ON MATHIAS GENERIC SETS

PETER A. CHOLAK, DAMIR D. DZHAFAROV, AND JEFFRY L. HIRST

Abstract. We present some results about generics for computable Mathias forcing. The n-generics and weak n-generics in this setting form a strict hierarchy as in the case of Cohen forcing. We analyze the complexity of the Mathias forcing relation, and show that if G is any n-generic with $n \geq 3$ then it satisfies the jump property $G^{(n-1)} = G' \oplus \emptyset^{(n)}$. We prove that every such G has generalized high degree, and so cannot have even Cohen 1-generic degree. On the other hand, we show that G, together with any bi-immune $A \leq_T \emptyset^{(n-1)}$, computes a Cohen n-generic.

1. Introduction

Forcing has been a central technique in computability theory since it was introduced (in the form we now call Cohen forcing) by Kleene and Post to exhibit a degree strictly between $\mathbf{0}$ and $\mathbf{0}'$. The study of the algorithmic properties of Cohen generic sets, and of the structure of their degrees, has long been a rich source of problems and results. In the present paper, we propose to undertake a similar investigation of generic sets for (computable) Mathias forcing, and present some of our initial results in this direction.

Mathias forcing was perhaps first used in computability theory by Soare in [11] to build an infinite set with no subset of strictly higher degree. Subsequently, it became a prominent tool for constructing infinite homogeneous sets for computable colorings of pairs of integers, as in Seetapun and Slaman [9], Cholak, Jockusch, and Slaman [2], and Dzhafarov and Jockusch [4]. It has also found applications in algorithmic randomness, in Binns, Kjos-Hanssen, Lerman, and Solomon [1].

We show below that a number of results for Cohen generics hold also for Mathias generics, and that a number of others do not. The main point of distinction is that neither the set of conditions, nor the forcing relation is computable, so many usual techniques do not carry over. We begin with background in Section 2, and present some preliminary results in Section 3. In Section 4 we characterize the complexity of the forcing relation, and in Section 5 we prove a number of results about the degrees of Mathias generic sets, and about their relationship to Cohen generic degrees. We indicate questions along the way we hope will be addressed in future work.

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2. Definitions

We assume familiarity with the terminology particular to Cohen forcing in computability theory. (For background on computability theory, see [10]. For background on Cohen generic sets, see Section 1.24 of [3].) The definition of the Mathias forcing partial order is standard, but its formalization in the setting of computability theory requires some care. A slightly different presentation is given in [1, Section 6], over which ours has the benefit of reducing the complexity of the set of conditions from Σ^0_3 to Π^0_2.

Definition 2.1.

(1) A *(computable Mathias) pre-condition* is a pair (D, E) where D is a finite set, E is a computable set, and $\max D < \min E$.
(2) A *(computable Mathias) condition* is a pre-condition (D, E), such that E is infinite.
(3) A pre-condition (D^*, E^*) *extends* a pre-condition (D, E), written $(D^*, E^*) \leq (D, E)$, if $D \subseteq D^* \subseteq D \cup E$ and $E^* \subseteq E$.
(4) A set A *satisfies* a pre-condition (D, E) if $D \subseteq A \subseteq D \cup E$.

By an *index* for a pre-condition (D, E) we shall mean a pair (d, e) such that d is the canonical index of D and $E = \{x : \Phi_e(x) \downarrow = 1\}$. By adopting the convention that for all x, if $\Phi_e(x) \downarrow$ then $\Phi_e(y) \downarrow \in \{0, 1\}$ for all $y \leq x$, it follows that Φ_e is total if E is infinite, i.e., if (D, E) is a condition. Of course, if E is finite then Φ_e may only be defined on a proper initial segment of ω.

The definition makes the set of all indices Π^0_1. However, we can pass to a computable subset containing an index for every pre-condition. Namely, define a strictly increasing computable function g by

$$\Phi_{g(d,e)}(x) = \begin{cases} 0 & \text{if } x \leq \max D_d, \\ \Phi_e(x) & \text{otherwise.} \end{cases}$$

Then the set of pairs of the form $(d, g(d, e))$ is computable, and each is an index for a pre-condition. Moreover, if (d, e) is an index as well, then it and $(d, g(d, e))$ index the same pre-condition. Formally, all references to pre-conditions in the sequel will be to indices from this set, and we shall treat D and E as numbers when convenient.

Note that whether one pre-condition extends another is a Π^0_2 question. By our convention about partial computable functions, the same question for conditions is seen to be Π^0_1.

