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Ramsey's theorem for trees: the polarized tree theorem and notions of stability

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Abstract We formulate a polarized version of Ramsey's theorem for trees. For exponents greater than 2, the reverse mathematics and computability theory associated with this theorem parallels that of its linear analog. For pairs, the situation is more complex. In particular, there are many reasonable notions of stability in the tree setting, complicating the analysis of the related results.

Keywords combinatorics · computability · Ramsey · polarized · reverse mathematics · stable

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1 Introduction

This paper continues the study of Ramsey's theorem for trees, from a computability theoretic and reverse mathematics point of view. For general background and notation in computability theory and reverse mathematics, see Soare [10] and Simpson [11], respectively.

Definition 1.1 Let $X \subseteq \mathbb{N}$ be infinite and $n,k \geq 1$.

1. We denote by $[X]^n$ the set of all n-element subsets of X.

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2. A function $f:[X]^n \to k$, where $k=\{0,1,\ldots,k-1\}$, is a *k-coloring* of $[X]^n$.
3. A set $H \subseteq \mathbb{N}$ is *homogeneous* for a k-coloring f of $[\mathbb{N}]^n$ if H is infinite and $f \restriction [H]^n$ is constant.

The following statement of the *linear* version of Ramsey's theorem can be made in RCA_0.

Ramsey's Theorem *Let $n,k \geq 1$.*

RT^n_k: *Every k-coloring $f:[\mathbb{N}]^n \to k$ has a homogeneous set.*
RT^n: *For all $k \geq 1$, RT^n_k.*
RT : *For all $n \geq 1$, RT^n.*

Chubb, Hirst, and McNicholl [4] considered a version of Ramsey's theorem for trees. Let $2^{<\mathbb{N}}$ denote the full binary tree of height ω; a *subtree* is any subset of $2^{<\mathbb{N}}$.

Definition 1.2 ([4]) Let $n,k \geq 1$.

1. We denote by $[2^{<\mathbb{N}}]^n$ the collection of *linearly ordered* subsets of $2^{<\mathbb{N}}$ of size n.
2. A subtree $S \subseteq 2^{<\mathbb{N}}$ is *(order) isomorphic* to $2^{<\mathbb{N}}$, written $S \cong 2^{<\mathbb{N}}$, if there is a bijection $g:2^{<\mathbb{N}} \to S$ such that for all $\sigma,\tau \in 2^{<\mathbb{N}}$, $\sigma \subseteq \tau$ if and only if $g(\sigma) \subseteq g(\tau)$.
3. A subtree $S \subseteq 2^{<\mathbb{N}}$ is *monochromatic* for a k-coloring $f:[2^{<\mathbb{N}}]^n \to k$ if $f \restriction [S]^n$ is constant. A *homogeneous set* for f is a monochromatic subtree $S \cong 2^{<\mathbb{N}}$.

Given $f:[2^{<\mathbb{N}}]^n \to k$ and $\{\sigma_0,\ldots,\sigma_{n-1}\} \in [2^{<\mathbb{N}}]^n$, we shall write $f(\sigma_0,\ldots,\sigma_{n-1})$ instead of $f(\{\sigma_0,\ldots,\sigma_{n-1}\})$ when $\sigma_0 \subseteq \cdots \subseteq \sigma_{n-1}$.

The following statement of the *tree* version of Ramsey's theorem can be made in RCA_0.

Tree Theorem *Let $n,k \geq 1$.*

TT^n_k: *Every k-coloring $f:[2^{<\mathbb{N}}]^n \to k$ has a homogeneous set.*
TT^n: *For all $k \geq 1$, TT^n_k.*
TT : *For all $n \geq 1$, TT^n.*

Dzhafarov and Hirst [6] considered a polarized version of Ramsey's theorem.

Definition 1.3 ([6], Definition 1.3) Let $n,k \geq 1$.

1. A *p-homogeneous set* for a k-coloring $f:[\mathbb{N}]^n \to k$ is a sequence $\langle H_0,\ldots,H_{n-1}\rangle$ of infinite sets such that for some $c<k$, $f(\{x_0,\ldots,x_{n-1}\})=c$ for all $\langle x_0,\ldots,x_{n-1}\rangle \in H_0 \times \cdots \times H_{n-1}$ with $x_i \neq x_j$ whenever $i \neq j$.
2. An *increasing p-homogeneous set* for f is a sequence $\langle H_0,\ldots,H_{n-1}\rangle$ of infinite sets such that for some $c<k$ and for all $\langle x_0,\ldots,x_{n-1}\rangle \in H_0 \times \cdots \times H_{n-1}$ with $x_0 < \cdots < x_{n-1}$, $f(\{x_0,\ldots,x_{n-1}\})=c$.

The following statements of *polarized* versions of (the linear) Ramsey's theorem can be made in RCA_0.

(Increasing) Polarized Theorem *Let $n,k \geq 1$.*

PT^n_k: *Every k-coloring* $f : [\mathbb{N}]^n \to k$ *has a p-homogeneous set.*
PT^n: *For every* $k \geq 1$, PT^n_k.
PT : *For every* $n \geq 1$, PT^n.
IPT^n_k: *Every k-coloring* $f : [\mathbb{N}]^n \to k$ *has an increasing p-homogeneous set.*
IPT^n: *For every* $k \geq 1$, IPT^n_k.
IPT : *For every* $n \geq 1$, IPT^n.

The question of whether the polarized form of Ramsey's theorem is weaker than the linear form was asked originally by Schmerl (personal communication), who was working on an application of the polarized theorem. Dzhafarov and Hirst [6] proved that, for fixed exponent, the two theorems are equivalent over RCA_0. Since we are interested in the ways that tree results differ from or resemble the results in the linear setting, it is natural to look at a polarized form of the Tree Theorem.

In order to formulate a polarized Ramsey theorem on trees, we must describe a way to interweave sequences of trees. One method to interweave a sequence of n trees, each isomorphic to $2^{<\mathbb{N}}$, is to construct a copy of $2^{<\mathbb{N}}$ with each node replaced by a linearly ordered sequence of n nodes containing one representative from each tree (in order). Another method, which we have chosen, is to divide a copy of $2^{<\mathbb{N}}$ up by levels. A polarized Ramsey theorem on trees can be formalized in each of these settings, and it can be shown that the results will be the same. We have chosen the formalization that we feel is easier to work with notationally.

Definition 1.4 Suppose S is a subtree of $2^{<\mathbb{N}}$ and $g : 2^{<\mathbb{N}} \to S$ is a bijection witnessing that S is order isomorphic to $2^{<\mathbb{N}}$. For $n \geq 1$, the sequence $S_0, \dots, S_{n-1}$ of *stratified subtrees (mod n)* is defined by

$$S_j = \{\sigma \in S \mid |g^{-1}(\sigma)| \equiv j \mod n\}$$

for each $j < n$. We write $S = \langle S_0, \dots, S_{n-1} \rangle$.

We can then define the analog of p-homogeneity in the tree setting, emulating Definition 1.3 of [6].

Definition 1.5 Let $n, k \geq 1$ and suppose $f : [2^{<\mathbb{N}}]^n \to k$.

1. A subtree $S = \langle S_0, \dots, S_{n-1} \rangle \cong 2^{<\mathbb{N}}$ is said to be a *p-homogeneous set* for f if there is some $c < k$ such that $f(\{\sigma_0, \dots, \sigma_{n-1}\}) = c$ for every set $\{\sigma_0, \dots, \sigma_{n-1}\} \in [2^{<\mathbb{N}}]^n$ satisfying $\langle \sigma_0, \dots, \sigma_{n-1} \rangle \in S_0 \times \cdots \times S_{n-1}$.
2. If the preceding holds just for subsets $\{\sigma_0, \dots, \sigma_{n-1}\}$ satisfying $\sigma_0 \subset \cdots \subset \sigma_{n-1}$, then we call S an *increasing* p-homogeneous set.

