COMPARING THE STRENGTH OF DIAGONALLY NON-RECURSIVE FUNCTIONS IN THE ABSENCE OF Σ_2^0 INDUCTION

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ABSTRACT. We prove that the statement "there is a k such that for every f there is a k-bounded diagonally non-recursive function relative to f" does not imply weak König's lemma over $\mathrm{RCA}_0+\mathrm{B}\Sigma_2^0$. This answers a question posed by Simpson. A recursion-theoretic consequence is that the classic fact that every k-bounded diagonally non-recursive function computes a 2-bounded diagonally non-recursive function may fail in the absence of $\mathrm{I}\Sigma_2^0$.

1. INTRODUCTION

It is a truth universally acknowledged, that a single man in possession of a good k-bounded diagonally non-recursive function, must be in want of a 2-bounded diagonally non-recursive function [2].

An enduring project in recursion theory is to determine the amount of induction necessary to prove its classic theorems, particularly those concerning the recursively enumerable sets. Post's problem and the Friedberg-Muchnik theorem [5, 23, 29], the Sacks splitting theorem [23, 29], the Sacks density theorem [16], the infinite injury method [6, 10, 11], and even the transitivity of Turing reducibility [17] have all been investigated. The non-standard methods developed in the course of these studies have been recently applied in *reverse mathematics*, an analysis of the logical strengths of ordinary mathematical statements in the context of second-order arithmetic, and led to solutions of several important open problems in the field. Remarkably, Chong, Slaman, and Yang proved that stable Ramsey's theorem for pairs is strictly weaker than Ramsey's theorem for pairs [9] and that Ramsey's theorem for pairs does not imply induction for Σ_2^0 predicates [8]. Furthermore, non-standard techniques are necessarily employed in proofs of conservativity results over systems with limited induction, such as the Π_1^1-conservativities of the cohesive principle and the chain-antichain principle over RCA_0 plus bounding for Σ_2^0 predicates [7]. Similarly, Corduan, Groszek, and Mileti proved what may be described as a dual conservativity result: an extension of RCA_0 by Π_1^1 axioms proves Ramsey's theorem for singletons on the complete binary tree if and only if the extension proves induction for Σ_2^0 predicates [13]. It follows that RCA_0 plus bounding for Σ_2^0 predicates does not prove Ramsey's theorem for singletons on the complete binary tree, which answers a question from [12]. For a comprehensive introduction to non-standard methods in recursion theory and reverse mathematics, we refer the reader to the recent survey by Chong, Li, and Yang [4].

Within this framework of reverse mathematics, we study the logical strengths of several statements asserting the existence of k-bounded diagonally non-recursive functions. Theorem 5 of Jockusch's classic analysis of diagonally non-recursive functions [21] states that every k-bounded diagonally non-recursive function computes a 2-bounded diagonally non-recursive function. The proof, which Jockusch attributes to Friedberg, is not uniform, and Jockusch proves that this is necessarily the case: Theorem 6 of [21] implies that if $k > 2$ then there is no uniform (i.e., Medvedev)

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reduction from the class of k-bounded diagonally non-recursive functions to the class of 2-bounded diagonally non-recursive functions. In a talk given at the 2001 Annual Meeting of the American Philosophical Association [26], Simpson asked if the reduction from k-bounded diagonally non-recursive functions to 2-bounded diagonally non-recursive functions can be implemented RCA_0. Specifically, he asked if the statement "there is a k such that for every X there a k-bounded diagonally non-recursive function relative to X" implies weak König's lemma over RCA_0. Our main result is that although the statement in question indeed implies weak König's lemma over RCA_0 plus induction for Σ_2^0 predicates, it does not imply weak König's lemma over RCA_0 plus bounding for Σ_2^0 predicates. Consequently, if induction for Σ_2^0 predicates fails, there may be k-bounded diagonally non-recursive functions (for some necessarily non-standard k) that do not compute 2-bounded diagonally non-recursive functions. This result expresses a sense in which induction for Σ_2^0 predicates is necessary to prove that every k-bounded diagonally non-recursive function computes a 2-bounded diagonally non-recursive function.

2. Background

We define the fragments of first-order and second-order arithmetic that we consider in this work. The standard references are Hájek and Pudlák's *Metamathematics of First-Order Arithmetic* [19] for fragments of first-order arithmetic and Simpson's *Subsystems of Second Order Arithmetic* [27] for fragments of second-order arithmetic in the context of reverse mathematics. Reverse mathematics is a foundational program, introduced by Friedman in [14], dedicated to characterizing the logical strengths of the classic theorems of mathematics when interpreted in second-order arithmetic. It is thus a fundamentally proof-theoretic endeavor, although its techniques are primarily recursion-theoretic. We encouragingly refer the interested reader to the introduction of Simpson's book for a hearty introduction to reverse mathematics and its metamathematical motivations.

We pause here to highlight one important notational convention. As is common when writing about reverse mathematics, throughout this work we use the symbol 'ω' to refer to the standard natural numbers and the symbol '$\mathbb{N}$' to refer to the first-order part of whatever structure is (often implicitly) under consideration.

2.1. **Fragments of first-order arithmetic.** The basic axioms of Peano arithmetic, here denoted PA^-, express that $\mathbb{N}$ is a discretely ordered commutative semi-ring with 1. Peano arithmetic, denoted PA, consists of PA^- plus the *induction scheme*, which consists of the universal closures of all formulas of the form

$$[\varphi(0) \wedge \forall n(\varphi(n) \to \varphi(n+1))] \to \forall n \varphi(n).$$

Fragments of PA are obtained by limiting the quantifier complexity of the formulas φ allowed in the induction scheme. For each $n \in \omega$, the Σ_n^0 (Π_n^0) induction scheme is the restriction of the induction scheme to Σ_n^0 (Π_n^0) formulas φ, and $\mathrm{I}\Sigma_n^0$ ($\mathrm{I}\Pi_n^0$) denotes the fragment of PA consisting of PA^- plus the Σ_n^0 (Π_n^0) induction scheme. We express induction for Δ_n^0 predicates via the Δ_n^0 induction scheme, which consists of the universal closures of all formulas of the form

$$\forall n(\varphi(n) \leftrightarrow \psi(n)) \to ([\varphi(0) \wedge \forall n(\varphi(n) \to \varphi(n+1))] \to \forall n \varphi(n)),$$

where φ is Σ_n^0 and ψ is Π_n^0. The fragment $\mathrm{I}\Delta_n^0$ is then PA^- plus the the Δ_n^0 induction scheme.

We also consider fragments of PA obtained by adding so-called *bounding schemes* (also called *collection schemes*). The Σ_n^0 (Π_n^0) *bounding scheme* consists of the universal closures of all formulas of the form

$$\forall a[(\forall n < a)(\exists m)\varphi(n, m) \to \exists b(\forall n < a)(\exists m < b)\varphi(n, m)]$$

where φ is Σ_n^0 (Π_n^0). The fragment $\mathrm{B}\Sigma_n^0$ ($\mathrm{B}\Pi_n^0$) is then $\mathrm{I}\Sigma_0^0$ plus the Σ_n^0 (Π_n^0) bounding scheme.

The following theorem summarizes the relationships among these fragments.

Theorem 2.1 (see [19] Theorem 2.4, [19] Theorem 2.5, and [28]). *Let $n \in \omega$. Over* PA^-:

- $\mathrm{I}\Sigma_n^0$ *and* $\mathrm{I}\Pi_n^0$ *are equivalent.*
- $\mathrm{B}\Sigma_{n+1}^0$ *and* $\mathrm{B}\Pi_n^0$ *are equivalent.*
- $\mathrm{I}\Sigma_{n+1}^0$ *is strictly stronger than* $\mathrm{B}\Sigma_{n+1}^0$, *which is strictly stronger than* $\mathrm{I}\Sigma_n^0$.
- *If $n \geq 2$, then $\mathrm{I}\Delta_n^0$ and $\mathrm{B}\Sigma_n^0$ are equivalent (the proof uses the totality of the exponential function, which is provable in $\mathrm{I}\Sigma_1^0$).*

A *cut* in a model $\mathbb{N}$ of PA^- is a set $I \subseteq \mathbb{N}$ such that $\forall n \forall m[(n \in I \wedge m < n) \to m \in I]$ and $\forall n(n \in I \to n + 1 \in I)$. A cut $I \subseteq \mathbb{N}$ is called *proper* if $I \neq \emptyset$ and $I \neq \mathbb{N}$. Definable proper cuts witness failures of induction. Suppose that $\mathbb{N} \models \mathrm{PA}^-$. If the induction axiom for φ fails in $\mathbb{N}$, then $\psi(n) = (\forall m < n)\varphi(m)$ defines a proper cut in $\mathbb{N}$, and if φ defines a proper cut in $\mathbb{N}$, then the induction axiom for φ fails in $\mathbb{N}$.

The following lemma, originally noticed by Friedman but by now part of the folklore, is key to many recursion-theoretic constructions in models with limited induction, including the main construction in this work.

Lemma 2.2. *If $\mathbb{N} \models \mathrm{B}\Sigma_2^0 + \neg\mathrm{I}\Sigma_2^0$, then there are a proper Σ_2^0 cut $I \subseteq \mathbb{N}$ and an increasing, cofinal function $c \colon I \to \mathbb{N}$ whose graph is Δ_2^0.*

Proof. Let $\varphi(n)$ be a Σ_2^0 formula witnessing the failure of $\mathrm{I}\Sigma_2^0$. That is, $\varphi(0) \wedge \forall n(\varphi(n) \to \varphi(n + 1)) \wedge \exists n \neg \varphi(n)$. Let $I = \{n : (\forall m < n)\varphi(m)\}$. I is a proper cut, and using $\mathrm{B}\Sigma_2^0$ one proves that I is Σ_2^0. Let θ be Π_1^0 such that $I = \{n : \exists m \theta(n, m)\}$. Define the function c by $c(n) = \mu m \theta(n, m)$ and observe that the graph of c is Δ_2^0. By $\mathrm{I}\Sigma_2^0$, if there is an m such that $\theta(n, m)$, then there is a least such m. Therefore $\mathrm{dom}(c) = I$. Furthermore, $\mathrm{ran}(c)$ is unbounded, for if $\exists b(\forall n \in I)(c(n) < b)$, then $\forall n(n \in I \leftrightarrow (\exists m < b)\theta(n, m))$, which constitutes a violation of $\mathrm{I}\Sigma_1^0$. If necessary, using $\mathrm{B}\Sigma_2^0$ we can dominate c by an increasing function with the same domain whose graph is still Δ_2^0. $\qquad \square$

2.2. Fragments of second-order arithmetic.

Full second-order arithmetic consists of PA^- plus the universal closures of the induction axiom

$$[0 \in X \wedge \forall n(n \in X \to n + 1 \in X)] \to \forall n(n \in X)$$

and the comprehension scheme

$$\exists X \forall n(n \in X \leftrightarrow \varphi(n)),$$

where φ is any formula in the language of second-order arithmetic in which X is not free. In the setting of second-order arithmetic, formulas may have free second-order parameters, and 'universal closure' means closure under both first-order and second-order universal quantifiers.

Fragments of second-order arithmetic are obtained by replacing the induction axiom by an induction scheme as in the first-order case and by limiting the comprehension scheme to formulas of a certain complexity. We emphasize again that in the second-order setting a formula may have free second-order parameters that are universally quantified in the corresponding induction axiom, hence an induction axiom holding in some second-order structure means that it holds relative to every second-order object in that structure. When studying reverse mathematics, we also produce fragments of second-order arithmetic by adding the statement of a well-known theorem to another fragment, as is the case in the system weak König's lemma described below. This work is concerned with the first two of the Big Five fragments of second-order arithmetic, *recursive comprehension axiom* (RCA_0) and *weak König's lemma* (WKL_0), as well as various fragments defined by statements asserting the existence of diagonally non-recursive functions.

RCA_0 is the fragment consisting of PA^-, the second-order Σ_1^0 induction scheme (which we still refer to as $\mathrm{I}\Sigma_1^0$ in this setting), and the Δ_1^0 comprehension scheme, which consists of the universal closures of all formulas of the form

$$\forall n(\varphi(n) \leftrightarrow \psi(n)) \to \exists X \forall n(n \in X \leftrightarrow \varphi(n)),$$

where φ is Σ_1^0, ψ is Π_1^0, and X is not free in φ.

The equivalences and implications of Theorem 2.1 hold over RCA_0 in the second-order setting. Most relevant to our purposes are that

- for all $n \in \omega$, $\mathrm{RCA}_0 + \mathrm{I}\Sigma_n^0$ and $\mathrm{RCA}_0 + \mathrm{I}\Pi_n^0$ are equivalent (in particular, $\mathrm{RCA}_0 \vdash \mathrm{I}\Pi_1^0$);
- $\mathrm{RCA}_0 + \mathrm{I}\Sigma_2^0$ is strictly stronger than $\mathrm{RCA}_0 + \mathrm{B}\Sigma_2^0$, which is strictly stronger than RCA_0; and
- $\mathrm{RCA}_0 + \mathrm{B}\Sigma_2^0 \vdash \mathrm{I}\Delta_2^0$ (in particular, models of $\mathrm{RCA}_0 + \mathrm{B}\Sigma_2^0$ have no Δ_2^0-definable cuts).

An important aid to working in RCA_0 is the fact that RCA_0 proves the bounded Σ_1^0 comprehension scheme (see [27] Theorem II.3.9), which consists of the universal closures of all formulas of the form

$$\forall n \exists X \forall i [i \in X \leftrightarrow (i < n \land \varphi(i))],$$

where φ is a Σ_1^0 formula in which X is not free. Contrastingly, adding the full Σ_1^0 comprehension scheme to RCA_0 is equivalent to adding comprehension for all arithmetical formulas and results in a stronger system denoted ACA_0 (see [27] Theorem III.1.3).

RCA_0 proves sufficient number-theoretic facts to implement the codings of sequences of numbers as numbers that are typical in recursion theory. See [27] Section II.2 for a carefully formalized development of such a coding. Thus in RCA_0 we can interpret the existence of the set $\mathbb{N}^{<\mathbb{N}}$ of all finite sequences (also called strings) and, more generally, give the usual definition of a *tree* as subset of $\mathbb{N}^{<\mathbb{N}}$ that is closed under initial segments. We now fix our notation and terminology concerning strings and trees. Let $k, s \in \mathbb{N}$, $\sigma, \tau \in \mathbb{N}^{<\mathbb{N}}$, $f \colon \mathbb{N} \to \mathbb{N}$ be a function, and $T \subseteq \mathbb{N}^{<\mathbb{N}}$ be a tree. Then

- $k^{<\mathbb{N}}$ is the set of strings over $\{0, 1, \ldots, k-1\}$, k^s is the set of strings in $k^{<\mathbb{N}}$ of length exactly s, and $k^{<s}$ is the set of strings in $k^{<\mathbb{N}}$ of length less than s;
- $|\sigma|$ is the length of σ;
- $\sigma \subseteq \tau$ means that σ is a substring of τ;
- $f \restriction n$ is the string $\langle f(0), f(1), \ldots, f(n-1) \rangle$;
- $\sigma \subseteq f$ means that σ is an initial segment of f (i.e., $\sigma = f \restriction |\sigma|$);
- f is a *path through* T if $\forall n (f \restriction n \in T)$.