In what follows, a Σ^0_n *set of conditions* refers to a Σ^0_n-definable set of pre-conditions, each of which is a condition. (Note that this is not the same as the set of all conditions satisfying a given Σ^0_n definition, as discussed further in the next section.) We call such a set *dense* if it contains an extension of every condition, and define what it means to meet or avoid such a set as usual.

Definition 2.2. Fix $n \in \omega$.

(1) A set G is *Mathias n-generic* if it meets or avoids every Σ^0_n set of conditions.
(2) A set G is *weakly Mathias n-generic* if it meets every dense such set.

We call a degree *generic* if it contains a set that is n-generic for all n.

It is easy to see that for every $n \geq 2$, there exists a Mathias n-generic $G \leq_T \emptyset^{(n)}$ (indeed, even $G' \leq_T \emptyset^{(n)}$). This is done just as in Cohen forcing (see [6, Lemma 2.6]), but as there is no computable listing of Σ^0_n sets of conditions, one goes through the Σ^0_n sets of pre-conditions and checks which of these consist of conditions alone. We pass to some other basic properties of generics. We refer to Mathias n-generics below simply as n-generics when no confusion is possible.

3. Basic results

Note that the set of all conditions is Π^0_2. Thus, the set of conditions satisfying a given Σ^0_n definition is Σ^0_n if $n \geq 3$, and Σ^0_3 otherwise. For $n < 3$, we may thus wish to consider the following stronger form of genericity, which has no analogue in the case of Cohen forcing.

Definition 3.1. A set G is *strongly n-generic* if, for every Σ^0_n-definable set of pre-conditions $\mathcal{P}$, either G satisfies some condition in $\mathcal{P}$ or G meets the set of conditions not extended by any condition in $\mathcal{P}$.

Proposition 3.2. *For $n \geq 3$, a set is strongly n-generic if and only if it is n-generic. For $n \leq 2$, a set is strongly n-generic if and only if it is* 3*-generic.*

Proof. Evidently, every strongly n-generic set is n-generic. Now suppose $\mathcal{P}$ is a Σ^0_n set of pre-conditions, and let $\mathcal{C}$ consist of all the conditions in $\mathcal{P}$. An infinite set meets or avoids $\mathcal{P}$ if and only if it meets or avoids $\mathcal{C}$, so every $\max\{n, 3\}$-generic set meets or avoids $\mathcal{P}$. For $n \geq 3$, this means that every n-generic set is strongly n-generic, and for $n \leq 2$ that every 3-generic set is strongly n-generic.

It remains to show that every strongly 0-generic set is 3-generic. Let $\mathcal{C}$ be a given Σ^0_3 set of conditions, and let R be a computable relation such that (D, E) belongs to $\mathcal{C}$ if and only if $(\exists a)(\forall x)(\exists y)R(D, E, a, x, y)$. Define a strictly increasing computable function g by

$$\Phi_{g(D,E,a)}(x) = \begin{cases} \Phi_E(x) & \text{if } (\exists y)R(D, E, a, x, y) \text{ and } \Phi_E(x)\downarrow, \\ \uparrow & \text{otherwise,} \end{cases}$$

and let $\mathcal{P}$ be the computable set of all pre-conditions of the form $(D, g(D, E, a))$. If $(D, E) \in \mathcal{C}$ then Φ_E is total and so there is an a such that $\Phi_{g(D,E,a)} = \Phi_E$. If, on the other hand, (D, E) is a pre-condition not in $\mathcal{C}$ then for each a there is an x such that $\Phi_{g(D,E,a)}(x)\uparrow$. Thus, the members of $\mathcal{C}$ are precisely the conditions in $\mathcal{P}$, so an infinite set meets or avoids $\mathcal{C}$ if and only if it meets or avoids $\mathcal{P}$. In particular, every strongly 0-generic set meets or avoids $\mathcal{C}$. □

As a consequence, we shall restrict ourselves to 3-genericity or higher from now on, or at most weak 2-genericity. Without further qualification, n below will always be a number ≥ 3.

