids and vector spaces, and finding connected component decompositions of graphs with finitely many components. Thus minECT is Weihrauch equivalent to all these principles, ECT is strictly Weihrauch weaker, and all of them are provably equivalent over RCA_0.

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