Weak König's lemma (WKL) is the statement "every infinite subtree of $2^{<\mathbb{N}}$ has an infinite path," and WKL_0 is the fragment $\mathrm{RCA}_0 + \mathrm{WKL}$. WKL_0 captures compactness arguments, and WKL is equivalent to many classical theorems over RCA_0. For example, the equivalence of WKL with the Heine-Borel compactness of $[0, 1]$, the extreme value theorem, Gödel's completeness theorem, and Brouwer's fixed point theorem can all be found in [27].

Suppressing the basic relations and functions, a structure in the language of second-order arithmetic is officially a pair $(\mathbb{N}, \mathcal{S})$, where the first-order part $\mathbb{N}$ is some set and the second-order part $\mathcal{S}$ is a collection of subsets of $\mathbb{N}$. However, via the simple coding of pairs possible in RCA_0 and the identification of a function $f \colon \mathbb{N} \to \mathbb{N}$ with its graph $\{\langle n, m \rangle : f(n) = m\}$, one immediately sees that it is equivalent to consider structures in which the second-order part is a collection of functions $f \colon \mathbb{N} \to \mathbb{N}$. Thus we use the functional variant of second-order structures because it is the more natural setting for our study.

2.3. **Turing reducibility and Turing functionals.** The standard definition of Turing reducibility in RCA_0 is [27] Definition VII.1.4, which essentially says that $Y \leq_{\mathrm{T}} X$ if Y is both r.e. and co-r.e. in X.

Definition 2.3 ([27] Definition VII.1.4). Fix a universal lightface Π_1^0 formula $\pi(e, m, X)$ with exactly the displayed variables free. For $X, Y \subseteq \mathbb{N}$, we say that Y *Turing reduces to* X ($Y \leq_{\mathrm{T}} X$) if there are $e_0, e_1 \in \mathbb{N}$ such that, for all m, $m \in Y \leftrightarrow \pi(e_0, m, X)$ and $m \notin Y \leftrightarrow \pi(e_1, m, X)$.

Note that in the preceding definition $m \in Y \leftrightarrow \neg\pi(e_1, m, X)$, so $\neg\pi(e_1, m, X)$ is a Σ_1^0 formula essentially witnessing that Y is r.e. in X. Extending this notion, we can formalize statements

involving recursive functionals as used in [30] Section III.1. For example, given $e \in \mathbb{N}$ we write $\Phi_e^f(n) = m$ to represent a formula asserting that there is a coded sequence of configurations of the eth Turing machine that starts with the machine's initial configuration for input n, ends with the machine's output configuration for output m, and is such that each configuration in the sequence follows from the previous one by the rules of the machine when equipped with oracle f. In this way we think of Φ_e^f as a partial f-recursive function as usual, and for two functions $f, g \colon \mathbb{N} \to \mathbb{N}$, $g \leq_{\mathrm{T}} f$ if and only if there is an e such that $g = \Phi_e^f$. We make the familiar definitions that $\Phi_e^f(n) \downarrow$ if there is an m such that $\Phi_e^f(n) = m$ and that $\Phi_e^f(n) \uparrow$ otherwise. Similarly, $\Phi_{e,s}^f(n) \downarrow$ if $\Phi_e^f(n) \downarrow$ within s computational steps and $\Phi_{e,s}^f(n) \uparrow$ otherwise. We follow the usual convention that the number of steps in a computation relative to a partial oracle is bounded by the first position where the oracle is undefined, such as with computations of the form $\Phi_e^\sigma(n)$ and $\Phi_e^{f \oplus \sigma}(n)$, where σ is some finite string.

The following notion will be useful to verify $B\Sigma_2^0$ when constructing models.

Definition 2.4. We say that Y *is low relative to* X if $\Phi_e^Y(e) \downarrow$ is equivalent to a $\Delta_2^0(X)$ statement.

Lemma 2.5 ([4] Proposition 4.14)**.** *If Y is low relative to X and $B\Sigma_2^0$ holds relative to X then $B\Sigma_2^0$ also holds relative to Y.*

2.4. **Diagonally non-recursive functions in the formal setting.** We now introduce the statements expressing the existence of diagonally non-recursive functions that are the main focus of this paper.

Definition 2.6. Let f and g be functions $\mathbb{N} \to \mathbb{N}$, and let $k \in \mathbb{N}$.
- The function g is *k-bounded* if $\mathrm{ran}(g) \subseteq \{0, 1, \ldots, k-1\}$.
- The function g is *diagonally non-recursive relative to* f (g is $\mathrm{DNR}(f)$ for short) if $\forall e(g(e) \neq \Phi_e^f(e))$.
- The function g is *k-bounded diagonally non-recursive relative to* f (g is $\mathrm{DNR}(k, f)$ for short) if g is k-bounded and $\mathrm{DNR}(f)$.

In a slight overloading of notation we also let $\mathrm{DNR}(f)$ denote the formal statement "there is a g that is $\mathrm{DNR}(f)$" and let $\mathrm{DNR}(k, f)$ denote the formal statement "there is a g that is $\mathrm{DNR}(k, f)$."

It is well-known that WKL and $\forall f \mathrm{DNR}(k, f)$ are equivalent over RCA_0 for every fixed $k \in \omega$ with $k \geq 2$. WKL and $\forall f \mathrm{DNR}(2, f)$ are equivalent by the classic work of Jockusch and Soare [22], and $\forall f \mathrm{DNR}(2, f)$ and $\forall f \mathrm{DNR}(k, f)$ are equivalent because if $k \in \omega$ and $k \geq 2$, then the proof of [21] Theorem 5 can be unwound in RCA_0. It is also well-known that $\forall f \mathrm{DNR}(f)$ is strictly weaker than WKL over RCA_0. In fact, $\forall f \mathrm{DNR}(f)$ is strictly weaker than WWKL [1], which is strictly weaker than WKL [31]. The purpose of this work is to analyze the strengths of the statements $\exists k \forall f \mathrm{DNR}(k, f)$ and $\forall f \exists k \mathrm{DNR}(k, f)$ over RCA_0. With a little care, it is possible to implement the proof of [21] Theorem 5 in $\mathrm{RCA}_0 + I\Sigma_2^0$. Hence the statements WKL, $\exists k \forall f \mathrm{DNR}(k, f)$, and $\forall f \exists k \mathrm{DNR}(k, f)$ are all equivalent over $\mathrm{RCA}_0 + I\Sigma_2^0$.

Theorem 2.7. $\mathrm{RCA}_0 + I\Sigma_2^0 + \forall f \exists k \mathrm{DNR}(k, f) \vdash \forall f \mathrm{DNR}(2, f)$.

Proof. Suppose g is $\mathrm{DNR}(2^k, f)$, and think of 2^k as the set of strings over $\{0, 1\}$ of length k. Define a partial f-computable function b by

$$b(n) = \begin{cases} 0 & \text{if } \Phi_n^f(n) = 0 \\ 1 & \text{if } \Phi_n^f(n) > 0 \\ \uparrow & \text{if } \Phi_n^f(n)\uparrow, \end{cases}$$

let $\ell \colon \mathbb{N}^k \to \mathbb{N}$ be a partial computable function such that

$$(\forall \vec{n} \in \mathbb{N}^k)(\forall x \in \mathbb{N})(\Phi_{\ell(\vec{n})}^f(x) = \langle b(n_0), b(n_1), \ldots, b(n_{k-1}) \rangle),$$

and let $h \colon k \times \mathbb{N}^k \to 2$ be the partial g-computable function defined by the equation

$$g(\ell(\vec{n})) = \langle h(0, \vec{n}), h(1, \vec{n}), \ldots, h(k-1, \vec{n}) \rangle.$$

By the Π^0_2 least element principle, a consequence of $\mathrm{RCA}_0 + \mathrm{I}\Sigma^0_2$, let i be least such that

$$(\forall \vec{n} \in \mathbb{N}^i)(\exists \vec{m} \in \mathbb{N}^{k-i})(\forall j < k - i)(h(i + j, \vec{n}^\frown\vec{m}) = \Phi^f_{m_j}(m_j)).$$

Notice that $i > 0$, for otherwise we would have an $\vec{m} \in \mathbb{N}^k$ such that $(\forall j < k)(h(j, \vec{m}) = \Phi^f_{m_j}(m_j) = b(m_j))$, in which case $g(\ell(\vec{m})) = \Phi^f_{\ell(\vec{m})}(\ell(\vec{m}))$, contradicting that g is $\mathrm{DNR}(2^k, f)$. Fix $\vec{n} \in \mathbb{N}^{i-1}$ such that $(\forall \vec{m} \in \mathbb{N}^{k-i+1})(\exists j < k - i + 1)(h(i + j, \vec{n}^\frown\vec{m}) \neq \Phi^f_{m_j}(m_j))$. We can now describe a $\mathrm{DNR}(2, f)$ function that is $\leq_\mathrm{T} f \oplus g$. Given $x \in \mathbb{N}$, search for an $\vec{m} \in \mathbb{N}^{k-i}$ such that $(\forall j < k - i)(h(i + j, \vec{n}^\frown x^\frown\vec{m}) = \Phi^f_{m_j}(m_j))$, then output $h(i - 1, \vec{n}^\frown x^\frown\vec{m})$. Such an $\vec{m}$ exists by choice of i, and $h(i - 1, \vec{n}^\frown x^\frown\vec{m}) \neq \Phi^f_x(x)$ by choice of $\vec{n}$. $\qquad\square$

In Section 4, we show that $\mathrm{RCA}_0 + \mathrm{B}\Sigma^0_2$ does not suffice to prove the equivalences of WKL, $\exists k \forall f \mathrm{DNR}(k, f)$, and $\forall f \exists k \mathrm{DNR}(k, f)$. Specifically, we prove

- Theorem 4.9: $\mathrm{RCA}_0 + \mathrm{B}\Sigma^0_2 + \forall f \exists k \mathrm{DNR}(k, f) \nvdash \exists k \forall f \mathrm{DNR}(k, f)$, and
- Theorem 4.10: $\mathrm{RCA}_0 + \mathrm{B}\Sigma^0_2 + \exists k \forall f \mathrm{DNR}(k, f) \nvdash \mathrm{WKL}$.

Hence, over $\mathrm{RCA}_0 + \mathrm{B}\Sigma^0_2$, $\forall f \exists k \mathrm{DNR}(k, f)$ is strictly weaker than $\exists k \forall f \mathrm{DNR}(k, f)$, which is strictly weaker than WKL. These results are, in a sense, as strong as possible. It is of course natural to ask if there is a reversal of Theorem 2.7. That is, it is natural to ask if $\mathrm{RCA}_0 \vdash (\exists k \forall f \mathrm{DNR}(k, f) \to \mathrm{WKL}) \to \mathrm{I}\Sigma^0_2$. However, this is readily seen not to be the case because $\mathrm{WKL}_0 \nvdash \mathrm{I}\Sigma^0_2$. In fact, $\mathrm{WKL}_0 \nvdash \mathrm{B}\Sigma^0_2$. No reversal over $\mathrm{RCA}_0 + \mathrm{B}\Sigma^0_2$ is possible either. That is, $\mathrm{RCA}_0 + \mathrm{B}\Sigma^0_2 \nvdash (\exists k \forall f \mathrm{DNR}(k, f) \to \mathrm{WKL}) \to \mathrm{I}\Sigma^0_2$. This is because $\mathrm{WKL}_0 + \mathrm{B}\Sigma^0_2 \nvdash \mathrm{I}\Sigma^0_2$. These comments all follow from the facts that WKL_0 is Π^1_1-conservative over RCA_0 (see [27] Corollary IX.2.6) and that $\mathrm{WKL}_0 + \mathrm{B}\Sigma^0_2$ is Π^1_1-conservative over $\mathrm{RCA}_0 + \mathrm{B}\Sigma^0_2$ (see [18] or adapt the proof of [27] Corollary IX.2.6).

3. A LITTLE COMBINATORICS OF TREES

In this short section we isolate two facts concerning the combinatorics of finite trees. These facts appear in [1], but we repeat them here for the sake of completeness and because it is important for our purposes to emphasize that the proofs are formalizable in the first-order fragment $\mathrm{I}\Sigma^0_1$ and hence in RCA_0.

Definition 3.1 (see [1] Definition 2.3).
- The *trunk* of a finite tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ is the longest $\sigma \in T$ such that every element of T is comparable with σ.
- A finite tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ with trunk σ is $\geq n$-*branching* if every $\tau \supseteq \sigma$ in T that is not a leaf has at least n immediate successors.