Proposition 3.3. *Every n-generic set is weakly n-generic, and every weakly n-generic set is $(n-1)$-generic.*

Proof. The first implication is clear. For the second, let a Σ^0_{n-1} set $\mathcal{C}$ of conditions be given. Let $\mathcal{D}$ be the class of all conditions that are either in $\mathcal{C}$ or else have no extension in $\mathcal{C}$, which is clearly dense. If $n \geq 4$, then $\mathcal{D}$ is easily seen to be Σ^0_n (actually Π^0_{n-1}) as saying a condition (D, E) has no extension in $\mathcal{C}$ is written

$$\forall (D^*, E^*)[[(D^*, E^*) \text{ is a condition } \wedge (D^*, E^*) \leq (D, E)] \implies (D^*, E^*) \notin \mathcal{C}].$$

If $n = 3$, this makes $\mathcal{D}$ appear to be Σ^0_4 but since $\mathcal{C}$ is a set of conditions only, we can re-write the antecedent of the above implication as

$$D \subseteq D^* \subset D \cup E \wedge (\forall x)[\Phi_{E^*}(x)\downarrow= 1 \wedge \Phi_E(x)\downarrow \implies \Phi_E(x) = 1]$$

to obtain an equivalent Σ^0_3 definition. In either case, then, a weakly n-generic set must meet $\mathcal{D}$, and hence must either meet or avoid $\mathcal{C}$. $\square$

The proof of the following proposition is straightforward. (The first half is proved much like its analogue in the Cohen case. See, e.g., [8], Corollary 2.7.)

Proposition 3.4. *Every weakly n-generic set G is hyperimmune relative to $\emptyset^{(n-1)}$. If G is n-generic, then its degree forms a minimal pair with $\mathbf{0}^{(n-1)}$.*

Corollary 3.5. *Not every n-generic set is weakly $(n+1)$-generic.*

Proof. Take any n-generic $G \leq_T \emptyset^{(n)}$. Then G is not hyperimmune relative to $\emptyset^{(n+1)}$, and so cannot be weakly $(n+1)$-generic. $\square$

We shall separate weakly n-generic sets from n-generic sets in Section 5, thereby obtaining a strictly increasing sequence of genericity notions

$$\text{weakly 3-generic} \Longleftarrow \text{3-generic} \Longleftarrow \text{weakly 4-generic} \Longleftarrow \cdots$$

as in the case of Cohen forcing. In many other respects, however, the two types of genericity are very different. For instance, as noted in [2, Section 4.1], every Mathias generic G is cohesive, i.e., satisfies $G \subseteq^* W$ or $G \subseteq^* \overline{W}$ for every computably enumerable set W. In particular, if we write $G = G_0 \oplus G_1$ then one of G_0 or G_1 is finite. This is false for Cohen generics, which, by an analogue of van Lambalgen's theorem due to Yu [12, Proposition 2.2], have relatively n-generic halves. Thus, no Mathias generic can be even Cohen 1-generic.

Question 3.6. What form of van Lambalgen's theorem holds for Mathias forcing?

Another basic fact is that every Mathias n-generic G is high, i.e., satisfies $G' \geq_T \emptyset''$. (See [1], Corollary 6.7, or [2], Section 5.1 for a proof.) By contrast, it is a well-known result of Jockusch [6, Lemma 2.6] that every Cohen n-generic set G satisfies $G^{(n)} \equiv_T G \oplus \emptyset^{(n)}$. As no high G can satisfy $G'' \leq_T G \oplus \emptyset''$, it follows that no Mathias generic can have even Cohen 2-generic degree. This does not prevent a Mathias n-generic from having Cohen 1-generic degree, as there are high 1-generic sets, but we show this does not happen either in Corollary 5.6.

4. The forcing relation

Much of the discrepancy between Mathias and Cohen genericity stems from the fact that the complexity of forcing a formula, defined below, does not agree with the complexity of the formula. Our forcing language here is the typical one of formal first-order arithmetic plus a set variable, X, and the epsilon relation, $\in$.

We regard every Σ^0_0 (i.e., bounded quantifier) formula φ with no free number variables as being written in disjunctive normal form according to some fixed effective procedure for doing so. Call a disjunct *valid* if the conjunction of all the equalities and inequalities in it is true, which can be checked computably. For each i (ranging over the number of valid disjuncts), let $P_{\varphi,i}$ be the set of all n such that $n \in X$ is a conjunct of the ith valid disjunct, and $N_{\varphi,i}$ the set of all n such that $n \notin X$ is a conjunct of the ith valid disjunct. Canonical indices for these sets can be determined uniformly effectively from an index for φ.

Definition 4.1. Let (D, E) be a condition and let $\varphi(X)$ be a formula with only the set variable X free. If φ is Σ^0_0, say (D, E) *forces* $\varphi(G)$, written $(D, E) \Vdash \varphi(G)$, if for some i, $P_{\varphi,i} \subseteq D$ and $N_{\varphi,i} \subseteq \overline{D \cup E}$. From here, extend the definition of

$(D,E) \Vdash \varphi(G)$ to arbitrary φ inductively according to the standard definition of strong forcing (e.g., as in [3], p. 100, footnote 22, items (iii)–(v)).