As in [6], we can formalize the *polarized tree theorem* in RCA_0.

(Increasing) Polarized Tree Theorem *Let* $n, k \geq 1$.

PTT^n_k: *Every k-coloring* $f : [2^{<\mathbb{N}}]^n \to k$ *has a p-homogeneous set.*
PTT^n: *For every* $k \geq 1$, PTT^n_k.
PTT : *For every* $n \geq 1$, PTT^n.
IPTT^n_k: *Every k-coloring* $f : [2^{<\mathbb{N}}]^n \to k$ *has an increasing p-homogeneous set.*

IPTT^n: *For every* $k \geq 1$, IPTT^n_k.
IPTT : *For every* $n \geq 1$, IPTT^n.

The following result is clear.

Proposition 1.6 (RCA_0) *For every* $n, k \geq 1$, $\mathsf{PTT}^n_k \rightarrow \mathsf{IPTT}^n_k$.

In this paper, we extend Chubb, Hirst, and McNicholl's results [4] for the tree version of Ramsey's theorem, with an emphasis on notions of stability for colorings of comparable pairs of strings. Our results show that, for exponent 3 and higher, the Polarized Tree Theorem and the Polarized Ramsey Theorem are equivalent over RCA_0. However, we shall see in the case of exponent 2 that the polarized versions of the Tree Theorem seem more problematic than the polarized versions of the linear Ramsey's Theorem. In particular, we shall see that the Polarized Tree Theorem for pairs is (apparently) weaker than the Tree Theorem for pairs. It could be that the linear form of Ramsey's theorem for pairs proves the Polarized Tree Theorem for pairs, and this could provide a way to connect linear results with the tree results for pairs. Conversely, the tree setting has exposed just how essential linearity is to many arguments about Ramsey's theorem. Identifying further examples of this phenomenon, as we do in this paper, could thus lead to a better understanding of the key differences between Ramsey's theorem and the Tree Theorem.

2 The polarized tree theorem (PTT)

We begin with a computability-theoretic investigation of the polarized version of the Tree Theorem. Dzhafarov and Hirst ([6], Remark 1.4) noted that $\mathsf{RT}^n \rightarrow \mathsf{PT}^n$ (over RCA_0), since every homogeneous set H for a k-coloring f of $[\mathbb{N}]^n$ computes the p-homogeneous set $\langle H, \dots, H \rangle$ (i.e., $H_i = H$ for all $0 \leq i < n$) for f. Since every tree T isomorphic to $2^{<\mathbb{N}}$ can be viewed as a sequence of stratified subtrees, the following is immediate.

Proposition 2.1 (RCA_0) *For every* $n, k \geq 1$, $\mathsf{TT}^n_k \rightarrow \mathsf{PTT}^n_k$.

The following observation of Chubb, Hirst, and McNicholl ([4], proof of Theorem 1.5) is also useful.

Remark 2.2 Every coloring $f : [\mathbb{N}]^n \rightarrow k$ induces a coloring $g : [2^{<\mathbb{N}}]^n \rightarrow k$ defined by $g(\sigma_0, \dots, \sigma_{n-1}) = f(|\sigma_0|, \dots, |\sigma_{n-1}|)$ for all $\{\sigma_0, \dots, \sigma_{n-1}\} \in [2^{<\mathbb{N}}]^n$. Every homogeneous set $T \cong 2^{<\mathbb{N}}$ for g computes an infinite homogeneous set (in the linear sense) for f. Similarly, every p-homogeneous set $T = \langle T_0, \dots, T_{n-1} \rangle \cong 2^{<\mathbb{N}}$ for g computes a p-homogeneous set (in the linear sense of [6]) for f.

In his 1972 paper [9], Jockusch studied the complexity, in terms of Turing degree and arithmetic definability, of homogeneous sets for computable finite colorings of $[\mathbb{N}]^n$.

Theorem 2.3 ([9], Theorem 5.5, Theorem 5.6, Theorem 5.7) *Fix* $k \geq 1$.

1. *For all* $n \geq 1$, *every computable* $f : [\mathbb{N}]^n \rightarrow k$ *has a* Π^0_n *homogeneous set.*

2. *For all $n \geq 1$, every computable $f : [\mathbb{N}]^n \to k$ has a homogeneous set A such that $A' \leq_T 0^{(n)}$.*
3. *For every $n \geq 2$, there exists a computable $f : [\mathbb{N}]^{n+1} \to 2$ such that for every homogeneous set A, $0^{(n-1)} \leq_T A$.*

Dzhafarov and Hirst ([6], Theorems 2.1 and 2.3) showed that the analogues of Theorem 2.3 (1) and (2) hold for p-homogeneous sets, and that the analogue of Theorem 2.3 (3) holds for increasing p-homogeneous sets. By Remark 2.2, the analogue of (3) (and, in fact, several other existential results) also holds in the tree setting. In addition, Chubb, Hirst, and McNicholl ([4], Theorem 2.7) showed that the analogue of Theorem 2.3 (1) holds for finite computable colorings of $[2^{<\mathbb{N}}]^n$. The analogue of Theorem 2.3 (2) for finite computable colorings of $[2^{<\mathbb{N}}]^n$ also follows immediately from their results. Since a proof is not given explicitly in [4], we provide a sketch, as Theorem 2.5, for completeness.

We first note that the proofs of Theorem 2.3 (1) and (2) are by induction on n. In the inductive step, Jockusch uses the fact ([9], Lemma 5.4) that for every computable finite coloring of $[\mathbb{N}]^{n+1}$, there is an infinite set A such that $A' \leq_T 0''$ and for all $a_0 < a_1 < \cdots < a_{n-1} < b_1, b_2$ in A, $f(a_0, a_1, \ldots, a_{n-1}, b_1) = f(a_0, a_1, \ldots, a_{n-1}, b_2)$. Chubb, Hirst, and McNicholl proved the following analogous result for $[2^{<\mathbb{N}}]^{n+1}$ (while they stated their result for $n > 1$, their proof also works for $n = 1$).

Lemma 2.4 ([4], Lemma 2.6) *Suppose that $n, k \geq 1$ and $f : [2^{<\mathbb{N}}]^{n+1} \to k$ is computable. There is a tree T which is isomorphic to $2^{<\mathbb{N}}$ such that the following hold:*

1. *$T' \leq_T 0''$.*
2. *If $\sigma_0, \ldots, \sigma_{n-1}$ is a sequence of n comparable elements of T and τ_1 and τ_2 are extensions of σ_{n-1} in T, then $f(\sigma_0, \ldots, \sigma_{n-1}, \tau_1) = f(\sigma_0, \ldots, \sigma_{n-1}, \tau_2)$.*

Theorem 2.5 (implicit in [4]) *For all $n, k \geq 1$, every computable $f : [2^{<\mathbb{N}}]^n \to k$ has a homogeneous set S such that $S' \leq_T 0^{(n)}$.*

Proof We proceed by induction on n. For $n = 1$, the result is essentially Theorem 1.2 of [4] that for all k, TT^1_k is provable in $\mathsf{RCA}_0 + \Sigma^0_2$-$\mathsf{IND}$. It is easy to adapt the proof of this theorem to show that every computable $f : 2^{<\mathbb{N}} \to k$ has a computable homogeneous set, and this argument clearly relativizes.