Lemma 3.2 ($\mathrm{I}\Sigma^0_1$; see [1] Lemma 2.5). *Let $m \geq 1$, let $T \subseteq \mathbb{N}^{<\mathbb{N}}$ be a finite, $\geq 2m$-branching tree with trunk σ, and let P_0 and P_1 be finite trees such that $T \subseteq P_0 \cup P_1$. Then there is a $\geq m$-branching tree $S \subseteq T$ with trunk σ such that* $\mathrm{leaves}(S) \subseteq \mathrm{leaves}(T)$ *and either $S \subseteq P_0$ or $S \subseteq P_1$.*

Proof. For the purposes of this proof, define $\mathrm{depth}(T, \sigma) = \max\{|\tau| - |\sigma| : \tau \in T\}$ for a finite tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ with trunk σ. We prove the lemma by induction on $\mathrm{depth}(T, \sigma)$. If $\mathrm{depth}(T, \sigma) = 0$, then $T = \sigma$ (we identify σ with $\{\tau : \tau \subseteq \sigma\}$ for simplicity). Thus $T \subseteq P_0 \cup P_1$ implies that $\sigma \in P_i$ for some $i < 2$, which implies that $T \subseteq P_i$. Now suppose that $\mathrm{depth}(T, \sigma) = n + 1$. Let $(\tau_j : j < 2m)$ be the first $2m$ immediate successors of σ in T, and for each $j < 2m$, let $T_j = \{\tau \in T : \tau \supseteq \tau_j\}$. For each $j < 2m$, T_j is a $\geq 2m$-branching tree with trunk τ_j, $\mathrm{depth}(T_j, \tau_j) \leq n$, and $T_j \subseteq P_0 \cup P_1$. By induction, for each $j < 2m$ there are an $i_j < 2$ and a $\geq m$-branching subtree $S_j \subseteq T_j$ with trunk

τ_j such that leaves(S_j) $\subseteq$ leaves(T_j) and $S_j \subseteq P_{i_j}$. There is then an $i < 2$ such that $i_j = i$ for at least m of the i_j. Let $S = \bigcup \{S_j : j < 2m \wedge i_j = i\}$. Then S is a desired $\geq m$-branching subtree of T with trunk σ such that leaves(S) $\subseteq$ leaves(T) and $S \subseteq P_i$. $\qquad\square$

Lemma 3.3 ($I\Sigma_1^0$; see [1] Lemma 2.6). *Let $m, n \geq 1$, let T be a finite, $\geq m2^{n-1}$-branching tree with trunk σ, and let $(P_i : i < n)$ be finite trees such that $T \subseteq \bigcup_{i<n} P_i$. Then there are an $i < n$ and a $\geq m$-branching tree $S \subseteq T$ with trunk σ such that leaves(S) $\subseteq$ leaves(T) and $S \subseteq P_i$.*

Proof. By induction on n. The case $n = 1$ is trivial. Suppose that T is a finite, $\geq m2^n$-branching tree with trunk σ such that $T \subseteq \bigcup_{i<n+1} P_i$. By Lemma 3.2, there is a $\geq m2^{n-1}$-branching tree $S \subseteq T$ with trunk σ such that leaves(S) $\subseteq$ leaves(T) and either $S \subseteq \bigcup_{i<n} P_i$ or $S \subseteq P_n$. If $S \subseteq P_n$ we are done. If $S \subseteq \bigcup_{i<n} P_i$, then by induction there are an $i < n$ and a $\geq m$-branching tree $S_0 \subseteq S$ with trunk σ such that leaves(S_0) $\subseteq$ leaves(S) $\subseteq$ leaves(T) and $S_0 \subseteq P_i$ as desired. $\qquad\square$

4. Low $\mathrm{DNR}(k, f)$ functions that avoid $\mathrm{DNR}(b, h)$ functions

Consider a countable model $M = (\mathbb{N}, \mathcal{S})$ of $\mathrm{RCA}_0 + B\Sigma_2^0$ with a proper Σ_2^0 cut. Let $f \in \mathcal{S}$, $n \in \omega$, $\vec{h}$ an n-tuple of elements of $\mathcal{S}$, and $\vec{b}$ an n-tuple of elements of $\mathbb{N}$ be such that $(\forall i < n)(h_i \leq_T f)$ and $(\forall i < n)(f$ computes no $\mathrm{DNR}(b_i, h_i)$ function). Our goal is to produce a function g (outside of $\mathcal{S}$) that is $\mathrm{DNR}(k, f)$ for some $k \in \mathbb{N}$ but is such that that $f \oplus g$ computes no $\mathrm{DNR}(b_i, h_i)$ function for any $i < n$.

In RCA_0, define the function $K(b, s)$ by $K(b, 0) = 2$ and $K(b, s+1) = K(b, s)2^{s^2+b+1}$. Our main technical result is the following theorem.

Theorem 4.1. *Let*
- *$M = (\mathbb{N}, \mathcal{S})$ be a countable model of $\mathrm{RCA}_0 + B\Sigma_2^0$ with a proper Σ_2^0 cut I;*
- *$n \in \omega$, $f \in \mathcal{S}$, $\vec{h}$ an n-tuple of elements of $\mathcal{S}$, and $\vec{b}$ an n-tuple of elements of $\mathbb{N}$ be such that*
  - *$(\forall i < n)(h_i \leq_T f)$ and*
  - *$(\forall i < n)(f$ computes no $\mathrm{DNR}(b_i, h_i)$ function);*
- *$b_{\max} = \max \vec{b}$;*
- *$k_0 \in \mathbb{N}$ be such that $(\forall i \in I)(k_0 > i)$;*
- *$k = K(b_{\max}, k_0)$.*

Then there is a $\mathrm{DNR}(k, f)$ function g such that $f \oplus g$ is low relative to f and such that $f \oplus g$ computes no $\mathrm{DNR}(b_i, h_i)$ function for any $i < n$.

The conclusion that $f \oplus g$ is low relative to f in Theorem 4.1 ensures that $f \oplus g$ preserves $B\Sigma_2^0$.

Before we continue with the proof of Theorem 4.1, we point out that its simplest case provides an interesting example concerning recursion theory in models with limited induction.

Corollary 4.2. *If $\mathbb{N}$ satisfies $B\Sigma_2^0$ but not $I\Sigma_2^0$, then there is a $k \in \mathbb{N}$ and a k-bounded diagonally non-recursive function that computes no 2-bounded diagonally non-recursive function.*

In their proof of [1] Theorem 2.1, Ambos-Spies et al. construct a diagonally non-recursive function $g\colon \omega \to \omega$ (with necessarily unbounded range) that computes no 2-bounded diagonally non-recursive function. In fact, given a recursive h, they construct a diagonally non-recursive g that computes no h-bounded diagonally non-recursive function. Our proof of Theorem 4.1 is essentially the proof of [1] Theorem 2.1 implemented inside of a Σ_2^0 cut as provided by Lemma 2.2. With this strategy, the Ambos-Spies et al. construction is completed in a bounded number of steps, thereby producing a diagonally non-recursive function g with bounded range that does not compute a 2-bounded diagonally non-recursive function.

We build a function g satisfying the conclusion of Theorem 4.1 in a sequence of finite extensions. Throughout the construction, we maintain a coded finite set D of divergent computations according to the following definition. Fix k as in the statement of Theorem 4.1. Henceforth and through the proof of Theorem 4.1, all strings are elements of $k^{<\mathbb{N}}$ and all trees are subtrees of $k^{<\mathbb{N}}$.

Definition 4.3. Let $f\colon \mathbb{N} \to \mathbb{N}$ be a function.

- A string σ *admits $\geq m$-branching f-convergence for $\langle e, x\rangle$* if there is a finite $\geq m$-branching tree T with trunk σ such that $(\forall \alpha \in \mathrm{leaves}(T))(\Phi_e^{f \oplus \alpha}(x)\downarrow)$.
- A string σ *forces $\geq m$-branching f-divergence for $\langle e, x\rangle$* if σ does not admit $\geq m$-branching f-convergence for $\langle e, x\rangle$.
- Let D be a finite coded subset of $\mathbb{N}$. A string σ *forces $\geq m$-branching f-divergence for D* if σ forces $\geq m$-branching f-divergence for every $\langle e, x\rangle \in D$.

The following lemma is essentially Lemma 2.8 in [1].

Lemma 4.4 (RCA$_0$; see [1] Lemma 2.8)**.** *Suppose σ is a string and D is a finite coded set such that σ forces $\geq m$-branching f-divergence for D, where $m2^{|D|} < k$. Then every $\geq m2^{|D|}$-branching tree with trunk σ has a leaf that forces $\geq m$-branching f-divergence for D.*

Proof. Suppose σ forces $\geq m$-branching f-divergence for D, suppose T is a finite $\geq m2^{|D|}$-branching tree with trunk σ, and suppose for a contradiction that no leaf of T forces $\geq m$-branching f-divergence for D. Enumerate D as $D = \{\langle e_i, x_i\rangle : i < |D|\}$, and, using bounded Σ_1^0 comprehension, define a function $j\colon \mathrm{leaves}(T) \to |D|$ by letting $j(\alpha)$ be least such that α admits $\geq m$-branching f-convergence for $\langle e_{j(\alpha)}, x_{j(\alpha)}\rangle$. For each $i < |D|$, let P_i be the tree consisting of the strings in T extendible to an $\alpha \in \mathrm{leaves}(T)$ with $j(\alpha) = i$. Then $T \subseteq \bigcup_{i < |D|} P_i$, so by Lemma 3.3 there is a tree $T' \subseteq T$ that has trunk σ, is $\geq m$-branching, and is contained P_i for some $i < |D|$. For each $\alpha \in \mathrm{leaves}(T')$, let T_α be a $\geq m$-branching tree with trunk α such that $(\forall \beta \in \mathrm{leaves}(T_\alpha))(\Phi_{e_i}^{f \oplus \beta}(x_i)\downarrow)$. Then $T' \cup \bigcup_{\alpha \in \mathrm{leaves}(T')} T_\alpha$ witnesses that σ admits $\geq m$-branching f-convergence for $\langle e_i, x_i\rangle$, a contradiction. $\square$

The construction proceeds in stages. In stages $s \equiv 0 \mod n+2$, we satisfy requirements ensuring that g is total. In stages $s \equiv i + 1 \mod n + 2$ for $i < n$, we satisfy blocks of requirements ensuring that $f \oplus g$ computes no DNR(b_i, h_i) function. In stages $s \equiv n + 1 \mod n + 2$, we satisfy blocks of requirements ensuring that $f \oplus g$ is low relative to f. In the end, g satisfies $\mathrm{ran}(g) \subseteq k$ because we only consider extensions by strings $\sigma \in k^{<\mathbb{N}}$, and g is diagonally non-recursive relative to f because we ensure the divergence of $\Phi_{e_0}^{f \oplus g}(e_0)$, where e_0 is an index such that $(\Phi_{e_0}^{f \oplus g}(e_0)\downarrow) \leftrightarrow \exists e(g(e) = \Phi_e^f(e))$. To make the non-DNR$(b_i, h_i)$ requirements more manageable, we condense a block of non-DNR(b_i, h_i) requirements into a single requirement.

Definition 4.5. Let $b \in \mathbb{N}$, let $f\colon \mathbb{N} \to b$, and let $h\colon \mathbb{N} \to \mathbb{N}$. We say that f is *eventually* DNR(b, h) if $\exists n(\forall e > n)(f(e) \neq \Phi_e^h(e))$.

Define a primitive recursive function $d\colon \mathbb{N}^3 \to \mathbb{N}$ such that, given functions $h \leq_T f$, an index e such that $\forall x(\Phi_e^f(x) = \Phi_x^h(x))$, and bounds a and b, $d(e, a, b)$ is an index for a program such that, for every $\ell \in \mathbb{N}$ and function g, $\Phi_{d(e,a,b)}^{f \oplus g}(\ell)$ searches for a pair $\langle i, s\rangle$ such that $i < a$, $\Phi_{i,s}^{f \oplus g}(\ell) < b$, and $\neg(\exists \ell_0 < \ell)(\Phi_{i,\ell}^{f \oplus g}(\ell_0) = \Phi_{\ell_0,\ell}^h(\ell_0))$. If such a pair is found, then $\Phi_{d(e,a,b)}^{f \oplus g}(\ell) = \Phi_{i,s}^{f \oplus g}(\ell)$ for the first such pair. Otherwise, $\Phi_{d(e,a,b)}^{f \oplus g}(\ell)\uparrow$.

Lemma 4.6 (RCA$_0$)**.** *For any functions f, g, h and $e, a, b \in \mathbb{N}$ as above, if there is an $i < a$ such that $\Phi_i^{f \oplus g}$ is DNR(b, h), then $\Phi_{d(e,a,b)}^{f \oplus g}$ is eventually DNR(b, h).*

Proof. The fact that $\Phi_i^{f \oplus g}$ is DNR(b, h) for some $i < a$ ensures that $\Phi_{d(e,a,b)}^{f \oplus g}$ is total. The finite set $X = \{j < a : \exists \ell, s(\Phi_{j,s}^{f \oplus g}(\ell) = \Phi_{\ell,s}^h(\ell))\}$ exists by bounded Σ_1^0 comprehension. BΣ_1^0 then provides a bound N such that $(\forall j \in X)(\exists \ell, s < N)(\Phi_{j,s}^{f \oplus g}(\ell) = \Phi_{\ell,s}^h(\ell))$. To show that $\Phi_{d(e,a,b)}^{f \oplus g}$ is eventually DNR(b, h), we show that $(\forall \ell > N)(\Phi_{d(e,a,b)}^{f \oplus g}(\ell) \neq \Phi_\ell^h(\ell))$. Suppose for a contradiction that $\ell > N$

and that $\Phi_{d(e,a,b)}^{f\oplus g}(\ell) = \Phi_\ell^h(\ell)$. By the definition of $d(e,a,b)$, there is a $j < a$ such that $\Phi_{d(e,a,b)}^{f\oplus g}(\ell) = \Phi_j^{f\oplus g}(\ell)$ and $\neg(\exists \ell_0 < \ell)(\Phi_{j,\ell}^{f\oplus g}(\ell_0) = \Phi_{\ell_0,\ell}^h(\ell_0))$. The equation $\Phi_j^{f\oplus g}(\ell) = \Phi_{d(e,a,b)}^{f\oplus g}(\ell) = \Phi_\ell^h(\ell)$ implies that $j \in X$ and hence that there are $\ell_0, s < N$ such that $\Phi_{j,s}^{f\oplus g}(\ell_0) = \Phi_{\ell_0,s}^h(\ell_0)$. Now, $N < \ell$ and therefore $(\exists \ell_0 < \ell)(\Phi_{j,\ell}^{f\oplus g}(\ell_0) = \Phi_{\ell_0,\ell}^h(\ell_0))$, a contradiction. $\square$

Lemma 4.7 and Lemma 4.8 below aid the construction of a function witnessing Theorem 4.1. Their proofs are straightforward if one assumes IΣ_2^0. The more complicated arguments below are necessary for our purposes because they are compatible with B$\Sigma_2^0 + \neg$IΣ_2^0.

Lemma 4.7 (RCA$_0$ + BΣ_2^0). *Let f and h be functions and let $b \in \mathbb{N}$ be such that f computes no* DNR(b,h) *function. Let σ be a string, let D be a finite coded set, and let $m \in \mathbb{N}$ be such that σ forces $\geq m$-branching f-divergence for D, where $m2^{|D|+b} < k$. Then for every finite coded set E there are a string $\sigma' \supseteq \sigma$ and a finite coded set $D' \supseteq D$ such that*

(i) $|D'| \leq |D| + |E|$,
(ii) σ' *forces* $\geq m2^{|D|+b}$-*branching f-divergence for D', and*
(iii) *for each $e \in E$, $(\exists \ell > |\sigma|)(\Phi_e^{f\oplus\sigma'}(\ell) \geq b)$, $(\exists \ell > |\sigma|)(\Phi_e^{f\oplus\sigma'}(\ell) = \Phi_\ell^h(\ell))$, or $\exists \ell(\langle e, \ell \rangle \in D')$.*

Proof. We prove the lemma in WKL$_0$+BΣ_2^0, which suffices because WKL$_0$+BΣ_2^0 is Π_1^1-conservative over RCA$_0$ + BΣ_2^0 (see [18] or adapt the proof of [27] Corollary IX.2.6). We thus construct an infinite tree T with trunk σ such that every infinite path through T has an initial segment σ' and a corresponding set D' that satisfy the conclusion of the lemma.