Remark 4.2. If $\varphi(X)$ is Σ^0_0 with only the set variable X free and A is a set then $\varphi(A)$ holds if and only if there is an i such that $P_{\varphi,i} \subseteq A$ and $N_{\varphi,i} \subseteq \overline{A}$. Hence, $(D,E) \Vdash \varphi(G)$ if and only if $\varphi(D \cup F)$ holds for all finite $F \subset E$.

Lemma 4.3. *Let (D,E) be a condition and let $\varphi(X)$ be a formula in exactly one free set variable.*

1. *If φ is Σ^0_0 with no free number variables then the relation $(D,E) \Vdash \varphi(G)$ is computable.*
2. *If φ is Π^0_1, Σ^0_1, or Σ^0_2, then so is the relation $(D,E) \Vdash \varphi(G)$.*
3. *For $n \geq 2$, if φ is Π^0_n then the relation of $(D,E) \Vdash \varphi(G)$ is Π^0_{n+1}.*
4. *For $n \geq 3$, if φ is Σ^0_n then the relation $(D,E) \Vdash \varphi(G)$ is Σ^0_{n+1}.*

Proof. We first prove 1. If φ is as hypothesized and $\varphi(D \cup F)$ does not hold for some finite $F \subset E$, then neither does $\varphi(D \cup (F \cap (\bigcup_i P_{\varphi,i} \cup N_{\varphi,i})))$. So by Remark 4.2, we have that $(D,E) \Vdash \varphi(G)$ if and only if $\varphi(D \cup F)$ holds for all finite $F \subset E \cap (\bigcup_i P_{\varphi,i} \cup N_{\varphi,i})$, which can be checked computably.

For 2, suppose that $\varphi(X) \equiv (\forall x)\theta(x,X)$, where θ is Σ^0_0. We claim that (D,E) forces $\varphi(G)$ if and only if $\theta(a, D \cup F)$ holds for all a and all finite $F \subset E$, which makes the forcing relation Π^0_1. The right to left implication is clear. For the other, suppose there is an a and a finite $F \subset E$ such that $\theta(a, D \cup F)$ does not hold. Writing $\theta_a(X)$ for the formula $\theta(a,X)$, let $D^* = D \cup F$ and

$$E^* = \{x \in E : x > \max D \cup F \cup \bigcup_i P_{\theta_a,i} \cup N_{\theta_a,i}\},$$

so that (D^*, E^*) is a condition extending (D,E). Then if (D^{**}, E^{**}) is any extension of (D^*, E^*), we have that

$$D^{**} \cap (\bigcup_i P_{\theta_a,i} \cup N_{\theta_a,i})) = (D \cup F) \cap (\bigcup_i P_{\theta_a,i} \cup N_{\theta_a,i})),$$

and so $\theta(a, D^{**})$ cannot force $\theta(a,G)$. Thus (D,E) does not force $\varphi(G)$. The rest of 2 follows immediately, since forcing a formula that is Σ^0_1 over another formula is Σ^0_1 over the complexity of forcing that formula.

We next prove 3 for $n = 2$. Suppose that $\varphi(G) \equiv (\forall x)(\exists y)\theta(x,y,X)$ where θ is Σ^0_0. Our claim is that $(D,E) \Vdash \varphi(G)$ if and only if, for every a and every condition (D^*, E^*) extending (D,E), there is a finite $F \subset E^*$ and a number $k > \max F$ such that

$$(D^* \cup F, \{x \in E^* : x > k\}) \Vdash (\exists y)\theta(a,y,G), \tag{1}$$

which is a Π^0_3 definition. Since the condition on the left side of (1) extends (D^*, E^*), this definition clearly implies forcing. For the opposite direction, suppose $(D,E) \Vdash \varphi(G)$ and fix any a and $(D^*, E^*) \leq (D,E)$. Then by definition, there is a b and a condition (D^{**}, E^{**}) extending (D^*, E^*) that forces $\theta(a,b,G)$. Write $\theta_{a,b}(X) = \theta(a,b,X)$, and let $F \subset E^*$ be such that $D^{**} = D^* \cup F$. Since $\theta_{a,b}(D^* \cup F)$ holds, we must have $P_{\theta_{a,b},i} \subseteq D^* \cup F$ and $N_{\theta_{a,b},i} \cap (D^* \cup F) = \emptyset$ for some i. Thus, if we let $k = \max N_{\theta_{a,b},i}$, we obtain (1).