Next, assume that the result and all its relativizations hold for some $n \geq 1$. Let $T_0 \cong 2^{<\mathbb{N}}$ be arbitrary, and suppose $f : [T_0]^{n+1} \to k$ is T_0-computable. Let $T \cong 2^{<\mathbb{N}}$ be as given by Lemma 2.4 relativized to T_0, so that $T \subseteq T_0$ and $(T_0 \oplus T)' \leq_T T_0''$. Define $\hat{f} : [T]^n \to k$ as follows: given a sequence $\sigma_0 \subseteq \cdots \subseteq \sigma_{n-1}$ of comparable elements in T, let σ_n be the least extension of σ_{n-1} in T, and let $\hat{f}(\sigma_0, \ldots, \sigma_{n-1}) = f(\sigma_0, \ldots, \sigma_{n-1}, \sigma_n)$. Note that $\hat{f} \leq_T f \oplus T \leq_T T_0 \oplus T$. By the inductive hypothesis, relativized to $T_0 \oplus T$, choose a homogeneous set $S \subseteq T$ for $\hat{f}$ such that $S' \leq_T (T_0 \oplus T)^{(n)}$. Then S is clearly homogeneous for f, and we have $S' \leq_T T_0^{(n+1)}$, as desired. □

By Proposition 2.1 and the preceding comments, we immediately have the following.

Theorem 2.6 *Fix $n, k \geq 1$.*

1. *Every computable $f : [2^{<\mathbb{N}}]^n \to k$ has a Π^0_n p-homogeneous set.*
2. *Every computable $f : [2^{<\mathbb{N}}]^n \to k$ has a p-homogeneous set whose jump is computable in $0^{(n)}$.*
3. *There exists a computable $f : [2^{<\mathbb{N}}]^{n+1} \to 2$ such that every increasing p-homogeneous set computes $0^{(n-1)}$.*

3 Notions of stability

Recall that in the linear setting, a k-coloring $f : [\mathbb{N}]^2 \to k$ is *stable* if for all $a \in \mathbb{N}$, there exists $b_0 \in \mathbb{N}$ such that for all $b \geq b_0$, $f(a,b_0) = f(a,b)$. The stable versions of Ramsey's theorem in the linear and (increasing) polarized linear settings have been studied in, for example, [1], [8], and [6].

Definition 3.1 (RCA_0) Let $n,k \geq 1$.

SRT^2_k : Every stable k-coloring $f : [\mathbb{N}]^2 \to k$ has a homogeneous set.
SPT^2_k : Every stable k-coloring $f : [\mathbb{N}]^2 \to k$ has a p-homogeneous set.
SIPT^2_k: Every stable k-coloring $f : [\mathbb{N}]^2 \to k$ has an increasing p-homogeneous set.

SRT^2, SPT^2, and SIPT^2 are then defined in the obvious way.

There are apparently several ways to formulate a notion of stability for colorings of $[2^{<\mathbb{N}}]^2$. In the definition below, the strongest version of stability corresponds to moving from a stable coloring of $[\mathbb{N}]^2$ (the linear setting) to the induced coloring of $[2^{<\mathbb{N}}]^2$ as in Remark 2.2. The weakest version corresponds to the most obvious rephrasing of the linear notion of stability in the tree setting.

Definition 3.2 (RCA_0) Let $k \geq 1$ and $f : [2^{<\mathbb{N}}]^2 \to k$. We say that f is

1. *1-stable* if for every $\sigma \in 2^{<\mathbb{N}}$ there exists $c < k$ and $n \geq |\sigma|$ such that $f(\sigma,\tau) = c$ for all $\tau \supset \sigma$ with $|\tau| \geq n$.
2. *2-stable* if for every $\sigma \in 2^{<\mathbb{N}}$ there is an $n \geq |\sigma|$ such that for every extension $\tau \supset \sigma$ of length n, $f(\sigma,\rho) = f(\sigma,\tau)$ for every $\rho \supseteq \tau$.
3. *3-stable* if for each $\sigma \in 2^{<\mathbb{N}}$ there exists $c < k$ such that for every $\sigma' \supseteq \sigma$ there exists $\tau \supset \sigma'$ with $f(\sigma,\rho) = c$ for all $\rho \supseteq \tau$.
4. *4-stable* if for each $\sigma \in 2^{<\mathbb{N}}$ and each $\sigma' \supseteq \sigma$, there exists $\tau \supset \sigma'$ such that $f(\sigma,\rho) = f(\sigma,\tau)$ for all $\rho \supseteq \tau$.
5. *5-stable* if for every $\sigma \in 2^{<\mathbb{N}}$ there is a $\sigma' \supset \sigma$ such that $f(\sigma,\tau) = f(\sigma,\sigma')$ for all $\tau \supseteq \sigma'$.
6. *6-stable* if for every $\sigma \in 2^{<\mathbb{N}}$ we can find a $\sigma' \supset \sigma$ and a $c < k$ such that for all subtrees T extending σ' which are isomorphic to $2^{<\mathbb{N}}$, there is a $\tau \in T$ such that $f(\sigma,\tau) = c$.

Intuitively, we can think of the various notions of stability in the following way. Given $f : [2^{<\mathbb{N}}]^2 \to k$, any fixed $\sigma \in 2^{<\mathbb{N}}$ induces a coloring f_σ of the (singleton) nodes of the tree extending σ, namely, for $\tau \supseteq \sigma$, $f_\sigma(\tau) = f(\sigma,\tau)$. If f is 5-stable, then every such induced coloring has at least one monochromatic "cone". If f is 4-stable, then monochromatic cones are dense in the ordering above σ. If f is 3-stable, then for each σ there is a single color such that the monochromatic cones of that color are dense in the ordering above σ. If f is 2-stable, then there is a level of $2^{<\mathbb{N}}$ such that each cone rooted at that level is colored the same as its root, and if f is 1-stable then the color of each of these cones is the same.

We first present some results about the relationships between various notions of stability. Note that while 6-stability may appear to be a weaker notion, it is in fact equivalent to 5-stability.

Theorem 3.3 (RCA_0) *Let $k \geq 1$. A k-coloring $f : [2^{<\mathbb{N}}]^2 \to k$ is 5-stable if and only if it is 6-stable.*

Proof Working throughout in RCA_0, first suppose f is 5-stable and fix σ. Applying the definition of 5-stable, choose $\sigma' \supset \sigma$ so that for all $\tau \supseteq \sigma'$, $f(\sigma,\tau) = f(\sigma,\sigma')$. Set $c = f(\sigma,\sigma')$. Then for any subtree T of extensions of σ', every $\tau \in T$ satisfies $f(\sigma,\tau) = c$. Hence f is 6-stable.

To prove the contrapositive of the converse, suppose f is not 5-stable and fix σ such that for every $\sigma' \supset \sigma$ there is a $\tau \supseteq \sigma'$ such that $f(\sigma,\tau) \neq f(\sigma,\sigma')$. Note that for any $\sigma' \supset \sigma$ and any $c < k$, either $f(\sigma,\sigma') \neq c$ or there is a $\tau \supset \sigma'$ such that $f(\sigma,\tau) \neq c$. Now fix $\sigma' \supset \sigma$ and $c < k$. Choose $\tau_{\langle\,\rangle}$ to be the least (proper or improper) extension of σ' such that $f(\sigma,\tau) \neq c$. If τ_α has been selected, then for each $i \in \{0,1\}$ let $\tau_{\alpha^\frown i}$ be the least extension of $\tau_\alpha{}^\frown i$ such that $f(\sigma,\tau_{\alpha^\frown i}) \neq c$. RCA_0 suffices to prove that $T = \{\tau_\alpha \mid \alpha \in 2^{<\mathbb{N}}\}$ exists and that for every $\tau \in T$, $f(\sigma,\tau) \neq c$. Hence, f is not 6-stable. □

The following relationships between the remaining notions of stability are obvious.