The tree T grows in stages ($T_s : s \in \mathbb{N}$). In order to describe the growth of T, we represent T_s as $T_s = \bigcup_{\tau \in R_s} T_s(\tau)$, where $R_s \subseteq k^{<\mathbb{N}}$ is finite and, for each $\tau \in R_s$, $T_s(\tau)$ is the tree $\tau^\frown k^{<t}$ for some $t \in \mathbb{N}$. Notice that $T_s(\tau)$ has trunk τ. As the construction proceeds, the component trees $T_s(\tau)$ are either *alive*, in which case they are extended, or *dead*, in which case they are not extended. If $T_s(\tau)$ is dead, then no string that is a proper extension of a leaf of $T_s(\tau)$ is ever added to T. During the course of the construction, a component tree $T_s(\tau)$ may be rewritten as a union of new component trees $\bigcup_{\eta \in \text{leaves}(T_s(\tau))} T_{s+1}(\eta)$, where $T_{s+1}(\eta) = \eta$ for each $\eta \in \text{leaves}(T_s(\tau))$, to allow the branches of $T_s(\tau)$ to grow according to different criteria. In this situation, when we update R_s to R_{s+1}, we remove τ and add the elements of leaves$(T_s(\tau))$. To each $\tau \in R_s$ we also associate a finite set $M(\tau)$ of requirements that have been met.

At stage 0, let $R_0 = \{\sigma\}$, $T_0(\sigma) = \sigma$, and $M(\sigma) = \emptyset$. $T_0(\sigma)$ is alive at stage 0.

At the beginning of stage $s + 1$, we have T_s represented as $T_s = \bigcup_{\tau \in R_s} T_s(\tau)$, and we have the corresponding sequence of met requirements $(M(\tau) : \tau \in R_s)$. For each $\tau \in R_s$ do the following:

(a) If $T_s(\tau)$ is dead, put τ in R_{s+1} and let $T_{s+1}(\tau) = T_s(\tau)$. $T_{s+1}(\tau)$ is dead.
(b) If $T_s(\tau)$ is alive:
    (i) If there are a $t \leq s$, a $\tau' \in R_t$ with $\tau' \subseteq \tau$, an $\langle e, x \rangle \in D$, and a $\geq m$-branching tree $S \subseteq k^{<s}$ with trunk τ' such that $(\forall \alpha \in \text{leaves}(S))(\Phi_e^{f\oplus\alpha}(x)\downarrow)$ (i.e., we learn at stage s that τ' admits $\geq m$-branching f-convergence for some $\langle e, x \rangle \in D$), then put τ in R_{s+1} and let $T_{s+1}(\tau) = T_s(\tau)$. $T_{s+1}(\tau)$ is dead.
    (ii) If (i) fails and there are an $e \in E \setminus M(\tau)$, an x with $|\sigma| < x \leq s$, and a $\geq m2^{|D|}$-branching tree $S \subseteq T_s(\tau)$ with trunk τ such that either $(\forall \alpha \in \text{leaves}(S))(\Phi_e^{f\oplus\alpha}(x) = \Phi_{x,s}^h(x))$ or $(\forall \alpha \in \text{leaves}(S))(\Phi_e^{f\oplus\alpha}(x) \geq b)$, then choose the least such e, the least such x for the chosen e, and the least such S for the chosen e and x. Put all leaves of $T_s(\tau)$ in R_{s+1}. Let $T_{s+1}(\eta) = \eta$ and $M(\eta) = M(\tau) \cup \{e\}$ for all $\eta \in \text{leaves}(T_s(\tau))$. If $\eta \in \text{leaves}(T_s(\tau))$ extends a leaf of S, then $T_{s+1}(\eta)$ is alive; otherwise $T_{s+1}(\eta)$ is dead.
    (iii) If (i) and (ii) fail, then put τ in R_{s+1} and let $T_{s+1}(\tau) = T_s(\tau)$. $T_{s+1}(\tau)$ is alive.

Finally, for each $\tau \in R_{s+1}$ with $T_{s+1}(\tau)$ alive, grow $T_{s+1}(\tau)$ by extending each $\alpha \in \text{leaves}(T_{s+1}(\tau))$ to $\alpha^\frown n$ for every $n < k$. This concludes stage $s + 1$.

The tree T is an $(f \oplus h)$-recursive subtree of $k^{<\mathbb{N}}$ because every $\alpha \in k^{<\mathbb{N}}$ is either in $T_s(\tau)$ for some τ in some R_s or properly extends some leaf of $T_s(\tau)$ for some τ in some R_s where $T_s(\tau)$ is dead.

Claim. T *is infinite.*

Proof of Claim. We prove that at the end of every stage s there is some $\tau \in R_s$ such that $T_s(\tau)$ is alive. $T_s(\tau)$ thus grows at the end of stage s, and therefore new strings are added to T at every stage. Hence T is infinite.

For each s, let Q_s be the tree of strings extendible to a $\tau \in R_s$ such that either $T_s(\tau)$ is alive or $T_s(\tau)$ died by item (b) part (i) at some stage $\leq s$. $\mathrm{I}\Sigma^0_1$ suffices to prove that each Q_s is $\geq m2^{|D|}$-branching with trunk σ. This is because a $\tau \in \mathrm{leaves}(Q_s)$ is extended in Q_{s+1} only by the result of acting according to item (b) part (ii) for τ at stage $s + 1$, in which case the subtree of $T_s(\tau)$ consisting of strings extendible to an $\eta \in \mathrm{leaves}(T_s(\tau))$ with $T_{s+1}(\eta)$ alive is $\geq m2^{|D|}$-branching with trunk τ. Thus in Q_{s+1}, τ is appended by a $\geq m2^{|D|}$-branching tree.

Suppose for a contradiction that, at some stage s, $T_s(\tau)$ is dead for all $\tau \in R_s$. Thus each $T_s(\tau)$ for $\tau \in \mathrm{leaves}(Q_s)$ died by item (b) part (i) at some stage $\leq s$. For each $\tau \in \mathrm{leaves}(Q_s)$, let $\tau' \subseteq \tau$ be such that τ' admits $\geq m$-branching f-convergence for some $\langle e, x \rangle \in D$ as in item (b) part (i) at the time of $T_s(\tau)$'s death. Let $R = \{\tau' : (\tau \in \mathrm{leaves}(Q_s)) \wedge \neg(\exists \eta \in \mathrm{leaves}(Q_s))(\eta' \subset \tau')\}$. Let S be the tree of strings extendible to some $\tau' \in R$. S is $\geq m2^{|D|}$-branching with trunk σ, but no leaf of S forces $\geq m$-branching f-divergence for D. This contradicts Lemma 4.4. $\square$

By WKL_0, let p be an infinite path through T. Using bounded Σ^0_1 comprehension, let $s \in \mathbb{N}$ and $\sigma' \subset p$ maximize $|M(\sigma')|$ over all $s \in \mathbb{N}$ and $\sigma' \in R_s$ with $\sigma' \subset p$. Observe that the construction never acts on $T_t(\sigma')$ according to item (b) part (i) or item (b) part (ii) at any stage $t > s$. If the construction acts at stage $t > s$ according to item (b) part (i), then $T_t(\sigma')$ dies and p could not be a path through T. If the construction acts at stage $t + 1 > s$ according to item (b) part (ii), then p must extend some $\eta \in \mathrm{leaves}(T_t(\sigma'))$ with $T_{t+1}(\eta)$ alive, and $|M(\eta)| > |M(\sigma')|$ for all such η. This contradicts the choice of σ' and s. It follows that $\sigma' \in R_t$ for all stages $t \geq s$.

To find D', we define a $\ell_{e,t} \in \mathbb{N}$ for every $e \in E \setminus M(\sigma')$ and every $t > s$ as follows. Let $\ell_{e,t}$ be least $> |\sigma|$ such that no tree $S \subseteq T_t(\sigma')$ with trunk σ' witnesses that σ' admits $\geq m2^{|D|+b}$-branching f-convergence for $\langle e, \ell_{e,t} \rangle$.

Claim. $(\forall e \in E \setminus M(\sigma'))(\exists t > s)(\forall t' > t)(\ell_{e,t'} = \ell_{e,t})$.

Proof of Claim. Let $e \in E \setminus M(\sigma')$. The numbers $\ell_{e,t}$ are increasing in t, so if $(\exists t)(\forall t' > t)(\ell_{e,t'} = \ell_{e,t})$ fails, then it must be that $\lim_{t \to \infty} \ell_{e,t} = \infty$. Thus suppose for a contradiction that $\lim_{t \to \infty} \ell_{e,t} = \infty$. We then compute an eventually $\mathrm{DNR}(b, h)$ function from f, contradicting the hypothesis that f computes no $\mathrm{DNR}(b, h)$ function and hence no eventually $\mathrm{DNR}(b, h)$ function.

Given $x \in \mathbb{N}$, if $x \leq |\sigma|$ then output 0. If $x > |\sigma|$, run the construction to a stage $t > s$ such that $t, \ell_{e,t} > x$ and there are an $i < b$ and a $\geq m2^{|D|}$-branching tree $S \subseteq T_t(\sigma')$ such that $(\forall \alpha \in \mathrm{leaves}(S))(\Phi_e^{f \oplus \alpha}(x) = i)$. Then output the least such i. This procedure describes a b-valued partial f-recursive function Φ^f. To see that $\Phi^f(x)$ converges for $x > |\sigma|$, observe that there is a $t > s, x$ such that $\ell_{e,t} > x$ because $\lim_{t \to \infty} \ell_{e,t} = \infty$ and that at such a stage t, by the definition of $\ell_{e,t}$, there must be a tree $S' \subseteq T_t(\sigma')$ with trunk σ' witnessing that σ' admits $\geq m2^{|D|+b}$-branching f-convergence for $\langle e, x \rangle$. For each $i < b$, let P_i be the tree consisting of the strings in S' that are extendible to an $\alpha \in \mathrm{leaves}(S')$ such that $\Phi_e^{f \oplus \alpha}(x) = i$, and let P_b be the tree consisting of the strings in S' that are extendible to an $\alpha \in \mathrm{leaves}(S')$ such that $\Phi_e^{f \oplus \alpha}(x) \geq b$. Then $S' \subseteq \bigcup_{i < b+1} P_i$, so by Lemma 3.3 there is a $\geq m2^{|D|}$-branching tree $S \subseteq S'$ with trunk σ' such that $S \subseteq P_i$ for some $i < b + 1$. If $S \subseteq P_b$, then the construction would have acted on $T_{t'}(\sigma')$ according to item (b) part (ii) at some stage $t' > s$, contradicting the choice of s. Thus $S \subseteq P_i$ for some $i < b$. Thus there are indeed a stage $t > s$ with $t, \ell_{e,t} > x$, an $i < b$, and a

$\geq m2^{|D|}$-branching tree $S \subseteq T_t(\sigma')$ such that $(\forall \alpha \in \text{leaves}(S))(\Phi_e^{f \oplus \alpha}(x) = i)$. So Φ^f is total. To see that $\Phi^f(x) \neq \Phi_x^h(x)$ for all $x > |\sigma|$, suppose for a contradiction that $x > |\sigma|$ is such that $\Phi^f(x) = \Phi_x^h(x)$. By the definition of $\Phi^f(x)$, there are a stage $t > s$ and a $\geq m2^{|D|}$-branching tree $S \subseteq T_t(\sigma')$ such that $(\forall \alpha \in \text{leaves}(S))(\Phi_e^{f \oplus \alpha}(x) = \Phi_x^h(x))$. Then the construction would have acted on $T_{t'}(\sigma')$ according to item (b) part (ii) at some stage $t' > s$, contradicting the choice of s. Thus Φ^f is eventually $\text{DNR}(b, h)$, contradicting that f computes no such function. Therefore we cannot have $\lim_{t \to \infty} \ell_{e,t} = \infty$, hence $(\exists t)(\forall t' > t)(\ell_{e,t'} = \ell_{e,t})$ as desired. $\qquad\square$

Applying $B\Sigma_2^0$ to the claim, we have that, in fact, $(\exists t_0 > s)(\forall e \in E \setminus M(\sigma'))(\forall t' > t_0)(\ell_{e,t'} = \ell_{e,t_0})$. For each $e \in E \setminus M(\sigma')$, let $\ell_e = \ell_{e,t_0}$. Then let $D' = D \cup \{\langle e, \ell_e \rangle : e \in E \setminus M(\sigma')\}$. We show that σ' and D' satisfy the conclusion of the lemma. The inequality $|D'| \leq |D| + |E|$ is clear.

First, σ' forces $\geq m$-branching f-divergence (and hence $\geq m2^{|D|+b}$-branching f-divergence) for D, otherwise the construction would act according to item (b) part (i) at some stage past s. To see that σ' forces $\geq m2^{|D|+b}$-branching f-divergence for each of the $\langle e, \ell_e \rangle$ with $e \in E \setminus M(\sigma')$, suppose not and let $\langle e, \ell_e \rangle$, with $e \in E \setminus M(\sigma')$, and S, a tree with trunk σ', be such that S witnesses that σ' admits $\geq m2^{|D|+b}$-branching f-convergence for $\langle e, \ell_e \rangle$. As the construction never acts on $T_t(\sigma')$ for $t > s$, there is a stage $t > t_0$ with $S \subseteq T_t(\sigma')$. Thus at stage $t + 1$ there is a tree $S \subseteq T_t(\sigma')$ with trunk σ' witnessing that σ' admits $\geq m2^{|D|+b}$-branching f-convergence for $\langle e, \ell_e \rangle = \langle e, \ell_{e,t} \rangle$, contradicting the choice of $\ell_{e,t}$.