To complete the proof, we prove 3 and 4 for $n \geq 3$ by simultaneous induction on n. Clearly, 3 for $n-1$ implies 4 for n, so we already have 4 for $n = 3$. Now assume 4 for some $n \geq 3$. The definition of forcing a Π^0_{n+1} statement is easily seen to be Π^0_2

over the relation of forcing a Σ^0_n statement, and hence Π^0_{n+2} by hypothesis. Thus, 3 holds for $n+1$. ◻

We shall see in Corollary 5.2 in the next section that the complexity bounds in parts 3 and 4 of the lemma cannot be lowered to Σ^0_n and Π^0_n, respectively. As a consequence, n-generics only decide all Σ^0_{n-1} formulas, and not necessarily all Σ^0_n formulas.

Proposition 4.4. *Let G be n-generic, and for $m \leq n$ let $\varphi(X)$ be a Σ^0_m or Π^0_m formula in exactly one free set variable. If (D,E) is any condition satisfied by G that forces $\varphi(G)$, then $\varphi(G)$ holds.*

Proof. If $m = 0$, then φ holds of any set satisfying (D,E), whether it is generic or not. If $m > 0$ and the result holds for Π^0_{m-1} formulas, it also clearly holds for Σ^0_m formulas. Thus, we only need to show that if $m > 0$ and the result holds for Σ^0_{m-1} formulas then it also holds for Π^0_m formulas. To this end, suppose $\varphi(X) \equiv (\forall x)\theta(x,X)$, where θ is Σ^0_{m-1}. For each a, let $\mathcal{C}_a$ be the set of all conditions forcing $\theta(a,X)$, which has complexity at most Σ^0_n by Lemma 4.3. Hence, G meets or avoids each $\mathcal{C}_a$. But if G were to avoid some $\mathcal{C}_a$, say via a condition (D^*,E^*), then (D^*,E^*) would force $\neg\theta(a,G)$, and then (D,E) and (D^*,E^*) would have a common extension forcing $\theta(a,G)$ and $\neg\theta(a,G)$. Thus, G meets every $\mathcal{C}_a$, so $\theta(a,G)$ holds for all a by hypothesis, meaning $\varphi(G)$ holds. ◻

Remark 4.5. It is not difficult to see that if $\varphi(G)$ is the negation of a Σ^0_m formula then any condition (D,E) forcing $\varphi(G)$ forces an equivalent Π^0_m formula. Thus, if G is n-generic and satisfies such a condition, then $\varphi(G)$ holds.

5. Degrees of Mathias generics

We begin here with a jump property for Mathias generics similar to that of Jockusch for Cohen generics. It follows that the degrees $\mathbf{d}$ satisfying $\mathbf{d}^{(n-1)} = \mathbf{d}' \cup \mathbf{0}^{(n-1)}$ yield a strict hierarchy of subclasses of the high degrees.

Theorem 5.1. *For all $n \geq 2$, if G is n-generic then $G^{(n-1)} \equiv_T G' \oplus \emptyset^{(n)}$.*

Proof. That $G^{(n-1)} \geq_T G' \oplus \emptyset^{(n)}$ follows from the fact that G is high, as discussed above. That $G^{(n-1)} \leq_T G' \oplus \emptyset^{(n)}$ is trivial for $n = 2$. To show it for $n \geq 3$, we wish to decide every $\Sigma^{0,G}_{n-1}$ sentence using $G' \oplus \emptyset^{(n)}$. Let $\varphi_0(X), \varphi_1(X), \ldots,$ be a computable enumeration of all Σ^0_{n-1} sentences in exactly one free set variable, and for each i let $\mathcal{C}_i$ be the set of conditions forcing $\varphi_i(G)$, and $\mathcal{D}_i$ the set of conditions forcing $\neg\varphi_i(G)$. Then $\mathcal{D}_i$ is the set of conditions with no extension in $\mathcal{C}_i$, so if G meets $\mathcal{C}_i$ it cannot also meet $\mathcal{D}_i$. On the other hand, if G avoids $\mathcal{C}_i$ then it meets $\mathcal{D}_i$ by definition. Now by Lemma 4.3, each $\mathcal{C}_i$ is Σ^0_n since $n \geq 3$, and so it is met or avoided by G. Thus, for each i, either G meets $\mathcal{C}_i$, in which case $\varphi_i(G)$ holds by Proposition 4.4, or else G meets $\mathcal{D}_i$, in which case $\neg\varphi_i(G)$ holds by Remark 4.5. To conclude the proof, we observe that $G' \oplus \emptyset^{(n)}$ can decide, uniformly in i, whether G meets $\mathcal{C}_i$ or $\mathcal{D}_i$. Indeed, from a given i, indices for $\mathcal{C}_i$ and $\mathcal{D}_i$ (as a Σ^0_n set and a Π^0_n set, respectively) can be found uniformly computably, and then $\emptyset^{(n)}$ has only to produce these sets until a condition in one is found that is satisfied by G, which can in turn be determined by G'. ◻