Proposition 3.4 (RCA_0) *Let $k \geq 1$ and $f : [2^{<\mathbb{N}}]^2 \to k$. Then*

f is 1-stable $\to$ f is 2-stable $\to$ f is 4-stable $\to$ f is 5-stable

and

f is 1-stable $\to$ f is 3-stable $\to$ f is 4-stable $\to$ f is 5-stable.

While there appears to be no obvious relationship between 2-stability and 3-stability, when combined we obtain a partial converse to Proposition 3.4.

Theorem 3.5 (RCA_0) *Let $k \geq 1$. For every k-coloring f of $[2^{<\mathbb{N}}]^2$, f is 1-stable if and only if f is both 2-stable and 3-stable.*

Proof We work in RCA_0. Let $f : [2^{<\mathbb{N}}]^2 \to k$. It is clear that if f is 1-stable, then f is both 2-stable and 3-stable. So assume that f is both 2-stable and 3-stable. Let $\sigma \in 2^{<\mathbb{N}}$ be given. Since f is 2-stable, we may fix $n > |\sigma|$ and $m = 2^{n-|\sigma|} - 1$ such that if $\tau_0, \dots, \tau_m$ are all the extensions of σ of length n, then for all i, $0 \leq i \leq m$, and for all $\tau \supseteq \tau_i$, $f(\sigma,\tau) = f(\sigma,\tau_i)$. Since f is 3-stable, fix $c < k$ such that for all $\sigma' \supseteq \sigma$, there exists $\tau \supseteq \sigma'$ such that $f(\sigma,\rho) = c$ for all $\rho \supseteq \tau$.

Let $\tau \supset \sigma$ with $|\tau| \geq n$, and fix i, $0 \leq i \leq m$, such that $\sigma \subseteq \tau_i \subseteq \tau$. By 2-stability, $f(\sigma,\tau_i) = f(\sigma,\tau)$. By 3-stability, we may fix $\tau' \supseteq \tau$ such that $f(\sigma,\rho) = c$ for all $\rho \supseteq \tau'$. But $\tau' \supseteq \tau \supseteq \tau_i$, so $c = f(\sigma,\tau') = f(\sigma,\tau_i)$, and thus $f(\sigma,\tau) = c$, as desired. Hence f is 1-stable. □

While the property of being 1-stable or 2-stable is preserved under subtrees, colorings which are 3-stable do not necessarily have this preservation property. This seems to be a barrier to extending our proof of Theorem 3.14, below, beyond 2-stability.

Proposition 3.6 *Let $k \geq 1$. If $f : [2^{<\mathbb{N}}]^2 \to k$ is 1-stable and T is a subtree of $2^{<\mathbb{N}}$ order isomorphic to $2^{<\mathbb{N}}$, then $f \upharpoonright [T]^2$ is also 1-stable. The same statement holds for 2-stable.*

Proof This is immediate from the definitions of 1-stable and 2-stable. In each case, for each σ, the choice of n for the full tree works for any subtree. □

Proposition 3.7 *There is a 3-stable coloring $f : [2^{<\mathbb{N}}]^2 \to 3$ and a subtree T order isomorphic to $2^{<\mathbb{N}}$ such that $f \upharpoonright [T]^2$ is not 5-stable (and consequently is not 3-stable).*

Proof The coloring f is defined in terms of a coloring $h : 2^{<\mathbb{N}} \to 3$ on single nodes. When the length of a string $\sigma \in 2^{<\mathbb{N}}$ is even, we can group the values of σ in consecutive pairs and consider the mod 4 representation. For example, the string 001110 corresponds to 032.

Define h for $\sigma \in 2^{<\mathbb{N}}$ by:

$h(\sigma) = 0$ if $\sigma = \langle\, \rangle$,
$h(\sigma) = 1$ if $|\sigma|$ is odd,
$h(\sigma) = 1$ if $|\sigma|$ is even and 2 or 3 appears in the mod 4 representation,
$h(\sigma) = 0$ if $|\sigma|$ is even, no 2 or 3 appears in the mod 4 representation, and the last digit of σ is 0, and
$h(\sigma) = 2$ if $|\sigma|$ is even, no 2 or 3 appears in the mod 4 representation, and the last digit of σ is 1.

For $\sigma \subset \tau$, define $f(\sigma, \tau) = h(\tau)$. It is not hard to show that for every $\sigma' \supset \sigma$, there is a $\tau \supseteq \sigma'$ such that $f(\sigma, \rho) = 1$ for all $\rho \supseteq \tau$, so f is 3-stable. Also, the subtree of $2^{<\mathbb{N}}$ consisting of those nodes not colored 1 is order isomorphic to $2^{<\mathbb{N}}$, and the restriction of f to this subtree is not 5-stable. □

We can define stable versions of the Tree Theorem and the (Increasing) Polarized Tree Theorem in RCA_0. The most obvious relationships among the various statements are given below.

Definition 3.8 (Stable Tree Theorems) (RCA_0) Let $i \in \{1,2,3,4,5\}$ and $k \geq 1$.

$\mathsf{S}^i\mathsf{TT}^2_k$: Every i-stable $f : [2^{<\mathbb{N}}]^2 \to k$ has a homogeneous set.
$\mathsf{S}^i\mathsf{PTT}^2_k$: Every i-stable $f : [2^{<\mathbb{N}}]^2 \to k$ has a p-homogeneous set.
$\mathsf{S}^i\mathsf{IPTT}^2_k$: Every i-stable $f : [2^{<\mathbb{N}}]^2 \to k$ has an increasing p-homogeneous set.

$\mathsf{S}^i\mathsf{TT}^2$, $\mathsf{S}^i\mathsf{PTT}^2$, and $\mathsf{S}^i\mathsf{IPTT}^2$ are then defined in the obvious way.

Corollary 3.9 (Corollary to Proposition 3.4) RCA_0 *proves that for each $k \geq 1$,*

$$\mathsf{S}^5\mathsf{TT}^2_k \to \mathsf{S}^4\mathsf{TT}^2_k \to \mathsf{S}^2\mathsf{TT}^2_k \to \mathsf{S}^1\mathsf{TT}^2_k,$$

and

$$\mathsf{S}^4\mathsf{TT}^2_k \to \mathsf{S}^3\mathsf{TT}^2_k \to \mathsf{S}^1\mathsf{TT}^2_k.$$

The analogous statements hold for the polarized and increasing polarized versions of the Stable Tree Theorem.

Proposition 3.10 *For each $i \in \{1,2,3,4,5\}$, RCA_0 proves that for every $k \geq 1$,*

1. $\mathsf{PTT}^2_k \to \mathsf{S}^i\mathsf{PTT}^2_k$.
2. $\mathsf{IPTT}^2_k \to \mathsf{S}^i\mathsf{IPTT}^2_k$.
3. $\mathsf{S}^i\mathsf{TT}^2_k \to \mathsf{S}^i\mathsf{PTT}^2_k \to \mathsf{S}^i\mathsf{IPTT}^2_k$.

Proposition 3.11 *For each $i \in \{1,2,3,4,5\}$, RCA_0 proves that for every $k \geq 1$,*

1. $\mathsf{S}^i\mathsf{PTT}^2_k \to \mathsf{SPT}^2_k$
2. $\mathsf{S}^i\mathsf{IPTT}^2_k \to \mathsf{SIPT}^2_k$

Proof This follows from Remark 2.2; note that when $f : [\mathbb{N}]^2 \to k$ is stable, then the induced coloring of $[2^{<\mathbb{N}}]^2$ in Remark 2.2 is 1-stable, and hence i-stable for each $i \in \{1,2,3,4,5\}$. □

The next theorem shows that Proposition 2.2 of [6] holds for trees and leads to a proof that the 2-stable Tree Theorem and the 2-stable (Increasing) Polarized Tree Theorems are equivalent.