Finally, we show that, for each $e \in E$, either $(\exists \ell > |\sigma|)(\Phi_e^{f \oplus \sigma'}(\ell) \geq b)$, $(\exists \ell > |\sigma|)(\Phi_e^{f \oplus \sigma'}(\ell) = \Phi_\ell^h(\ell))$, or $\exists \ell(\langle e, \ell \rangle \in D')$. By the definition of D', if $e \in E \setminus M(\sigma')$ then there is an ℓ such that $\langle e, \ell \rangle \in D'$. Thus we need to show that if $e \in M(\sigma')$ then either $(\exists \ell > |\sigma|)(\Phi_e^{f \oplus \sigma'}(\ell) \geq b)$ or $(\exists \ell > |\sigma|)(\Phi_e^{f \oplus \sigma'}(\ell) = \Phi_\ell^h(\ell))$. Suppose that $e \in M(\sigma')$, and let $t + 1 \leq s$ be least such that $e \in M(\eta)$ for some $\eta \subseteq \sigma'$ with $\eta \in R_{t+1}$. Then e entered $M(\eta)$ at stage $t+1$ by an action according to item (b) part (ii). Thus at stage $t + 1$ there must have been a $\tau \in R_t$ with $\eta \in \text{leaves}(T_t(\tau))$ and a least x with $|\sigma| < x \leq t$ having a least $\geq m2^{|D|}$-branching tree $S \subseteq T_t(\tau)$ with trunk τ such that either $(\forall \alpha \in \text{leaves}(S))(\Phi_e^{f \oplus \alpha}(x) = \Phi_{x,t}^h(x))$ or $(\forall \alpha \in \text{leaves}(S))(\Phi_e^{f \oplus \alpha}(x) \geq b)$. Moreover, η must extend a leaf of S because $T_{t+1}(\eta)$ must be alive because η is an initial segment of a path through T. So if $(\forall \alpha \in \text{leaves}(S))(\Phi_e^{f \oplus \alpha}(x) = \Phi_{x,t}^h(x))$, then $(\exists \ell > |\sigma|)(\Phi_e^{f \oplus \sigma'}(\ell) = \Phi_\ell^h(\ell))$, and if $(\forall \alpha \in \text{leaves}(S))(\Phi_e^{f \oplus \alpha}(x) \geq b)$ then $(\exists \ell > |\sigma|)(\Phi_e^{f \oplus \sigma'}(\ell) \geq b)$. $\qquad\square$

Lemma 4.8 (RCA$_0$). *Let f be a function, let σ be a string, and let D be a finite coded set such that σ forces $\geq m$-branching f-divergence for D, where $m2^{|D|} < k$. Then for every finite coded set E there is a string $\sigma' \supseteq \sigma$ with the following property. Let $E' = \{e \in E : \Phi_e^{f \oplus \sigma'}(e) \uparrow\}$, and let e' be an index for a program such that, for any function g, $(\Phi_{e'}^g(e') \downarrow) \leftrightarrow (\exists e \in E')(\Phi_e^g(e) \downarrow)$. Then σ' forces $\geq m2^{|D|}$-branching f-divergence for $D \cup \{\langle e', e' \rangle\}$.*

Proof. The proof is similar to that of Lemma 4.7. We prove the lemma in WKL$_0$, which suffices because WKL$_0$ is Π_1^1-conservative over RCA$_0$ (see [27] Corollary IX.2.7). Thus we construct an infinite tree T with trunk σ such that every infinite path through T has an initial segment σ' that satisfies the conclusion of the lemma.

As in Lemma 4.7, T grows in stages $(T_s : s \in \mathbb{N})$, where $T_s = \bigcup_{\tau \in R_s} T_s(\tau)$ and, for each $\tau \in R_s$, $T_s(\tau)$ is $\tau^\frown k^{<t}$ for some $t \in \mathbb{N}$. The component trees are either alive or dead, as before. To every $s \in \mathbb{N}$ and $\tau \in R_s$ we associate the set $E(\tau) = \{e \in E : \Phi_e^{f \oplus \tau}(e) \uparrow\}$ and the index $e(\tau)$, where $(\Phi_{e(\tau)}^g(e(\tau)) \downarrow) \leftrightarrow (\exists e \in E(\tau))(\Phi_e^g(e) \downarrow)$.

At stage 0, let $R_0 = \{\sigma\}$ and $T_0(\sigma) = \sigma$. $T_0(\sigma)$ is alive at stage 0.

At the beginning of stage $s + 1$ we have T_s represented as $T_s = \bigcup_{\tau \in R_s} T_s(\tau)$, and we have the corresponding auxiliary information $(E(\tau) : \tau \in R_s)$ and $(e(\tau) : \tau \in R_s)$. For each $\tau \in R_s$ do the following:

(a) If $T_s(\tau)$ is dead, put τ in R_{s+1} and let $T_{s+1}(\tau) = T_s(\tau)$. $T_{s+1}(\tau)$ is dead.

(b) If $T_s(\tau)$ is alive:

   (i) If there are a $t \leq s$, a $\tau' \in R_t$ with $\tau' \subseteq \tau$, an $\langle e, x \rangle \in D$, and a $\geq m$-branching tree $S \subseteq k^{<s}$ with trunk τ' such that $(\forall \alpha \in \text{leaves}(S))(\Phi_e^{f \oplus \alpha}(x) \downarrow)$ (i.e., we learn at stage s that τ' admits $\geq m$-branching f-convergence for $\langle e, x \rangle \in D$), then put τ in R_{s+1} and let $T_{s+1}(\tau) = T_s(\tau)$. $T_{s+1}(\tau)$ is dead.

   (ii) If (i) fails and there is a $\geq m2^{|D|}$-branching tree $S \subseteq T_s(\tau)$ with trunk τ such that $(\forall \alpha \in \text{leaves}(S))(\Phi_{e(\tau)}^{f \oplus \alpha}(e(\tau)) \downarrow)$, then choose the first such S. Put all leaves of $T_s(\tau)$ in R_{s+1}, and let $T_{s+1}(\eta) = \eta$ for all $\eta \in \text{leaves}(T_s(\tau))$. If $\eta \in \text{leaves}(T_s(\tau))$ extends a leaf of S, then $T_{s+1}(\eta)$ is alive; otherwise $T_{s+1}(\eta)$ is dead.

   (iii) If (i) and (ii) fail, then put τ in R_{s+1} and let $T_{s+1}(\tau) = T_s(\tau)$. $T_{s+1}(\tau)$ is alive.

Finally, for each $\tau \in R_{s+1}$ with $T_{s+1}(\tau)$ alive, grow $T_{s+1}(\tau)$ by extending each $\alpha \in \text{leaves}(T_{s+1}(\tau))$ to $\alpha^\frown n$ for every $n < k$. This concludes stage $s + 1$.

The tree T is an infinite f-recursive subtree of $k^{<\mathbb{N}}$ by arguments similar to those in the proof of Lemma 4.7. By WKL_0, let p be an infinite path through T. Using bounded Σ_1^0 comprehension, let $s \in \mathbb{N}$ and $\sigma' \subset p$ minimize $|E(\sigma')|$ over all $s \in \mathbb{N}$ and $\sigma' \in R_s$ with $\sigma' \subset p$. Then σ' satisfies the conclusion of the lemma. Note that the corresponding e' is $e(\sigma')$. As in the proof of Lemma 4.7, the construction never acts on $T_t(\sigma')$ according to item (b) part (i) or item (b) part (ii) at any stage $t > s$. Consequently, σ' forces $\geq m$-branching f-divergence (and hence $\geq m2^{|D|}$-branching f-divergence) for D because otherwise the construction would act according to item (b) part (i) at some stage past s. Similarly, σ' forces $\geq m2^{|D|}$-branching f-divergence for $\langle e', e' \rangle = \langle e(\sigma'), e(\sigma') \rangle$ because otherwise the construction would act on $T_t(\sigma')$ according to item (b) part (ii) at some stage $t > s$. Thus σ' forces $\geq m2^{|D|}$-branching f-divergence for $D \cup \{\langle e', e' \rangle\}$. $\qquad\square$

Proof of Theorem 4.1. Let M, I, n, f, $\vec{h}$, $\vec{b}$, $b_{\max}$, k_0, and k be as in the statement of Theorem 4.1. For each $i < n$, fix an index w_i such that $\forall x(\Phi_{w_i}^f(x) = \Phi_x^{h_i}(x))$. The proof of Lemma 2.2 shows that there is an increasing, cofinal function $c \colon I \to \mathbb{N}$ whose graph is Δ_2^0.

We build a $\Delta_2^0(f)$ sequence $(\langle \sigma_s, D_s \rangle : s \in J)$ in stages, where $J \subseteq I$ is a $\Sigma_2^0(f)$ cut determined during the course of the construction. In the end, we set $g = \bigcup_{s \in J} \sigma_s$. Let e_0 be an index such that, for any g and x, $(\Phi_{e_0}^{f \oplus g}(x) \downarrow) \leftrightarrow \exists e(g(e) = \Phi_e^f(e))$. At stage 0, set $\sigma_0 = \emptyset$ and set $D_0 = \{\langle e_0, e_0 \rangle\}$.

- At stage $s + 1 \equiv 0 \mod n + 2$, search for the least $\sigma_{s+1} \supseteq \sigma_s$ such that $|\sigma_{s+1}| > \max\{c(s), D_s\}$ (where here D_s is interpreted as the number coding the set D_s) and that σ_{s+1} forces $\geq K(b_{\max}, s+1)$-branching f-divergence for D_s. Let $D_{s+1} = D_s$.
- At stage $s + 1 \equiv i + 1 \mod n + 2$ for an $i < n$, search for the least pair $\langle \sigma', D' \rangle$ as in the conclusion of Lemma 4.7 for f, $h = h_i$, $b = b_i$, $\sigma = \sigma_s$, $D = D_s$, $m = K(b_{\max}, s)$, and $E = \{d(w_i, |\sigma_t|, b_i) : t \leq s\}$. Let $\sigma_{s+1} = \sigma'$ and let $D_{s+1} = D'$.
- At stage $s + 1 \equiv n + 1 \mod n + 2$, search for the least σ' as in the conclusion of Lemma 4.8 for f, $\sigma = \sigma_s$, $D = D_s$, $m = K(b_{\max}, s)$, and $E = \{t : t \leq |\sigma_s|\}$. Let $\sigma_{s+1} = \sigma'$ and let $D_{s+1} = D \cup \{\langle e', e' \rangle\}$.

Let J be the set of $s \in \mathbb{N}$ such that the construction reaches stage s. That is, J is the set of $s \in \mathbb{N}$ for which there is a sequence $(\langle \sigma_j, D_j \rangle : j \leq s)$ where $\sigma_0 = \emptyset$, $D_0 = \{\langle e_0, e_0 \rangle\}$, and, for all $j < s$, $\langle \sigma_{j+1}, D_{j+1} \rangle$ follows from $\langle \sigma_j, D_j \rangle$ according to the rules of the construction. Checking whether $\langle \sigma_{j+1}, D_{j+1} \rangle$ follows from $\langle \sigma_j, D_j \rangle$ is $\Delta_2^0(f)$, so J is $\Sigma_2^0(f)$.

Clearly J is downward closed. To see $J \subseteq I$, let $s \in J$ and let $n_0 < n + 2$ be such that $s - n_0 \equiv 0 \mod n + 2$. Then $s - n_0$ must be in I because $c(s - n_0)$ must be defined in order for $s - n_0$ to be in J. Hence $s \in I$ because I is a cut and $n_0 \in \omega$.

Notice that at stage $s + 1$ at most $s + 1$ elements are added to D_{s+1}. Therefore, for all $s \in J$, $|D_s| \leq 1 + \sum_{j \leq s} j = \frac{1}{2}(s^2 + s) + 1 \leq s^2 + 1$ (the '+1' is because $|D_0| = 1$, not $|D_0| = 0$).

Claim. *If $s \in J$, then σ_s forces $\geq K(b_{\max}, s)$-branching f-divergence for D_s.*

Proof of Claim. Let $(\langle \sigma_j, D_j \rangle : j \leq s)$ be a witness to $s \in J$. We prove the claim by Π_1^0 induction on $j \leq s$. To see that $\sigma_0 = \emptyset$ forces ≥ 2-branching f-divergence for $D_0 = \{\langle e_0, e_0 \rangle\}$, consider a ≥ 2-branching tree T with trunk $\emptyset$. Let t be the height of T, and, by bounded Σ_1^0 comprehension, let $X = \{e < t : \Phi_e^f(e) \downarrow\}$. As T is ≥ 2-branching, we can find an $\alpha \in \text{leaves}(T)$ such that $(\forall e \in X)(\alpha(e) \neq \Phi_e^f(e))$. Then $\Phi_{e_0}^{f \oplus \alpha}(e_0) \uparrow$, showing that T does not witness that $\emptyset$ admits ≥ 2-branching f-convergence for $\langle e_0, e_0 \rangle$, as desired.

Now suppose that $j < s$ and that σ_j forces $\geq K(b_{\max}, j)$-branching f-divergence for D_j.

- If $j + 1 \equiv 0 \mod n + 2$, then σ_{j+1} forces $\geq K(b_{\max}, j+1)$-branching f-divergence for D_{j+1} by definition.
- If $j + 1 \equiv i + 1 \mod n + 2$ for an $i < n$, then σ_{j+1} forces $\geq K(b_{\max}, j)2^{|D_j| + b_i}$-branching f-divergence for D_{j+1} by definition (refer to the statement of Lemma 4.7). As $|D_j| + b_i \leq j^2 + b_{\max} + 1$, σ_{j+1} forces $\geq K(b_{\max}, j+1)$-branching f-divergence for D_{j+1}.
- If $j + 1 \equiv n + 1 \mod n + 2$, then σ_{j+1} forces $\geq K(b_{\max}, j)2^{|D_j|}$-branching f-divergence for D_{j+1} by definition (refer to the statement of Lemma 4.8). As $|D_j| \leq j^2 + b_{\max} + 1$, σ_{j+1} forces $\geq K(b_{\max}, j+1)$-branching f-divergence for D_{j+1}. $\qquad \square$

Claim. *J is a cut.*

Proof of Claim. We have seen that J is downward closed. We need to show that $\forall s(s \in J \rightarrow s+1 \in J)$. So suppose $s \in J$. By the previous claim, σ_s forces $\geq K(b_{\max}, s)$-branching f-divergence for D_s.