Corollary 5.2. *For every $n \geq 2$ there is a Π^0_n formula in exactly one free set variable, the relation of forcing which is not Π^0_n. For every $n \geq 3$ there is a Σ^0_n formula in exactly one free set variable, the relation of forcing which is not Σ^0_n.*

Proof. It suffices to prove the second part, as it implies the first by the proof of Lemma 4.3. For consistency with Theorem 5.1, we fix $n \geq 4$ and prove the result for $n-1$. If forcing every Σ^0_{n-1} formula were Σ^0_{n-1}, then the proof of the theorem could be carried out computably in $G' \oplus \emptyset^{(n-1)}$ instead of $G' \oplus \emptyset^{(n)}$. Hence, we would have $G^{(n-1)} \equiv_T G' \oplus \emptyset^{(n-1)}$, contradicting that G must be high. □

The following result is the analogue of Theorem 2.3 of Kurtz [8] that every $A >_T \emptyset^{(n-1)}$ hyperimmune relative to $\emptyset^{(n-1)}$ is Turing equivalent to the $(n-1)$st jump of a weakly Cohen n-generic set. The proof, although mostly similar, requires a few important modifications. The main problem is in coding A into $G^{(n-2)}$, which, in the case of Cohen forcing, is done by appending long blocks of 1s to the strings under construction. As the infinite part of a Mathias condition can be made very sparse, we cannot use the same idea here. We highlight the changes below, and only sketch the rest of the details. Recall that a set is *co-immune* if its complement has no infinite computable subset.

Proposition 5.3. *If $A >_T \emptyset^{(n-1)}$ is hyperimmune relative to $\emptyset^{(n-1)}$, then $A \equiv_T G^{(n-2)}$ for some weakly n-generic set G.*

Proof. Computably in A, we build a sequence $(D_0, E_0) \geq (D_1, E_1) \geq \cdots$ of conditions, beginning with $(D_0, E_0) = (\emptyset, \omega)$. Let $\mathcal{C}_0, \mathcal{C}_1, \ldots$ be a listing of all Σ^0_n sets of pre-conditions, and fixing a $\emptyset^{(n-1)}$-computable enumeration of each $\mathcal{C}_i$, let $\mathcal{C}_{i,s}$ be the set of all pre-conditions enumerated into $\mathcal{C}_i$ by stage $p_A(s)$. We may assume that $\langle D, E\rangle \leq s$ for all $(D, E) \in \mathcal{C}_{i,s}$. Let $B_0, B_1, \ldots$ be a uniformly $\emptyset^{(n-1)}$-computable sequence of pairwise disjoint co-immune sets. Say $\mathcal{C}_i$ *requires attention* at stage s if there exists $b \leq p_A(s)$ in $B_i \cap E_s$ and a condition (D, E) in $\mathcal{C}_{i,s}$ extending $(D_s \cup \{b\}, \{x \in E_s : x > b\})$.

At stage s, assume (D_s, E_s) is given. If there is no $i \leq s$ such that $\mathcal{C}_i$ requires attention at stage s, set $(D_{s+1}, E_{s+1}) = (D_s, E_s)$. Otherwise, fix the least such i. Choose the least corresponding b and earliest enumerated extension (D, E) in $\mathcal{C}_{i,s}$, and let $(D^*, E^*) = (D, E)$. Then obtain (D^{**}, E^{**}) from (D^*, E^*) by forcing the jump, in the usual manner. Finally, let k be the number of stages $t < s$ such that $(D_t, E_t) \neq (D_{t+1}, E_{t+1})$, and let $(D^{***}, E^{***}) = (D^{**} \cup \{b\}, \{x \in E^{**} : x > b\})$, where b is the least element of $B_{A(k)} \cap E^{**}$. If $\langle D^{***}, E^{***}\rangle \leq s+1$, set $(D_{s+1}, E_{s+1}) = (D^{***}, E^{***})$, and otherwise set $(D_{s+1}, E_{s+1}) = (D_s, E_s)$.