Theorem 3.12 *For every $k \geq 1$ and every 2-stable $f : [2^{<\mathbb{N}}]^2 \to k$, every increasing p-homogeneous set for f computes a homogeneous one.*

Proof Let $f : [2^{<\mathbb{N}}]^2 \to k$ be 2-stable, and let $S = \langle S_0, S_1 \rangle \cong 2^{<\mathbb{N}}$ be an increasing p-homogeneous set for f with color $c < k$. Let $S = \{\sigma_\tau \mid \tau \in 2^{<\mathbb{N}}\}$, so that $S_0 = \{\sigma_\tau \mid |\tau| \equiv 0 \mod 2\}$ and $S_1 = \{\sigma_\tau \mid |\tau| \equiv 1 \mod 2\}$. We construct a homogeneous subtree $R = \{p_\tau \mid \tau \in 2^{<\mathbb{N}}\}$ isomorphic to $2^{<\mathbb{N}}$ such that $R \subseteq S_0$ by enumerating R in "increasing order", using S as an oracle.

Let $p_{\langle\,\rangle} = \sigma_{\langle\,\rangle}$. At stage $n+1$, $n \geq 0$, we assume that R has been defined through height n; i.e., assume we have $R_n = \{p_\tau \mid \tau \in 2^{\leq n}\} \subseteq S_0$, where $2^{\leq n}$ denotes the full binary tree of height n, and assume that for all $\alpha \subset \beta$ in R_n, $f(\alpha,\beta) = c$.

Given a leaf $p_\tau \in R_n$, we define $p_{\tau^\frown 0}$ and $p_{\tau^\frown 1}$. Note that because f is defined only on pairs of comparable strings, we need only be sure that for all $\rho \subseteq p_\tau$, $\rho \in R_n$, $f(\rho, p_{\tau^\frown 0}) = f(\rho, p_{\tau^\frown 1}) = c$.

By the 2-stability of f, for each $\alpha \subseteq \tau$, there is a level n_α such that for any extension σ of p_α of length n_α, $f(p_\alpha,\sigma) = f(p_\alpha,\sigma')$ for every σ' extending σ. This implies that for any extension σ of p_α of length greater than or equal to n_α, and for every σ' extending σ, we have $f(p_\alpha,\sigma) = f(p_\alpha,\sigma')$. If σ is an extension of p_α of length at least n_α and $\sigma \in S_0$, then there is a $\sigma' \supseteq \sigma$ with $\sigma' \in S_1$. By the p-homogeneity of S, we know $f(p_\alpha,\sigma') = c$, so since $f(p_\alpha,\sigma) = f(p_\alpha,\sigma')$, we have $f(p_\alpha,\sigma) = c$ also. Summarizing, if σ is an extension of p_α of length at least n_α and $\sigma \in S_0$, then $f(p_\alpha,\sigma) = c$. In particular, for every $\sigma \supset p_\tau$ in S_0 of length at least $\max\{n_\alpha \mid \alpha \subseteq \tau\}$, we have $f(\rho,\sigma) = c$ for all $\rho \subseteq p_\tau$.

We may therefore successfully search S-computably for the least incomparable strings $\tau_0, \tau_1 \in 2^{<\mathbb{N}}$ such that $|\tau_0| \equiv |\tau_1| \equiv 0 \mod 2$ (so that $\sigma_{\tau_0}, \sigma_{\tau_1} \in S_0$), σ_{τ_0} and σ_{τ_1} extend p_τ, and for all $\rho \subseteq p_\tau$ in R_n, $f(\rho,\sigma_{\tau_0}) = f(\rho,\sigma_{\tau_1}) = c$. Define $p_{\tau^\frown 0} = \sigma_{\tau_0}$ and $p_{\tau^\frown 1} = \sigma_{\tau_1}$. Finally, define

$$R_{n+1} = R_n \cup \{p_{\tau^\frown i} \mid \tau \in 2^{<\mathbb{N}} \wedge |\tau| = n \wedge i \in \{0,1\}\}.$$

The set $R = \cup_{n\in\mathbb{N}} R_n$ is S-computable and homogeneous for f.

□

Note that $B\Sigma^0_2$ suffices to formalize the proof of Theorem 3.12. Since every 1-stable coloring is 2-stable, the next corollary follows immediately.

Corollary 3.13 *For every $k \geq 1$ and every 1-stable $f : [2^{<\mathbb{N}}]^2 \to k$, every increasing p-homogeneous set for f computes a homogeneous one.*

The following reverse mathematics results also follow from Theorem 3.12.

Theorem 3.14 *For $i \in \{1,2\}$, RCA_0 proves*

1. *For every $k \geq 1$, $\mathsf{S}^i\mathsf{TT}^2_k \leftrightarrow \mathsf{S}^i\mathsf{PTT}^2_k \leftrightarrow \mathsf{S}^i\mathsf{IPTT}^2_k$.*
2. *$\mathsf{S}^i\mathsf{TT}^2 \leftrightarrow \mathsf{S}^i\mathsf{PTT}^2 \leftrightarrow \mathsf{S}^i\mathsf{IPTT}^2$.*

Proof Let $i \in \{1,2\}$. We work in RCA_0. By Proposition 3.10, we need only show that $\mathsf{S}^i\mathsf{IPTT}^2_k \to \mathsf{S}^i\mathsf{TT}^2_k$. For $k = 1$, this is trivial. Assume $\mathsf{S}^i\mathsf{IPTT}^2_k$ and $k \geq 2$. First note that $\mathsf{S}^i\mathsf{IPTT}^2_k \to \mathsf{SIPT}^2_k$ by Proposition 3.11. By Theorem 3.5 of [6], $\mathsf{SIPT}^2_k \to D^2_k$, where D^2_k is the statement (in second order arithmetic) that for every stable $f : [\mathbb{N}]^2 \to k$, there exist an infinite set X and $c < k$ such that $\lim_s f(x,s) = c$ for all $x \in X$. By Chong, Lempp, and Yang ([3], Theorem 1.4), $D^2_k \to B\Sigma^0_2$. Hence we may assume $B\Sigma^0_2$. Thus, if we let $f : [2^{<\mathbb{N}}]^2 \to k$ be i-stable, f has a p-homogeneous set by $\mathsf{S}^i\mathsf{IPTT}^2_k$ and hence a homogeneous set by (the formalization of the proof of) Theorem 3.12. The second statement follows immediately. □

We have not been successful in proving a version of Theorem 3.12 for other versions of stability. Consequently we are interested in properties that characterize 1-stable and 2-stable colorings. As already noted, Propositions 3.6 and 3.7 describe a property that is common to 1-stable and 2-stable colorings, but not 3-stable colorings.

Next we consider a potential idea for proving the equivalence of TT^2_2 and PTT^2_2. We require the following statements, which can be formalized in RCA_0.

Definition 3.15 (RCA_0)

COH: For every sequence $\langle X_i \mid i \in \mathbb{N}\rangle$, there exists an infinite set X such that for every $i \in \mathbb{N}$, either $X \subseteq^* X_i$ or $X \subseteq^* \overline{X_i}$.

ADS: For every linear order $\preceq$ on $\mathbb{N}$ there exists an infinite set $X \subseteq \mathbb{N}$ which, under $\preceq$, is either an ascending sequence or else a descending sequence.

Cholak, Jockusch, and Slaman ([1], corrigendum in [2]) showed that, in the linear case, Ramsey's theorem for pairs can be broken up into the stable version and a statement about cohesiveness.

Proposition 3.16 ([1], Lemma 7.11; see also [2]) $\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \leftrightarrow \mathsf{SRT}^2_2 + \mathsf{COH}$.