- If $s + 1 \equiv 0 \mod n + 2$, then consider the tree $\sigma_s{^\frown} K(b_{\max}, s+1)^{<\max\{c(s), D_s\}}$. It is $\geq K(b_{\max}, s)2^{|D_s|}$-branching with trunk σ_s, so by Lemma 4.4 it has a leaf that forces $\geq K(b_{\max}, s)$-branching f-divergence for D_s, and this leaf also forces $\geq K(b_{\max}, s + 1)$-branching f-divergence for D_s. Thus σ_{s+1} and D_{s+1} are defined.
- If $s + 1 \equiv i + 1 \mod n + 2$ for an $i < n$, then Lemma 4.7 applies. The previous claim and the inequality $K(b_{\max}, s)2^{|D_s| + b} < k$ show that the hypotheses of Lemma 4.7 are satisfied, where $K(b_{\max}, s)2^{|D_s| + b} < k$ because

$$K(b_{\max}, s)2^{|D_s| + b} \leq K(b_{\max}, s)2^{s^2 + b + 1} = K(b_{\max}, s + 1) < K(b_{\max}, k_0) = k,$$

with the strict inequality holding because $s + 1 \in I$ and therefore $s + 1 < k_0$. Thus σ_{s+1} and D_{s+1} are defined.
- If $s + 1 \equiv n + 1 \mod n + 2$, then Lemma 4.8 applies by an argument similar to the one in the previous item. Thus σ_{s+1} and D_{s+1} are defined. $\qquad \square$

Claim. *The function g is total.*

Proof of Claim. If g is not total, then there is a $t \in \mathbb{N}$ such that $(\forall s \in J)(|\sigma_s| < t \wedge D_s < t)$. By increasing t, we may assume that if $s \in J$ then $\sigma_s < t$ (i.e., σ_s is coded by a number $< t$). Thus $s \in J$ if and only if there is a sequence $(\langle \sigma_j, D_j \rangle : j \leq s) \leq \langle t, t \rangle^{s+1}$ (i.e., the sequence of $s + 1$ copies of $\langle t, t \rangle$), where $\sigma_0 = \emptyset$, $D_0 = \langle e_0, e_0 \rangle$, and, for all $j < s$, $\langle \sigma_{j+1}, D_{j+1} \rangle$ follows from $\langle \sigma_j, D_j \rangle$ according to the rules of the construction. This shows that J is $\Delta_2^0(f)$, which is a contradiction because by $B\Sigma_2^0$ there are no $\Delta_2^0(f)$ cuts. $\qquad \square$

Claim. *The function g is $\text{DNR}(k, f)$.*

Proof of Claim. The function g has range contained in k by the convention that all trees are subtrees of $k^{<\mathbb{N}}$. Suppose for a contradiction that $\exists e(g(e) = \Phi_e^f(e))$. Then $\Phi_{e_0}^{f \oplus g}(e_0) \downarrow$, so there is an initial

segment $\sigma \subset g$ such that $\Phi_{e_0}^{f \oplus \sigma}(e_0)\downarrow$. Let s be a stage with $\sigma_s \supseteq \sigma$. Then $\Phi_{e_0}^{f \oplus \sigma_s}(e_0)\downarrow$, but this is a contradiction because $\langle e_0, e_0 \rangle \in D_s$ and σ_s forces $\geq K(b_{\max}, s)$-branching f-divergence for D_s. $\square$

Claim. $f \oplus g$ *is low relative to* f.

Proof of Claim. To determine whether or not $\Phi_e^{f \oplus g}(e)\downarrow$, run the construction to a stage s with $s + 1 \equiv n + 1 \mod n + 2$ and $|\sigma_s| > e$. Then $\Phi_e^{f \oplus g}(e)\downarrow$ if and only if $\Phi_e^{f \oplus \sigma_{s+1}}(e)\downarrow$. Since the sequence $(\langle \sigma_s, D_s \rangle : s \in J)$ is $\Delta_2^0(f)$, it then follows that $f \oplus g$ is low relative to f. Clearly $(\Phi_e^{f \oplus \sigma_{s+1}}(e)\downarrow) \to (\Phi_e^{f \oplus g}(e)\downarrow)$. To see $(\Phi_e^{f \oplus \sigma_{s+1}}(e)\uparrow) \to (\Phi_e^{f \oplus g}(e)\uparrow)$, suppose for a contradiction that $\Phi_e^{f \oplus \sigma_{s+1}}(e)\uparrow$ but $\Phi_e^{f \oplus g}(e)\downarrow$. Let $\langle e', e' \rangle$ be the element added to D_{s+1} at stage $s + 1$. Then $\Phi_{e'}^{f \oplus g}(e')\downarrow$ because e is in E' (where E' is as in Lemma 4.8) and $\Phi_e^{f \oplus g}(e)\downarrow$. Let $r > s$ be a stage such that $\Phi_{e'}^{f \oplus \sigma_r}(e')\downarrow$. Clearly σ_r admits $\geq K(b_{\max}, r)$-branching f-convergence for $\langle e', e' \rangle$, but this contradicts that $\langle e', e' \rangle \in D_r$ and σ_r forces $\geq K(b_{\max}, r)$-branching f-divergence for D_r. $\square$

Claim. *For each* $i < n$, *the function* $f \oplus g$ *computes no* $\mathrm{DNR}(b_i, h_i)$ *function.*

Proof of Claim. Suppose for a contradiction that $\Phi_e^{f \oplus g}$ is $\mathrm{DNR}(b_i, h_i)$. Fix a stage t such that $|\sigma_t| > e$. By the previous claim, $\mathrm{B}\Sigma_2^0$ and hence $\mathrm{I}\Sigma_1^0$ holds relative to $f \oplus g$. Thus Lemma 4.6 applies to $f \oplus g$, so $\Phi_{d(w_i, |\sigma_t|, b_i)}^{f \oplus g}$ is eventually $\mathrm{DNR}(b_i, h_i)$. We show that in fact $\Phi_{d(w_i, |\sigma_t|, b_i)}^{f \oplus g}$ is not eventually $\mathrm{DNR}(b_i, h_i)$, giving the contradiction. Fix ℓ_0. We want to find an $\ell > \ell_0$ such that $\Phi_{d(w_i, |\sigma_t|, b_i)}^{f \oplus g}(\ell) \geq b_i$ or $\Phi_{d(w_i, |\sigma_t|, b_i)}^{f \oplus g}(\ell) = \Phi_\ell^{h_i}(\ell)$. Let $s + 1 > t$ be a stage with $s + 1 \equiv i + 1 \mod n + 2$ and $|\sigma_s| > \ell_0$. At stage $s + 1$, σ_{s+1} and D_{s+1} are defined to be as in the conclusion of Lemma 4.7 for an E with $d(w_i, |\sigma_t|, b_i) \in E$. The result is that $(\exists \ell > |\sigma_s|)(\Phi_{d(w_i, |\sigma_t|, b_i)}^{f \oplus \sigma_{s+1}}(\ell) \geq b_i)$, $(\exists \ell > |\sigma_s|)(\Phi_{d(w_i, |\sigma_t|, b_i)}^{f \oplus \sigma_{s+1}}(\ell) = \Phi_\ell^{h_i}(\ell))$, or $\exists \ell (\langle d(w_i, |\sigma_t|, b_i), \ell \rangle \in D_{s+1})$. If either of the first two alternatives hold, then we have our $\ell > \ell_0$ such that $\Phi_{d(w_i, |\sigma_t|, b_i)}^{f \oplus g}(\ell) \geq b_i$ or $\Phi_{d(w_i, |\sigma_t|, b_i)}^{f \oplus g}(\ell) = \Phi_\ell^{h_i}(\ell)$. If the third alternative holds, then $\Phi_{d(w_i, |\sigma_t|, b_i)}^{f \oplus g}(\ell)\uparrow$, again contradicting that $\Phi_{d(w_i, |\sigma_t|, b_i)}^{f \oplus g}$ is eventually $\mathrm{DNR}(b_i, h_i)$. To see that $\Phi_{d(w_i, |\sigma_t|, b_i)}^{f \oplus g}(\ell)\uparrow$, suppose instead that $\Phi_{d(w_i, |\sigma_t|, b_i)}^{f \oplus g}(\ell)\downarrow$ and let $r > s + 1$ be a stage such that $\Phi_{d(w_i, |\sigma_t|, b_i)}^{f \oplus \sigma_r}(\ell)\downarrow$. Clearly σ_r admits $\geq K(b_{\max}, r)$-branching f-convergence for $\langle d(w_i, |\sigma_t|, b_i), \ell \rangle$, but this contradicts that $\langle d(w_i, |\sigma_t|, b_i), \ell \rangle \in D_r$ and σ_r forces $\geq K(b_{\max}, r)$-branching f-divergence for D_r. $\square$

This concludes the proof of Theorem 4.1. $\square$

Theorem 4.9. $\mathrm{RCA}_0 + \mathrm{B}\Sigma_2^0 + \forall f \exists k \mathrm{DNR}(k, f) \nvdash \exists k \forall f \mathrm{DNR}(k, f)$.

Proof. We build a model of $\mathrm{RCA}_0 + \mathrm{B}\Sigma_2^0 + \forall f \exists k \mathrm{DNR}(k, f) + \neg \exists k \forall f \mathrm{DNR}(k, f)$ by iterating Theorem 4.1.

Let $\mathbb{N}$ be a countable first-order model of $\mathrm{B}\Sigma_2^0 + \neg \mathrm{I}\Sigma_2^0$. By Lemma 2.2, let I be a proper Σ_2^0 cut in $\mathbb{N}$. Fix $k_0 \in \mathbb{N}$ such that $(\forall i \in I)(k_0 > i)$.

Fix an increasing, cofinal sequence $(b_m : m \in \omega)$ of numbers in $\mathbb{N}$. We define a sequence $(f_m : m \in \omega)$ of functions $\mathbb{N} \to \mathbb{N}$ such that, for all $m \in \omega$,

   (i) $f_m \leq_\mathrm{T} f_{m+1}$;
   (ii) $(\mathbb{N}, \Delta_1^0(f_m)) \models \mathrm{RCA}_0 + \mathrm{B}\Sigma_2^0$;
   (iii) no $h \leq_\mathrm{T} f_m$ is $\mathrm{DNR}(b_{m_0}, f_{m_0})$ for any $m_0 \leq m$;
   (iv) for every $h \leq_\mathrm{T} f_m$, there are a $k \in \mathbb{N}$ and a $g \leq_\mathrm{T} f_{m+1}$ that is $\mathrm{DNR}(k, h)$.

Let $f_0 = 0$. The function f_0 is Δ_1^0, so items (ii) and (iii) hold for $m = 0$, with item (ii) holding because $\mathbb{N} \models \mathrm{B}\Sigma_2^0$. Suppose now that $(f_j : j < m + 1)$ satisfies items (i) and (iv) for all $j < m$ and satisfies items (ii) and (iii) for all $j < m + 1$. Then $M = (\mathbb{N}, \Delta_1^0(f_m))$, I, $n = m + 1$, $f = f_m$, $\vec{h} = (f_j : j < m + 1)$, $\vec{b} = (b_j : j < m + 1)$, and $k = K(b_{\max}, k_0)$ satisfy the hypotheses of

Theorem 4.1. Thus let g be as in the conclusion of Theorem 4.1, and let $f_{m+1} = f_m \oplus g$. Then item (i) holds for m and items (ii) and (iii) hold for $m + 1$, with item (ii) holding because f_{m+1} is low relative to f_m. Item (iv) holds for m because $g \leq_T f_{m+1}$ is DNR(k, f_m) and hence computes a DNR(k, h) function for every $h \leq_T f_m$.

Let $\mathcal{S} = \bigcup_{m \in \omega} \Delta_1^0(f_m)$. Then $(\mathbb{N}, \mathcal{S})$ models RCA$_0$ + BΣ_2^0 + $\forall f \exists k$DNR(k, f) + $\neg\exists k \forall f$DNR(k, f). BΣ_2^0 holds relative to every $h \in \mathcal{S}$ by item (ii), so $(\mathbb{N}, \mathcal{S}) \models$ RCA$_0$ + BΣ_2^0. We have that $(\mathbb{N}, \mathcal{S}) \models$ $\forall f \exists k$DNR(k, f) by item (iv). To see that $(\mathbb{N}, \mathcal{S}) \not\models \exists k \forall fDNR(k, f)$, let $k \in \mathbb{N}$ and let $b_{m_0} > k$. Then observe that no $h \in \mathcal{S}$ is DNR(b_{m_0}, f_{m_0}) (hence no $h \in \mathcal{S}$ is DNR(k, f_{m_0})) by item (iii). $\square$

Theorem 4.10. RCA$_0$ + BΣ_2^0 + $\exists k \forall f$DNR$(k, f) \nvdash$ WKL.

Proof. The proof is a simplification of the proof of Theorem 4.9. We build a model of RCA$_0$ + BΣ_2^0 + $\exists k \forall f$DNR(k, f) + $\neg\forall f$DNR$(2, f)$ by iterating Theorem 4.1. As $\forall f$DNR$(2, f)$ and WKL are equivalent over RCA$_0$, this also a model of RCA$_0$ + BΣ_2^0 + $\exists k \forall f$DNR(k, f) + $\neg$WKL.

Proceed as in the proof of Theorem 4.9, but fix $k = K(2, k_0)$ and ignore the sequence $(b_m : m \in \omega)$. Define a sequence $(f_m : m \in \omega)$ of functions $\mathbb{N} \to \mathbb{N}$ that satisfy items (i) and (ii) as before and satisfy the following modified versions of items (iii) and (iv):

(iii') no $h \leq_T f_m$ is DNR$(2, 0)$;
(iv') for every $h \leq_T f_m$, there is a $g \leq_T f_{m+1}$ that is DNR(k, h).

Now f_{m+1} is obtained from f_m by applying Theorem 4.1 to $M = (\mathbb{N}, \Delta_1^0(f_m))$, I, $n = 1$, $f = f_m$, $\vec{h} = (0)$, $\vec{b} = (2)$, and k. The witnessing model $(\mathbb{N}, \mathcal{S})$ is built from $(f_m : m \in \omega)$ as before. $\square$

Now that we know that the statements $\exists k \forall f$DNR(k, f) and $\forall f \exists k$DNR(k, f) do not imply WKL even over RCA$_0$ + BΣ_2^0, it is natural to ask if either statement implies weak weak König's lemma.

Question 4.11. Do either $\exists k \forall f$DNR(k, f) or $\forall f \exists k$DNR(k, f) imply WWKL over RCA$_0$ (or over RCA$_0$ + BΣ_2^0)?

5. Observations concerning the connection between diagonally non-recursive functions and graph colorings

Just as DNR(ℓ, f) trivially implies DNR(k, f) when $k \geq \ell$, so the existence of an ℓ-coloring of a graph trivially implies the existence of a k-coloring of that graph when $k \geq \ell$. This motivates the search for a connection between DNR functions and graph colorings. So far, our efforts in this area have produced more questions than answers.

Definition 5.1 (RCA$_0$). A *graph* $G = (V, E)$ consists of a set of vertices $V \subseteq \mathbb{N}$ and an irreflexive, symmetric relation $E \subseteq V \times V$ which indicates when two vertices are adjacent. Let G be a graph, and let $\ell \in \mathbb{N}$.

- An ℓ-*coloring* of G is a function $\chi \colon V \to \ell$ such that $(\forall u, v \in V)((u, v) \in E \to \chi(u) \neq \chi(v))$.
- G is (*globally*) ℓ-*colorable* if there is an ℓ-coloring of G.
- G is *locally* ℓ-*colorable* if for every finite $V_0 \subseteq V$, the induced subgraph $(V_0, E \cap (V_0 \times V_0))$ is ℓ-colorable.