By definition, the B_i must intersect every computable set infinitely often, and so the entire construction is A-computable. That $G = \bigcup_s D_s$ is weakly n-generic can be verified much like in Kurtz's proof, but using the $\emptyset^{(n-1)}$-computable function h where $h(s)$ is the least t so that for each (D, E) with $\langle D, E\rangle \leq s$ there exists $b \leq t$ in $B_i \cap E$ and $(D^*, E^*) \in \mathcal{C}_{i,t}$ extending $(D \cup \{b\}, \{x \in E : x > b\})$. That $G^{(n-2)} \leq_T A$ follows by Theorem 5.1 from G' being forced during the construction and thus being A-computable. Finally, to show $A \leq_T G^{(n-2)}$, let $s_0 < s_1 < \cdots$ be all the stages $s > 0$ such that $(D_{s-1}, E_{s-1}) \neq (D_s, E_s)$. The sequence $(D_{s_0}, E_{s_0}) > (D_{s_1}, E_{s_1}) \cdots$ can be computed by $G^{(n-2)}$ as follows. Given (D_{s_k}, E_{s_k}), the least $b \in G - D_{s_k}$ must belong to some B_i, and since $G^{(n-2)}$ computes $\emptyset^{(n-1)}$ it can tell which B_i. Then $G^{(n-2)}$ can produce $\mathcal{C}_i$ until the first (D^*, E^*) extending $(D_{s_k} \cup \{b\}, \{x \in$

$E_{s_k} : x > b\}$), and then obtain (D^{**}, E^{**}) from (D^*, E^*) by forcing the jump. By construction, G satisfies (D^{**}, E^{**}) and $(D_{s_{k+1}}, E_{s_{k+1}}) = (D^{**} \cup \{b\}, \{x \in E^{**} : x > b\})$ for the least $b \in G - D_{s_{k+1}}$. And this b is in B_1 or B_0 depending as k is or is not in B. □

Corollary 5.4. *Not every weakly n-generic set is n-generic.*

Proof. By the previous proposition, $\emptyset^{(n)} \equiv_T G^{(n-2)}$ for some weakly n-generic set G. By Theorem 5.1, if G were n-generic we would have $\emptyset^{(n+1)} \equiv_T G^{(n-1)} \equiv_T G' \oplus \emptyset^{(n)} \equiv_T \emptyset^{(n)}$, which cannot be. □

In spite of Theorem 5.1, we are still left with the possibility that some Mathias n-generic set has Cohen 1-generic degree. We now show that this cannot happen.

Theorem 5.5. *If G is n-generic then it has $\mathbf{GH}_1$ degree, i.e., $G' \equiv_T (G \oplus \emptyset')'$.*

Proof. A condition (D, E) forces $i \in (G \oplus \emptyset')'$ if there is a $\sigma \in 2^{<\omega}$ such that that $\Phi_i^\sigma(i)\downarrow$ and for all $x < |\sigma|$,

$$\begin{aligned} \sigma(x) = 1 \quad &\Longrightarrow \quad (D, E) \Vdash x \in G \oplus \emptyset'; \\ \sigma(x) = 0 \quad &\Longrightarrow \quad (D, E) \Vdash x \notin G \oplus \emptyset'. \end{aligned}$$

This is thus a Σ^0_2 relation, as forcing $x \in G \oplus \emptyset'$ and $x \notin G \oplus \emptyset'$ are Σ^0_1 and Π^0_1, respectively. We claim that (D, E) forcing $i \notin (G \oplus \emptyset')'$, i.e., $\neg(i \in (G \oplus \emptyset')')$, is equivalent to (D, E) having no finite extension that forces $i \in (G \oplus \emptyset')'$, and hence is Π^0_2. That forcing implies this fact is clear. In the other direction, suppose (D, E) does not force $i \notin (G \oplus \emptyset')'$, and so has an extension (D^*, E^*) that forces $i \notin (G \oplus \emptyset')'$. Let σ witness this fact, as above. Then if P and N consist of the $x < |\sigma|$ such that $\sigma(2x) = 1$ and $\sigma(2x) = 0$, respectively, σ witnesses that $(D \cup P, \{x \in E : x > \max P \cup N\})$ also forces $i \in (G \oplus \emptyset')'$.