The proof of the preceding result relies on the idea that, when X is a cohesive set (i.e., a set X that satisfies COH for an appropriate sequence $\langle X_i \mid i \in \mathbb{N}\rangle$) and $f : [\mathbb{N}]^2 \to 2$, $f \restriction [X]^2$ is stable. This fact motivates the next definition.

Definition 3.17 (RCA_0) Let $k \geq 1$ and $i \in \{1,2,3,4,5\}$.

$\mathsf{C}^i\mathsf{TT}^2_k$: For every $f : [2^{<\mathbb{N}}]^2 \to k$ there exists $T \cong 2^{<\mathbb{N}}$ such that $f \restriction [T]^2$ is i-stable.

The following result is easy to prove.

Proposition 3.18 *Let $k \geq 1$. For each $i \in \{1,2,3,4,5\}$,*

$$\mathsf{RCA}_0 \vdash \mathsf{TT}^2_k \leftrightarrow \mathsf{S}^i\mathsf{TT}^2_k + \mathsf{C}^i\mathsf{TT}^2_k.$$

Dzhafarov and Hirst ([6], Theorem 3.8) used Proposition 3.16 to show that, over RCA_0, PT^2_2 implies RT^2_2. Their proof showed that, over RCA_0, PT^2_2 implies ADS and relied on Hirschfeldt and Shore's result ([7], Proposition 2.10) that, over RCA_0, ADS implies COH. While emulating this idea for trees does not apparently establish the desired result, that over RCA_0, PTT^2_2 implies TT^2_2, it produces a partial step toward this result. However, it also again raises the possibility that there exist different forms of stability for trees with associated Ramsey theorems of different proof theoretic strengths. We begin with some definitions.

Definition 3.19

1. A *tree-linear* ordering on $T \subseteq 2^{<\mathbb{N}}$ is a reflexive, transitive, and antisymmetric relation $\preceq$ such that for all comparable $\sigma, \tau \in T$, either $\sigma \preceq \tau$ or $\tau \preceq \sigma$.
2. Given a tree-linear ordering $\preceq$ on $2^{<\mathbb{N}}$, we call $T \subseteq 2^{<\mathbb{N}}$ *ascending* for this ordering if for all $\sigma, \tau \in T$, $\sigma \subseteq \tau$ if and only if $\sigma \preceq \tau$, and we call S *descending* if instead $\sigma \subseteq \tau$ if and only if $\tau \preceq \sigma$.

The following statement of the tree version of ADS can be made in RCA_0.

Definition 3.20 (RCA_0)

TADS: For every tree-linear ordering $\preceq$ on $2^{<\mathbb{N}}$ there exists $T \cong 2^{<\mathbb{N}}$ which is either ascending or descending for this ordering.

The proof of the next proposition is motivated by Hirschfeldt and Shore's proof of Proposition 2.10 in [7].

Proposition 3.21 $\mathsf{RCA}_0 \vdash \mathsf{TADS} \rightarrow \mathsf{C}^4\mathsf{TT}^2_2$.

Proof Let $f : [2^{<\mathbb{N}}]^2 \to 2$ be given. Define a tree-linear ordering $\preceq$ on $2^{<\mathbb{N}}$ as follows: for $\sigma \subseteq \tau$, let $\sigma \preceq \tau$ if $\langle f(\rho,\sigma) \mid \rho \subseteq \sigma \rangle \leq_{\text{lex}} \langle f(\rho,\tau) \mid \rho \subseteq \tau \rangle$, and otherwise let $\tau \preceq \sigma$. Apply TADS to obtain $T \cong 2^{<\mathbb{N}}$ which is, say, ascending for $\preceq$ (the descending case being analogous). We claim that $f \upharpoonright [T]^2$ is 4-stable. Let $\sigma, \sigma' \in T$ with $\sigma' \supset \sigma$, and let $r_\sigma \in 2^{<\mathbb{N}}$ be the lexicographically greatest string of length $|\sigma|+1$ such that

$$(\exists \tau \supset \sigma')[\tau \in T \wedge r_\sigma \leq_{\text{lex}} \langle f(\rho,\tau) \mid \rho \subseteq \tau \rangle],$$

which exists because there are only finitely many strings of length $|\sigma|+1$ and because $0^{|\sigma|+1} \leq_{\text{lex}} \langle f(\rho,\tau') \mid \rho \subseteq \tau' \rangle$ for all $\tau' \supset \sigma'$. Fix the least corresponding τ. Since T is ascending, we must have $r_\sigma \leq_{\text{lex}} \langle f(\rho,\tau') \mid \rho \subseteq \tau' \rangle$, and hence also $r_\sigma \leq_{\text{lex}} \langle f(\rho,\tau') \mid \rho \subseteq \tau' \upharpoonright |\sigma|+1 \rangle$, for all $\tau' \supseteq \tau$ with $\tau' \in T$. But by our choice of r_σ, this means that $r_\sigma = \langle f(\rho,\tau') \mid \rho \subseteq \tau' \upharpoonright |\sigma|+1 \rangle$ for all $\tau' \supseteq \tau$ with $\tau' \in T$, because $\langle f(\rho,\tau') \mid \rho \subseteq \tau' \upharpoonright |\sigma|+1 \rangle \leq_{\text{lex}} \langle f(\rho,\tau') \mid \rho \subseteq \tau' \rangle$ for all τ'. Hence, for all $\tau' \supseteq \tau$ with $\tau' \in T$, we have $f(\sigma,\tau') = r_\sigma(|\sigma|)$. Since σ and σ' were chosen arbitrarily, this proves the claim. □

Proposition 3.22 $\mathsf{RCA}_0 \vdash \mathsf{PTT}^2_2 \to \mathsf{TADS}$.

Proof Fix a tree-linear ordering $\preceq$ on $2^{<\mathbb{N}}$. Define $f : [2^{<\mathbb{N}}]^2 \to 2$ by

$$f(\sigma,\tau) = \begin{cases} 0 \text{ if } \sigma \preceq \tau \\ 1 \text{ if } \tau \preceq \sigma \end{cases}$$

for all $\sigma \subseteq \tau$. Let $\langle S_0, S_1\rangle$ be a p-homogeneous set for f, as given by PTT^2_2. We define $T = \{t_\sigma \mid \sigma \in 2^{<\mathbb{N}}\}$ isomorphic to $2^{<\mathbb{N}}$ which is either ascending or descending for $\preceq$. Let $t_{\langle\,\rangle}$ be the bottom node of S_0, and suppose that for some $\sigma \supseteq \langle\,\rangle$ we have defined t_σ. Let $t_{\sigma^\frown 0}$ and $t_{\sigma^\frown 1}$ be the least incompatible extensions of σ in S_1 if $|\sigma|$ is even, in S_0 if $|\sigma|$ is odd. Then T exists by Δ^0_1-comprehension and clearly $T \cong 2^{<\mathbb{N}}$. Furthermore, by p-homogeneity there exists $c < 2$ such that for every $\sigma \in 2^{<\mathbb{N}}$ and $i < 2$, we have $f(t_\sigma, t_{\sigma^\frown i}) = c$, so by definition of f, either $t_\sigma \preceq t_{\sigma^\frown i}$ for all σ and i, or $t_{\sigma^\frown i} \preceq t_\sigma$ for all σ and i. Thus, T is either ascending or descending for $\preceq$, as desired. □

Corollary 3.23 $\mathsf{RCA}_0 \vdash \mathsf{PTT}^2_2 \leftrightarrow \mathsf{S}^4\mathsf{PTT}^2_2 + \mathsf{C}^4\mathsf{TT}^2_2$.