Let COL(ℓ, k, G) denote the formal statement that "if the graph G locally ℓ-colorable, then G is globally k-colorable." A classic compactness argument shows that a graph is ℓ-colorable if and only if it is locally ℓ-colorable. In the context of reverse mathematics, the following theorem expresses that this fact is equivalent to WKL over RCA$_0$.

Theorem 5.2 (see [20] Theorem 3.4)**.**

$$\text{RCA}_0 \vdash (\forall \ell \geq 2)(\text{WKL} \leftrightarrow \forall G\, \text{COL}(\ell, \ell, G)).$$

In [3], Bean gave an example of a recursive 3-colorable graph that has no recursive k-coloring for any $k \in \omega$. This result suggests that coloring a 3-colorable (or more generally ℓ-colorable) graph with any finite number of colors may also be difficult from the proof-theoretic point of view. To this end, Gasarch and Hirst proved the following theorem.

Theorem 5.3 ([15] Theorem 3).
$$\mathrm{RCA}_0 \vdash (\forall \ell \geq 2)(\mathrm{WKL} \leftrightarrow \forall G\, \mathrm{COL}(\ell, 2\ell - 1, G)).$$

Gasarch and Hirst then conjectured that the $(2\ell - 1)$ in their theorem can be replaced by any $k \geq \ell$.

Conjecture 5.4 ([15] Conjecture 4).
$$\mathrm{RCA}_0 \vdash (\forall \ell \geq 2)(\forall k \geq \ell)(\mathrm{WKL} \leftrightarrow \forall G\, \mathrm{COL}(\ell, k, G)).$$

In [24], Schmerl verified a weakened version of this conjecture in which ℓ and k are both fixed and standard.

Theorem 5.5 ([24] Theorem 1). *Fix $k, \ell \in \omega$ with $k \geq \ell \geq 2$. Then*
$$\mathrm{RCA}_0 \vdash \mathrm{WKL} \leftrightarrow \forall G\, \mathrm{COL}(\ell, k, G).$$

Schmerl connected two key ingredients to prove Theorem 5.5. The first ingredient is the *on-line coloring game* $\Gamma_d(\boldsymbol{K}, k)$, where $\boldsymbol{K}$ is a class of graphs and $d, k \in \mathbb{N}$. $\Gamma_d(\boldsymbol{K}, k)$ is a game between two players, $\forall$ and $\exists$. Player $\forall$ builds a graph in $\boldsymbol{K}$, and Player $\exists$ k-colors it. The game lasts for d rounds. In each round, $\forall$ and $\exists$ alternate plays as follows. Player $\forall$ goes first by adding a new vertex to the graph and connecting it to the existing vertices in such a way that the graph remains in $\boldsymbol{K}$. Player $\exists$ goes second and colors the new vertex with a color from $\{0, 1, \ldots, k - 1\}$. After d rounds, $\exists$ wins if she has produced a k-coloring of the graph enumerated by $\forall$. We say that the class $\boldsymbol{K}$ is *locally on-line k colorable* if for every $d \in \mathbb{N}$, $\exists$ has a winning strategy in $\Gamma_d(\boldsymbol{K}, k)$. (Assuming WKL, it then follows that $\exists$ has a winning strategy in the unbounded on-line coloring game $\Gamma(\boldsymbol{K}, k)$, where the two players continue for as long as $\forall$ keeps playing new vertices.)

The second ingredient can be found in [25] Lemma 2.3, where Schmerl isolates a recursion-theoretic principle similar to the negation of $\mathrm{DNR}(k, f)$. Fix $d \geq 1$, along with a primitive recursive d-tupling function $\mathbb{N}^d \to \mathbb{N}$ with associated primitive recursive projections $p_0, \ldots, p_{d-1} \colon \mathbb{N} \to \mathbb{N}$. Given a function f, define

$$\Delta_{i,d}^f(x) = \begin{cases} \Phi_{p_i(x)}^f(p_i(x)) & \text{if } \Phi_{p_j(x)}^f(p_j(x))\!\downarrow \text{ for all } j \leq i, \\ \uparrow & \text{otherwise.} \end{cases}$$

Write $D_{i+1,d}^f = \mathrm{dom}(\Delta_{i,d}^f)$, and set $D_{0,d}^f = \mathbb{N}$.

Definition 5.6. Consider a function $f \colon \mathbb{N} \to \mathbb{N}$ and a length d sequence $\vec{g}$ of functions $g_i \colon D_{i,d}^f \to \mathbb{N}$ for $i < d$.

- The sequence $\vec{g}$ is *depth d diagonally non-recursive relative to f* ($\vec{g}$ is $\mathrm{DNR}_d(f)$ for short) if $(\forall x)(\exists i < d)(x \in \mathrm{dom}(g_i) \land g_i(x) \neq \Delta_{i,d}^f(x))$.
- The sequence $\vec{g}$ is *k-bounded depth d diagonally non-recursive relative to f* ($\vec{g}$ is $\mathrm{DNR}_d(k, f)$ for short) if it is $\mathrm{DNR}_d(f)$ and each g_i is k-bounded.

Overloading notation as we did before, we let $\mathrm{DNR}_d(f)$ denote the formal statement "there is a $\vec{g}$ that is $\mathrm{DNR}_d(f)$," and we let $\mathrm{DNR}_d(k, f)$ denote the formal statement "there is a $\vec{g}$ that is $\mathrm{DNR}_d(k, f)$." Although different d-tupling schemes lead to different classes of $\mathrm{DNR}_d(f)$-functions, it is always possible to translate back and forth between any two such schemes. In particular, the principles $\mathrm{DNR}_d(f)$ and $\mathrm{DNR}_d(k, f)$ are unaffected by such choices. We therefore see that $\mathrm{DNR}(k, f)$ is equivalent to $\mathrm{DNR}_1(k, f)$ and that if $c > d$ then $\mathrm{DNR}_d(k, f)$ implies $\mathrm{DNR}_c(k, f)$. A further relation between these principles is given by the following lemma.

Lemma 5.7 (RCA$_0$). *Let $f\colon \mathbb{N} \to \mathbb{N}$ be a function, and let $c, d, k \in \mathbb{N}$ be positive. Then $\mathrm{DNR}_c(k^d, f)$ implies $\mathrm{DNR}_{cd}(k, f)$. In particular, $\mathrm{DNR}(k^d, f)$ implies $\mathrm{DNR}_d(k, f)$.*

Proof. Let $\vec{g}$ be $\mathrm{DNR}_c(k^d, f)$. We define a sequence $\vec{h}$ that is $\mathrm{DNR}_{cd}(k, f)$. Let $q_0, \ldots, q_{c-1}$ be the projections for the c-tupling function, and let $p_0, \ldots, p_{cd-1}$ be the projections for the cd-tupling function. Define a recursive function $s\colon \mathbb{N} \to \mathbb{N}$ so that for all $i < c$ and all $x, y \in \mathbb{N}$

$$\Phi^f_{q_i(s(x))}(y) = \sum_{j < d} k^j \Phi^f_{p_{di+j}(x)}(p_{di+j}(x)).$$

Now define $\vec{h} = (h_\ell)_{\ell < cd}$ so that each h_ℓ is k-bounded and so that the equation $g_i(s(x)) = \sum_{j<d} k^j h_{di+j}(x)$ holds for each $i < c$.

We show that $D^f_{i,cd} \subseteq \mathrm{dom}(h_i)$ for each $i < cd$, so that we may shrink the domain of each h_i to be exactly $D^f_{i,cd}$ (if necessary). First, observe that by the definition of $\vec{h}$, if $i < c$ and $s(x) \in \mathrm{dom}(g_i) = D^f_{i,c}$, then $(\forall j < d)(x \in \mathrm{dom}(h_{di+j}))$. Thus we fix $i < c$, $j < d$, and $x \in D^f_{di+j,cd}$, and we show that $s(x) \in D^f_{i,c}$. If $i = 0$, this follows from the definition $D^f_{0,c} = \mathbb{N}$, so we may assume $i > 0$. Now, $x \in D^f_{di+j,cd} = \mathrm{dom}(\Delta^f_{di+j-1,cd}) \subseteq \mathrm{dom}(\Delta^f_{di-1,cd})$ implies that $(\forall \ell \leq di - 1)(\Phi^f_{p_\ell(x)}(p_\ell(x)) \downarrow)$ and therefore that $(\forall m \leq i - 1)(\Phi^f_{q_m(s(x))}(q_m(s(x))) \downarrow)$. Thus $s(x) \in \mathrm{dom}(\Delta^f_{i-1,c}) = D^f_{i,c}$ as desired.

Now we show that $\vec{h}$ is indeed depth cd diagonally non-recursive relative to f. Let $x \in \mathbb{N}$, and let $i < c$ be such that $s(x) \in \mathrm{dom}(g_i)$ and $g_i(s(x)) \neq \Delta^f_{i,c}(s(x))$. Then $(\forall j < d)(x \in \mathrm{dom}(h_{di+j}))$ and $g_i(s(x)) = \sum_{j<d} k^j h_{di+j}(x)$. Thus if $(\forall i < d)(h_{di+j}(x) = \Phi^f_{p_{di+j}(x)}(p_{di+j}(x))$, we would have the contradiction $s(x) \in \mathrm{dom}(\Delta^f_{i,c})$ (as $s(x) \in D^f_{i,c}$ and $(\forall j < d)(\Phi^f_{p_{di+j}(x)}(p_{di+j}(x)) \downarrow)$) and

$$g_i(s(x)) = \sum_{j < d} k^j h_{di+j}(x) = \sum_{j < d} k^j \Phi^f_{p_{di+j}(x)}(p_{di+j}(x)) = \Phi^f_{q_i(s(x))}(q_i(s(x))) = \Delta^f_{i,c}(s(x)).$$

Thus there must be a $j < d$ such that $x \in \mathrm{dom}(h_{di+j})$ and $h_{di+j}(x) \neq \Delta^f_{di+j,cd}(x)$. $\qquad\square$

Schmerl considers colorings of graphs in classes $\boldsymbol{K}$ of a certain kind. The class $\boldsymbol{K}$ is a *universal class of graphs* if there is a set K of finite (coded) graphs such that a graph G belongs to $\boldsymbol{K}$ if and only if every finite induced subgraph of G is isomorphic to a graph in K. The class $\boldsymbol{K}$ is a *natural class of graphs* if it is moreover closed under disjoint sums. That is, if $G_0 = (V_0, E_0), G_1 = (V_1, E_1)$ are graphs in $\boldsymbol{K}$ with mutually disjoint vertex sets, then $G_0 + G_1 = (V_0 \cup V_1, E_0 \cup E_1)$ is also in $\boldsymbol{K}$. It then follows that the class $\boldsymbol{K}$ is closed under countable disjoint sums. For every positive integer ℓ, the locally ℓ-colorable graphs form a natural class of graphs.

The link between the on-line coloring games and the generalized DNR principles is the following result, which can be extracted from the proof of [25] Theorem 2.1.

Lemma 5.8 (RCA$_0$). *Let $f\colon \mathbb{N} \to \mathbb{N}$ be a function, $\boldsymbol{K}$ be a natural class of graphs, and $d, k \in \mathbb{N}$ be positive. If $\forall$ has a winning strategy in $\Gamma_d(\boldsymbol{K}, k)$ and $\mathrm{DNR}_d(k, f)$ fails, then there is an f-recursive graph from the class $\boldsymbol{K}$ that is not globally k-colorable. We may further require that the connected components of this graph have size at most d, and the graph construction is uniform in the parameters k, d and f.*

Lemma 5.8 has two immediate consequences.

Theorem 5.9 (RCA$_0$). *Let $\boldsymbol{K}$ be a natural class of graphs.*
- *For every $k \in \mathbb{N}$, if $\boldsymbol{K}$ is not locally on-line k-colorable but every graph in $\boldsymbol{K}$ is k-colorable, then $\exists d \forall f \mathrm{DNR}_d(k, f)$.*
- *If $\boldsymbol{K}$ is not locally on-line k-colorable for any $k \in \mathbb{N}$ but every graph in $\boldsymbol{K}$ is finitely colorable, then $\forall f \exists d \exists k \mathrm{DNR}_d(k, f)$.*

It is provable in RCA_0 that the natural class of forests (i.e., graphs without cycles) is not locally on-line k-colorable for any k. More precisely, one can recursively construct a strategy for $\forall$ in the game $\Gamma_{2^k}(\boldsymbol{K}, k)$, where $\boldsymbol{K}$ is the class of forests. Since forests are locally 2-colorable, it follows that for every $\ell \geq 2$, the natural class of locally ℓ-colorable graphs is likewise not locally on-line k-colorable for any k.

Corollary 5.10.
- $\mathrm{RCA}_0 \vdash \forall k (\forall G \, \mathrm{COL}(2, k, G) \to \exists d \forall f \mathrm{DNR}_d(k, f))$.
- $\mathrm{RCA}_0 \vdash \forall G \exists k \mathrm{COL}(2, k, G) \to \forall f \exists d \exists k \mathrm{DNR}_d(k, f))$.

When working over $\mathrm{RCA}_0 + \mathrm{I}\Sigma_2^0$, both $\exists k \forall f \mathrm{DNR}(k, f)$ and $\forall f \exists k \mathrm{DNR}(k, f)$ are equivalent to WKL by Theorem 2.7. A similar argument shows that $\exists d \exists k \forall f \mathrm{DNR}_d(k, f)$ and $\forall f \exists d \exists k \mathrm{DNR}_d(k, f)$ are likewise equivalent to WKL over $\mathrm{RCA}_0 + \mathrm{I}\Sigma_2^0$. It follows from Corollary 5.10 that Conjecture 5.4 is true with $\mathrm{RCA}_0 + \mathrm{I}\Sigma_2^0$ in place of RCA_0.

Corollary 5.11. $\mathrm{RCA}_0 + \mathrm{I}\Sigma_2^0 \vdash (\forall \ell \geq 2)(\forall k \geq \ell)(\mathrm{WKL} \leftrightarrow \forall G \, \mathrm{COL}(\ell, k, G))$.

The relationships among diagonally non-recursive functions, depth d diagonally non-recursive sequences, and graph colorings need further clarification.

Question 5.12.
- Are $\exists k \exists d \forall f \mathrm{DNR}_d(k, f)$ and $\exists k \forall G \, \mathrm{COL}(2, k, G)$ equivalent over RCA_0 (or over $\mathrm{RCA}_0 + \mathrm{B}\Sigma_2^0$)?
- Are $\forall f \exists k \exists d \mathrm{DNR}_d(k, f)$ and $\forall G \exists k \mathrm{COL}(2, k, G)$ equivalent over RCA_0 (or over $\mathrm{RCA}_0 + \mathrm{B}\Sigma_2^0$)?