We now show that $G' \geq_T (G \oplus \emptyset')'$. Let $\mathcal{C}_i$ be the set of conditions that force $i \in (G \oplus \emptyset')'$, and $\mathcal{D}_i$ the set of conditions that force $i \notin (G \oplus \emptyset')'$. Then $\mathcal{C}_i$ is Σ^0_3 and $\mathcal{D}_i$ is Π^0_2, and indices for them as such can be found uniformly from i. Each $\mathcal{C}_i$ must be either met or avoided by G, and as in the proof of Theorem 5.1, G meets $\mathcal{C}_i$ if and only if it does not meet $\mathcal{D}_i$. Which of the two is the case can be determined by G' since $G' \geq_T \emptyset''$ and $\mathcal{C}_i$ and $\mathcal{D}_i$ are both c.e. in $\emptyset''$. By Proposition 4.4, G' can thus determine whether $i \in (G \oplus \emptyset')'$, as desired. □

Recall that a degree $\mathbf{d}$ is $\mathbf{GL}_n$ if $\mathbf{d}^{(n)} = (\mathbf{d} \cup \mathbf{0}')^{(n-1)}$, and that no such degree can be $\mathbf{GH}_1$. It was shown by Jockusch and Posner [7, Corollary 7] that every $\overline{\mathbf{GL}}_2$ degree computes a Cohen 1-generic set. Hence, we obtain the following:

Corollary 5.6. *Every Mathias n-generic set has $\overline{\mathbf{GL}}_m$ degree for all $m \geq 1$. Hence, it is not of Cohen 1-generic degree, but does compute a Cohen 1-generic.*

We leave open the following question, which we have so far been unable to answer. Partial answers are given in the subsequent results.

Question 5.7. Does every Mathias n-generic set compute a Cohen n-generic set?

Theorem 5.8. *If G is Mathias n-generic, and $A \leq_T \emptyset^{(n-1)}$ is bi-immune (i.e., A and $\overline{A}$ are each co-immune), then $G \oplus A$ computes a Cohen n-generic.*

Proof. For every set $S = \{s_0 < s_1 < \cdots\}$, define $S_A = A(s_0)A(s_1)\cdots$, which is a string in $2^{<\omega}$ if S is finite, and a sequence in 2^ω otherwise. Now let $\mathcal{C}_0, \mathcal{C}_1, \ldots$ be a listing of all Σ^0_n subsets of $2^{<\omega}$, together with fixed $\emptyset^{(n-1)}$-computable enumerations. For each i, let $\mathcal{D}_i$ be the set of all conditions (D, E) such that the string D_A belongs to $\mathcal{C}_i$. Then $\mathcal{D}_i$ is a Σ^0_n set of conditions, and as such must be met or avoided by G. If G meets $\mathcal{D}_i$ then G_A, viewed as an element of 2^ω, meets $\mathcal{C}_i$. If G avoids $\mathcal{D}_i$, we claim that G_A must avoid $\mathcal{C}_i$. Indeed, suppose G avoids $\mathcal{D}_i$ via (D, E). Since A and $\overline{A}$ are co-immune, they intersect E infinitely often, and so if D_A had an extension τ in $\mathcal{C}_i$, we could make a finite extension (D^*, E^*) of (D, E) so that $D^*_A = \tau$. This extension would belong to $\mathcal{D}_i$, a contradiction. □

Thus, for example, the join of G with any non-computable $A \leq_T \emptyset'$ computes a Cohen n-generic, as every such A is bi-immune ([5], Corollary 5 (iii)).

Proposition 5.9. *If G is Mathias n-generic and H is Cohen n-generic then H is not many-one reducible to G.*

Proof. Seeking a contradiction, suppose f is a computable function such that $f(H) \subseteq G$ and $f(\overline{H}) \subseteq \overline{G}$. The set of conditions (D, E) with $E \subseteq \operatorname{ran}(f)$ is Σ^0_3-definable, and must be met by G else $G \cap \operatorname{ran}(f)$ would be finite and H would be computable. So fix a condition (D, E) in this set satisfied by G. For all $a > \max D$, we then have that $a \in G$ if and only if $a \in E$ and $f^{-1}(a) \subseteq H$. It follows that $G \leq_T H$, and hence that $G \equiv_T H$, contradicting our observation at the end of Section 3 that no Mathias n-generic can have Cohen n-generic degree. □

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Department of Mathematics, University of Notre Dame, Notre Dame, Indiana 46556 U.S.A.

E-mail address: cholak@nd.edu

Department of Mathematics, University of Notre Dame, Notre Dame, Indiana 46556 U.S.A.

E-mail address: ddzhafar@nd.edu

Department of Mathematical Sciences, Appalachian State University,, Boone, North Carolina 28608 U.S.A.

E-mail address: jlh@math.appstate.edu