One way, then, to prove the equivalence of PTT^2_2 with TT^2_2, would be to get Theorem 3.14 to work for 4-stability, i.e, to show that $\mathsf{S}^4\mathsf{PTT}^2_2$ is equivalent to $\mathsf{S}^4\mathsf{TT}^2_2$ over RCA_0. Another way would be to strengthen Proposition 3.21 by replacing $\mathsf{C}^4\mathsf{TT}^2_2$ with $\mathsf{C}^1\mathsf{TT}^2_2$ or $\mathsf{C}^2\mathsf{TT}^2_2$. We do not know if either of these approaches is viable.

4 Δ^0_2 upper bounds and the Stable Tree Theorem

The following result on Δ^0_2 upper bounds in the linear setting appears in [1] and is well known.

Proposition 4.1 ([1], Lemma 3.5) *Let $k \geq 1$. For any computable stable k-coloring f of $[\mathbb{N}]^2$, there are k disjoint Δ^0_2 sets A_i such that $\bigsqcup_{i<k} A_i = \mathbb{N}$ and any infinite subset of any A_i computes a homogeneous set for f.*

Since (as noted earlier) every homogeneous set computes a p-homogeneous one, this result also holds in the polarized linear setting (see [6], Theorem 2.1 (3)).

In the tree setting, we can investigate this result from the different points of view afforded by our various notions of stability. We first consider 1-stable colorings.

Definition 4.2 Suppose $k \geq 1$, $f : [2^{<\mathbb{N}}]^2 \to k$ is 1-stable and $c < k$.

1. We write $\lim_{1,\tau\uparrow} f(\sigma,\tau) = c$ if there is an $n \geq |\sigma|$ such that $\forall \tau \supset \sigma(|\tau| \geq n \to f(\sigma,\tau) = c)$.
2. We let $A^f_c = \{\sigma \in 2^{<\mathbb{N}} \mid \lim_{1,\tau\uparrow} f(\sigma,\tau) = c\}$.

Note that when $f : [2^{<\mathbb{N}}]^2 \to k$ is 1-stable, then each set A^f_c, $c < k$, is Δ^0_2, relative to f, and that $2^{<\mathbb{N}} = \bigsqcup_{c<k} A^f_c$.

Lemma 4.3 *Suppose $k \geq 1$ and $f : [2^{<\mathbb{N}}]^2 \to k$ is 1-stable. There is a coloring $f^* : 2^{<\mathbb{N}} \to k$ such that $f^* \leq_T f'$ (i.e. f^* is computable from the jump of f) and for all σ, $f^*(\sigma) = c$ if and only if $\lim_{1,\tau\uparrow} f(\sigma,\tau) = c$.*

Proof Suppose $f : [2^{<\mathbb{N}}]^2 \to k$ is 1-stable. Fix $\sigma \in 2^{<\mathbb{N}}$ and let $\tau_0, \tau_1, \ldots$ be some standard computable enumeration of the extensions of σ. Starting with $i = 0$, use the jump of f to determine if $(\forall \tau \supset \tau_i)[f(\sigma,\tau) = f(\sigma,\tau_i)]$. If the answer is yes, set $f^*(\sigma) = f(\sigma,\tau_i)$. Otherwise, increment i. Since f is 1-stable, this process always halts, and sets $f^*(\sigma)$ equal to $\lim_{1,\tau\uparrow} f(\sigma,\tau)$. □

Lemma 4.4 *Let $k \geq 1$. Suppose $f : [2^{<\mathbb{N}}]^2 \to k$ is 1-stable, $c < k$, and S is a subtree isomorphic to $2^{<\mathbb{N}}$ such that for all $\sigma \in S$, $\lim_{1,\tau\uparrow} f(\sigma,\tau) = c$. Then there is a subtree T of S which is computable from S, isomorphic to $2^{<\mathbb{N}}$, and homogeneous for f.*

Proof Suppose f, S, and c are as in the hypothesis of the lemma. Label the nodes of S as $\{s_\sigma \mid \sigma \in 2^{<\mathbb{N}}\}$ so that the function $h : 2^{<\mathbb{N}} \to S$ defined by $h(\sigma) = s_\sigma$ is an order isomorphism. Since S is order isomorphic to $2^{<\mathbb{N}}$, such a labeling is computable from S. Fix an enumeration (computable in S) of the nodes extending each node of S. Define $T = \{t_\sigma \mid \sigma \in 2^{<\mathbb{N}}\}$ as follows. Set $t_{\langle\,\rangle} = s_{\langle\,\rangle}$. If t_σ has been calculated, let $t_{\sigma^\frown 0}$ and $t_{\sigma^\frown 1}$ be the first pair of incomparable proper extensions of t_σ in S such that $(\forall \rho \subseteq \sigma)[f(t_\rho, t_{\sigma^\frown 0}) = f(t_\rho, t_{\sigma^\frown 1}) = c]$. Since f is 1-stable, t_σ exists for each $\sigma \in 2^{<\mathbb{N}}$, and $T = \{t_\sigma \mid \sigma \in 2^{<\mathbb{N}}\}$ is computable from S. By our construction, T is order isomorphic to $2^{<\mathbb{N}}$ and homogeneous for f. □

Corollary 4.5 *Let $k \geq 1$. For any computable 1-stable k-coloring of $[2^{<\mathbb{N}}]^2$, there are k disjoint Δ^0_2 subsets A^f_c, $c < k$, of $2^{<\mathbb{N}}$ with $\bigsqcup_{c<k} A^f_c = 2^{<\mathbb{N}}$ such that any subset $S \cong 2^{<\mathbb{N}}$ of any A^f_c computes a homogeneous set for f.*

Theorem 4.6 *Every computable 1-stable finite coloring of $[2^{<\mathbb{N}}]^2$ has a Δ^0_2 homogeneous set.*

Proof Suppose $f : [2^{<\mathbb{N}}]^2 \to k$ is a computable 1-stable coloring. Apply Corollary 4.5 to obtain a Δ^0_2 k-coloring $g : [2^{<\mathbb{N}}]^1 \to k$ such that any homogeneous set for g computes a homogenous set for f. By the proof of Theorem 1.2 of [4], relativized to g, there is a homogeneous set S for g with $S \leq_T g$. Let T be a homogeneous set for f with $T \leq_T S$. Clearly $T \leq_T S \leq_T g \leq_T 0'$, so T is Δ^0_2. □

Corollary 4.7 *Every computable 1-stable finite coloring of $[2^{<\mathbb{N}}]^2$ has a Δ^0_2 p-homogeneous set.*

Proof By Theorem 4.6 we can find a homogeneous set computable from $0'$. Taking alternating levels provides a p-homogeneous set computable from $0'$ and therefore Δ^0_2. □

Note that if f is a 2-stable k-coloring of $[2^{<\mathbb{N}}]^2$, then the sets A^f_c, $c < k$, no longer partition $2^{<\mathbb{N}}$. For example, if $f : [2^{<\mathbb{N}}]^2 \to 2$ is 2-stable, then $2^{<\mathbb{N}}$ is the disjoint union of three sets: A^f_0, A^f_1, and a "mixed" set

$$A^f_{0,1} = \{\sigma \in 2^{<\mathbb{N}} \mid (\forall n)(\forall i < 2)(\exists \tau \supset \sigma)[|\tau| > n \wedge f(\sigma,\tau) = i]\}.$$

It turns out that this mixed set does not necessarily have the property described in Corollary 4.5, that any subset $S \cong 2^{<\mathbb{N}}$ of it computes a homogeneous set for f, as the following theorem shows.