Note that the answers to both parts of Question 5.12 are positive over $\mathrm{RCA}_0 + \mathrm{I}\Sigma_2^0$ because the statements $\exists k \exists d \forall f \mathrm{DNR}_d(k, f)$, $\exists k \forall G \, \mathrm{COL}(2, k, G)$, $\forall f \exists k \exists d \mathrm{DNR}_d(k, f)$, and $\forall G \exists k \mathrm{COL}(2, k, G)$ are each equivalent to WKL over $\mathrm{RCA}_0 + \mathrm{I}\Sigma_2^0$.

While $\mathrm{DNR}(k^d, f)$ implies $\mathrm{DNR}_d(k, f)$ over RCA_0 by Lemma 5.7, it is not known whether the reverse implication holds.

Question 5.13.
- Are $\exists k \forall f \mathrm{DNR}(k, f)$ and $\exists d \exists k \forall f \mathrm{DNR}_d(k, f)$ equivalent over RCA_0 (or over $\mathrm{RCA}_0 + \mathrm{B}\Sigma_2^0$)?
- Are $\forall f \exists k \mathrm{DNR}(k, f)$ and $\forall f \exists d \exists k \mathrm{DNR}_d(k, f)$ equivalent over RCA_0 (or over $\mathrm{RCA}_0 + \mathrm{B}\Sigma_2^0$)?

Note that the answers to both parts of Question 5.13 are positive over $\mathrm{RCA}_0 + \mathrm{I}\Sigma_2^0$ because the statements $\exists k \forall f \mathrm{DNR}(k, f)$, $\exists d \exists k \forall f \mathrm{DNR}_d(k, f)$, $\forall f \exists k \mathrm{DNR}(k, f)$, and $\forall f \exists d \exists k \mathrm{DNR}_d(k, f)$ are each equivalent to WKL over $\mathrm{RCA}_0 + \mathrm{I}\Sigma_2^0$.

Similar to the diagonally non-recursive case, it is possible that $\forall \ell \forall G \exists k \mathrm{COL}(\ell, k, G)$ is strictly weaker than $\forall \ell \exists k \forall G \, \mathrm{COL}(\ell, k, G)$ over $\mathrm{RCA}_0 + \mathrm{B}\Sigma_2^0$. However, our techniques do not readily adapt to avoiding graph colorings because the construction of an eventually $\mathrm{DNR}(b, h)$ function given only an upper bound on the index of a $\mathrm{DNR}(b, h)$ function in Lemma 4.6 relies heavily on the homogeneity of diagonally non-recursive functions. If f_0 and f_1 are diagonally non-recursive functions, then another diagonally non-recursive function g can be obtained by choosing $g(n) \in \{f_0(n), f_1(n)\}$ for each n. However, if f_0 and f_1 are graph colorings, there is no reason to expect that a g chosen the same way is also a graph coloring.

Question 5.14. Does $\forall \ell \forall G \exists k \mathrm{COL}(\ell, k, G)$ imply $\forall \ell \exists k \forall G \, \mathrm{COL}(\ell, k, G)$ over RCA_0 (or over $\mathrm{RCA}_0 + \mathrm{B}\Sigma_2^0$)?

Note that the answer to Question 5.14 is positive over $\mathrm{RCA}_0 + \mathrm{I}\Sigma_2^0$ because the statement $\forall \ell \forall G \exists k \mathrm{COL}(\ell, k, G)$ and the statement $\forall \ell \exists k \forall G \, \mathrm{COL}(\ell, k, G)$ are both equivalent to WKL over $\mathrm{RCA}_0 + \mathrm{I}\Sigma_2^0$.

Motivated by Question 5.12, we conclude by exploring further relationships between diagonally non-recursive functions and graph colorings. First, we observe that the existence of k-bounded diagonally non-recursive functions does not suffice to ensure that locally k-colorable graphs are $(2k-1)$-colorable.

Proposition 5.15. $\mathrm{RCA}_0 + B\Sigma_2^0 \nvdash \forall k(\forall f \mathrm{DNR}(k, f) \to \forall G \, \mathrm{COL}(k, 2k-1, G))$.

Proof. If $\mathrm{RCA}_0 + B\Sigma_2^0 \vdash \forall k(\forall f \mathrm{DNR}(k, f) \to \forall G \, \mathrm{COL}(k, 2k-1, G))$, then also $\mathrm{RCA}_0 + B\Sigma_2^0 + \exists k \forall f \mathrm{DNR}(k, f) \vdash \exists k \forall G \, \mathrm{COL}(k, 2k-1, G)$. It would then follow from Theorem 5.3 that $\mathrm{RCA}_0 + B\Sigma_2^0 + \exists k \forall f \mathrm{DNR}(k, f) \vdash \mathrm{WKL}$, contradicting Theorem 4.10. So $\mathrm{RCA}_0 + B\Sigma_2^0 \nvdash \forall k(\forall f \mathrm{DNR}(k, f) \to \forall G \, \mathrm{COL}(k, 2k-1, G))$. $\square$

A $(2k-1)$-coloring of a graph G is also a $2k$-coloring of G, so asserting that every locally k-colorable graph G is $2k$-colorable is potentially weaker than asserting that it is $(2k-1)$-colorable. This situation raises the following question.

Question 5.16. Does RCA_0 (or $\mathrm{RCA}_0 + B\Sigma_2^0$) prove $\forall k(\forall f \mathrm{DNR}(k, f) \to \forall G \, \mathrm{COL}(k, 2k, G))$?

Note that because $\forall f \mathrm{DNR}(k, f)$ implies WKL over RCA_0 for any fixed $k \in \omega$, the answer to the question is positive when restricted to a fixed $k \in \omega$. Moreover, the answer to the question is also positive over $\mathrm{RCA}_0 + I\Sigma_2^0$ by Theorem 2.7. Although we have not answered this question in general, we can formulate an analog of Theorem 5.2 with $\exists k \forall f \mathrm{DNR}(k, f)$ replacing WKL by restricting the class of graphs.

Definition 5.17.
- A *complete k-partite graph* is a graph $G = (V, E)$ where V is a set of vertices of the form $V = \{v_{(i,n)} : i < k \wedge n \in \mathbb{N}\}$ and $E = \{(v_{(i,n)}, v_{(j,m)}) : i, j < k \wedge n, m \in \mathbb{N} \wedge i \neq j\}$.
- An *ornamented complete k-partite graph* is a graph $G = (V \cup W, E)$, where $(V, E \cap (V \times V))$ is a complete k-partite graph and every $w \in W$ is either isolated or adjacent to exactly one $v \in V$.

Proposition 5.18.

$\mathrm{RCA}_0 \vdash (\forall k \geq 2)(\forall f \mathrm{DNR}(k, f) \leftrightarrow$ *every ornamented complete k-partite graph is k-colorable*$)$.

Proof. Fix $k \in \mathbb{N}$.

For the forward direction, let $G = (V \cup W, E)$ be an ornamented complete k-partite graph, where $V = \{v_{(i,n)} : i < k \wedge n \in \mathbb{N}\}$ and $W = \{w_n : n \in \mathbb{N}\}$. Define a function $h \colon \mathbb{N} \to \mathbb{N}$ so that, for all $n, x \in \mathbb{N}$, $\Phi_{h(n)}^G(x) = i$ if there is an $m \in \mathbb{N}$ such that w_n is adjacent to $v_{(i,m)}$ (and $\Phi_{h(n)}^G(x) \uparrow$ otherwise). Let g be $\mathrm{DNR}(k, G)$. Define $\chi \colon V \to k$ by $\chi(v_{(i,n)}) = i$ and $\chi(w_n) = g(h(n))$. It is easy to verify that χ is a k-coloring of G.

For the backward direction, let $G_0 = (V, E_0)$ be a complete k-partite graph, and, given f, extend G_0 to the ornamented complete k-partite graph $G = (V \cup W, E)$, where $W = \{w_n : n \in \mathbb{N}\}$, by defining $(w_n, v_{(i,s)}) \in E$ if and only if $\Phi_{n,s}^f(n) = i$ and $(\forall t < s)(\Phi_{n,t}^f(n) \uparrow)$. Let χ be a k-coloring of G, and permute the colors so that $\chi(v_{(i,0)}) = i$ for each $i < k$. Then the function g defined by $g(n) = \chi(w_n)$ is $\mathrm{DNR}(k, f)$. $\square$

Say that a graph $G_0 = (V_0, E_0)$ embeds into a graph $G_1 = (V_1, E_1)$ if there is an injection $h \colon V_0 \to V_1$ such that $(\forall v, w \in V_0)((v, w) \in E_0 \to (h(v), h(w)) \in E_1)$. Notice that a graph is k-colorable if and only if it embeds into a complete k-partite graph. In fact, it is not hard to see that RCA_0 proves this fact. We can rephrase Question 5.16 in terms of embeddings as follows.

Question 5.19. Does RCA_0 (or $\mathrm{RCA}_0 + B\Sigma_2^0$) prove the following statement?

$\forall k(\forall f \mathrm{DNR}(k, f) \to$ every locally k-colorable graph

can be embedded into an ornamented complete $2k$-partite graph$)$

Again, the answer to Question 5.19 is positive over $\mathrm{RCA}_0 + \mathrm{I}\Sigma_2^0$ by Theorem 2.7.

Acknowledgement

References

[1] Klaus Ambos-Spies, Bjørn Kjos-Hanssen, Steffen Lempp, and Theodore A. Slaman, *Comparing DNR and WWKL*, Journal of Symbolic Logic **69** (2004), no. 4, 1089–1104.

[2] Jane Austen, *Pride and Prejudice*, T. Egerton, Whitehall, 1813.

[3] Dwight R. Bean, *Effective coloration*, Journal of Symbolic Logic **41** (1976), no. 2, 469–480.

[4] C. T. Chong, Wei Li, and Yue Yang, *Nonstandard models in recursion theory and reverse mathematics*, Bulletin of Symbolic Logic **20** (2014), no. 2, 170–200.

[5] C. T. Chong and K. J. Mourad, Σ_n *definable sets without* Σ_n *induction*, Transactions of the American Mathematical Society **334** (November 1992), no. 1, 349–363 (English).

[6] C. T. Chong, Lei Qian, Theodore A. Slaman, and Yue Yang, Σ_2 *induction and infinite injury priority arguments, part III: Prompt sets, minimal pairs and Shoenfield's conjecture*, Israel Journal of Mathematics **121** (2001), 1–28.

[7] C. T. Chong, Theodore A. Slaman, and Yue Yang, Π_1^1-*conservation of combinatorial principles weaker than Ramsey's theorem for pairs*, Advances in Mathematics **230** (2012), no. 3, 1060–1077.

[8] _____, *The inductive strength of Ramsey's theorem for pairs*, 2014. preprint.

[9] _____, *The metamathematics of stable Ramsey's theorem for pairs*, Journal of the American Mathematical Society **27** (2014), no. 3, 863–892.

[10] C. T. Chong and Yue Yang, Σ_2 *induction and infinite injury priority arguments, Part II Tame* Σ_2 *coding and the jump operator*, Annals of Pure and Applied Logic **87** (1997), no. 2, 103–116.

[11] _____, Σ_2 *induction and infinite injury priority argument, Part I: Maximal sets and the jump operator*, Journal of Symbolic Logic **63** (1998), no. 3, 797–814.

[12] Jennifer Chubb, Jeffry L. Hirst, and Timothy H. McNicholl, *Reverse mathematics, computability, and partitions of trees*, Journal of Symbolic Logic **74** (2009), no. 1, 201–215.

[13] Jared Corduan, Marcia J. Groszek, and Joseph R. Mileti, *Reverse mathematics and Ramsey's property for trees*, Journal of Symbolic Logic **75** (2010), no. 3, 945–954.

[14] Harvey Friedman, *Some systems of second order arithmetic and their use*, Proceedings of the International Congress of Mathematicians (Vancouver, B. C., 1974), Vol. 1, 1975, pp. 235–242.

[15] William Gasarch and Jeffry L. Hirst, *Reverse mathematics and recursive graph theory*, Mathematical Logic Quarterly **44** (1998), no. 4, 465–473.

[16] Marcia J. Groszek, Michael E. Mytilinaios, and Theodore A. Slaman, *The Sacks density theorem and* Σ_2-*bounding*, Journal of Symbolic Logic **61** (1996), no. 2, 450–467.

[17] Marcia J. Groszek and Theodore A. Slaman, *On Turing reducibility*, 1994. preprint.

[18] Petr Hájek, *Interpretability and fragments of arithmetic*, Arithmetic, proof theory, and computational complexity, 1993, pp. 185–196.

[19] Petr Hájek and Pavel Pudlák, *Metamathematics of First-Order Arithmetic*, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1998.

[20] Jeffry L. Hirst, *Marriage theorems and reverse mathematics*, Logic and computation (Pittsburgh, PA, 1987), 1990, pp. 181–196.

[21] Carl G. Jockusch Jr., *Degrees of functions with no fixed points*, Logic, Methodology and Philosophy of Science VIII (1989), 191–201.

[22] Carl G. Jockusch Jr. and Robert I. Soare, Π_1^0 *classes and degrees of theories*, Transactions of the American Mathematical Society **173** (1972), 33–56.

[23] Michael E. Mytilinaios, *Finite injury and* Σ_1-*induction*, Journal of Symbolic Logic **54** (1989), no. 1, 38–49.

[24] James H. Schmerl, *Graph coloring and reverse mathematics*, Mathematical Logic Quarterly **46** (2000), no. 4, 543–548.

[25] _____, *Reverse mathematics and graph coloring: eliminating diagonalization*, Reverse mathematics 2001, 2005, pp. 331–348.

[26] Stephen G. Simpson, *Why the recursion theorists ought to thank me*, 2001.

[27] _____, *Subsystems of Second Order Arithmetic*, Cambridge University Press, 2009.

[28] Theodore A. Slaman, Σ_n-*Bounding and* Δ_n-*Induction*, Proceedings of the American Mathematical Society **132** (August 2004), no. 8, 2449–2456 (English).

[29] Theodore A. Slaman and W. Hugh Woodin, Σ_1-*Collection and the finite injury priority method*, Mathematical logic and applications, 1989, pp. 178–188.

[30] Robert I. Soare, *Recursively Enumerable Sets and Degrees*, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1987.

[31] Xiaokang Yu and Stephen G. Simpson, *Measure theory and weak König's lemma*, Archive for Mathematical Logic **30** (1990), no. 3, 171–180.

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