Theorem 4.8 *There exists a 2-stable computable $f : [2^{<\mathbb{N}}]^2 \to 2$ and a subtree $T \cong 2^{<\mathbb{N}}$ of $A^f_{0,1}$ which computes no homogeneous set for f.*

Proof We build a 2-stable $f : [2^{<\mathbb{N}}]^2 \to 2$ with $A^f_{0,1} = 2^{<\mathbb{N}}$ such that f has no computable homogeneous set. Since $A^f_{0,1} = 2^{<\mathbb{N}}$ is computable, the result follows immediately.

Suppose $g : [\mathbb{N}]^2 \to 2$ is a computable stable coloring of pairs of natural numbers which has no computable homogeneous set (such a coloring exists by Proposition 2.14 of [8]). Define $f : [2^{<\mathbb{N}}]^2 \to 2$ by

$$f(\sigma,\tau) = \begin{cases} g(|\sigma|,|\tau|) & \text{if } \tau \supseteq \sigma^\frown 1, \\ 1 - g(|\sigma|,|\tau|) & \text{if } \tau \supseteq \sigma^\frown 0. \end{cases}$$

To see that f is 2-stable, fix σ and, by stability of g, choose n_0 so large that for all $m \geq n_0$, $g(|\sigma|,m) = g(|\sigma|,n_0)$. Thus, for all $\tau \supset \sigma$ with $|\tau| \geq n_0$, and for all $\rho \supseteq \tau$, $f(\sigma,\rho) = f(\sigma,\tau)$. Thus, f is 2-stable. Furthermore, when n_0, ρ, and τ are as above, either $\tau \supseteq \sigma^\frown 1$, in which case $f(\sigma,\rho) = f(\sigma,\tau) = g(|\sigma|,n_0)$, or $\tau \supseteq \sigma^\frown 0$, in which case $f(\sigma,\rho) = f(\sigma,\tau) = 1 - g(|\sigma|,n_0)$. Since both options must occur, $\sigma \in A^f_{0,1}$, and since σ was arbitrary, $A^f_{0,1} = 2^{<\mathbb{N}}$.

Now we will show that every homogeneous set for f computes a homogeneous set for g. Let $T \cong 2^{<\mathbb{N}}$ be a homogeneous set for f, and let $\sigma_1, \sigma_2, \ldots$ enumerate the leftmost path in T. Consider the sets

$$H_0 = \{|\sigma_i| \mid i \in \mathbb{N} \wedge \sigma_{i+1} \supseteq \sigma_i^\frown 0\} \text{ and } H_1 = \{|\sigma_i| \mid i \in \mathbb{N} \wedge \sigma_{i+1} \supseteq \sigma_i^\frown 1\}.$$

Note that both of these sets are computable from T, and at least one of them is infinite and thereby homogeneous for g, so our claim follows. Finally, since g has no computable homogeneous set, neither does f. □

Even so, we can modify the argument of Theorem 4.6 to obtain the result for 5-stable colorings.

Definition 4.9 Let $k \geq 1$ and suppose $f : [2^{<\mathbb{N}}]^2 \to k$ is 5-stable. Fix an enumeration of the proper extensions of each node of $2^{<\mathbb{N}}$. Let τ_0 be the least node extending σ such that $f(\sigma,\tau) = f(\sigma,\tau_0)$ for all $\tau \supseteq \tau_0$. Then we write $\lim_{5,\tau\uparrow} f(\sigma,\tau) = f(\sigma,\tau_0)$, and $\rho_{\lim}(f,\sigma) = \tau_0$.

Recall that we can think of a 5-stable coloring as one in which the induced maps are eventually constant on the subtree above some node; the limiting value may depend on the choice of the subtree. In the preceding definition, the use of the enumeration makes the limiting value uniquely determined, and the $\rho_{\lim}$ function points to the defining root. Neither the limit nor the root function need be computable from f, but both are computable from the jump of f.

Lemma 4.10 *Suppose $k \geq 1$ and $f : [2^{<\mathbb{N}}]^2 \to k$ is 5-stable. Then we can find a subtree H isomorphic to $2^{<\mathbb{N}}$ and a function $f^* : H \to k$ such that*

(1) both H and f^* are computable from the jump of f,
(2) f is 1-stable on H, and
(3) for all $\sigma \in H$, $f^*(\sigma) = c$ if and only if $\lim_{1,\tau\uparrow} f(\sigma,\tau) = c$, relative to H.

Proof Suppose $f : [2^{<\mathbb{N}}]^2 \to k$ is 5-stable. Modifying the proof of Lemma 4.3, we construct f^* and $h = \{h_\sigma \mid \sigma \in 2^{<\mathbb{N}}\}$ simultaneously. Let $h_{\langle\,\rangle} = \langle\,\rangle$. Use the jump of f to evaluate $\lim_{5,\tau\uparrow} f(\langle\,\rangle,\tau)$ and set $f^*(h_{\langle\,\rangle}) = \lim_{5,\tau\uparrow} f(\langle\,\rangle,\tau)$. Suppose h_σ is defined and f^* is defined for all h_τ with $\tau \subset \sigma$. Use the jump of f to evaluate $\rho_{\lim}(f,h_\sigma)$ and set $h_{\sigma^\frown i} = \rho_{\lim}(f,h_\sigma)^\frown i$ for $i \in \{0,1\}$. For each $i \in \{0,1\}$, use the jump of f to evaluate $\lim_{5,\tau\uparrow} f(h_{\sigma^\frown i},\tau)$ and set $f^*(h_{\sigma^\frown i}) = \lim_{5,\tau\uparrow} f(h_{\sigma^\frown i},\tau)$. It is straightforward to verify that the construction yields f^* and H satisfying the statement of the lemma. □

Theorem 4.11 *Every computable 5-stable finite coloring of $[2^{<\mathbb{N}}]^2$ has a Δ^0_2 homogeneous set and a Δ^0_2 p-homogeneous set. Furthermore, this results holds for i-stable colorings for all $i \leq 5$.*

Proof Suppose f is 5-stable. Apply Lemma 4.10 to find f^* and H, then rerun the proofs of Lemma 4.4, Theorem 4.6, and Corollary 4.7, all relativized to H. The final sentence follows immediately from the fact that every i-stability previously defined implies 5-stability. □

5 Questions

Although the computability theory and reverse mathematics of the polarized tree theorem for triples and above exactly parallels the linear case, we have many questions concerning the results for pairs. We hope that resolving these questions may lead to a deeper understanding of Ramsey's theorem for pairs. Of particular note is the profusion of versions of stability in the tree setting. Our versions are somewhat ad hoc; certainly more concepts of stability could be formulated and explored. This leads us to ask:

Q1: What other forms of stability may be of interest? Is it possible to characterize all reasonable notions of stability in a systematic fashion?

Q2: Are there forms of stability that yield provably distinct results in reverse mathematics or computability theory? Specifically, does Theorem 3.12 fail for 3-stable colorings? Similarly, does Proposition 3.21 fail for 2-stable colorings?

It may be possible to prove results for trees that are open in the linear setting.

Q3: Can any of the one-way arrows in the diagram from Section 3 of [6] be reversed for trees?

Perhaps a statement about trees can be applied to deduce an apparently stronger statement in the linear setting. Examples of questions of this sort include:

Q4: Does $\mathsf{S}^i\mathsf{TT}^2$ imply IPT^2 for any $i \in \{1,2,3,4,5\}$? Does IPTT^2 imply RT^2?

As mentioned at the conclusion of Section 3, we do not know the answer to the following question.

Q5: Does PTT^2 imply TT^2? Does PTT^2_k imply TT^2_k?

This list of questions is certainly incomplete. Our work was motivated in part by questions posed at the workshop on Computability, Reverse Mathematics, and Combinatorics held at the Banff International Research Station in December of 2008 (see [5]). The list of open problems from that meeting could be used to generate many additional questions pertaining to polarized and stable tree theorems.

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