A NOTE ON COMPACTNESS OF COUNTABLE SETS

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Abstract. The Heine/Borel compactness theorem restricted to subsets of the rationals implies $\mathsf{WKL_0}$.

One of the earliest results in reverse mathematics was Friedman's theorem on the equivalence of $\mathsf{WKL_0}$ and the Heine/Borel theorem for the unit interval [2]. In June 1999, he asked if the restriction of the Heine/Borel theorem to countable closed subsets implied $\mathsf{WKL_0}$. The following theorem provides an affirmative answer, even in the case when the countable subsets consist entirely of rational numbers.

THEOREM 1 ($\mathsf{RCA_0}$). *The following are equivalent:*

1. $\mathsf{WKL_0}$
2. *Every closed set of rationals in* $[0,1]$ *is Heine/Borel compact.*

PROOF. Friedman [2] proved that $\mathsf{WKL_0}$ implies the Heine/Borel theorem for $[0,1]$. (See [3] for the proof.) For the result above, only the reversal requires proof. Emulating Simpson's reversal of the full Heine/Borel theorem ([3] pages 128–129), for each $s \in 2^{<\mathbb{N}}$, let

$$a_s = \sum_{i<\mathrm{lh}(s)} \frac{2s(i)}{3^{i+1}} \qquad \text{and} \qquad b_s = a_s + \frac{1}{3^{\mathrm{lh}(s)}}.$$

Also, let $c_s = \frac{1}{2}(a_s + b_s)$ and define the intervals

$$M_s = (c_s - \frac{1}{2 \cdot 3^{\mathrm{lh}(s)+1}}, c_s + \frac{1}{2 \cdot 3^{\mathrm{lh}(s)+1}}).$$

Note that for every s, the interval $[a_s, b_s]$ is divided into a left third $[a_{s^\frown\langle 0\rangle}, b_{s^\frown\langle 0\rangle}]$, a middle third M_s centered at c_s, and a right third $[a_{s^\frown\langle 1\rangle}, b_{s^\frown\langle 1\rangle}]$. Since $[a_{\langle\,\rangle}, b_{\langle\,\rangle}] = [0,1]$, these intervals correspond to those in the usual Cantor construction. As in [3], let

$$a'_s = a_s - \frac{1}{3^{\mathrm{lh}(s)+1}} \qquad \text{and} \qquad b'_s = b_s + \frac{1}{3^{\mathrm{lh}(s)+1}}.$$

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Meeting

The interval (a'_s, b'_s) is enough wider than (a_s, b_s) that it includes the points $\{c_t \mid t \supseteq s\}$ and every limit point of this set. Furthermore, if $t \not\supseteq s$, then $c_t \notin (a'_s, b'_s)$.

let T be a $0-1$ tree with no infinite paths. We will prove the desired reversal by showing that 2) implies that T is finite. Let $C = \{c_s \mid s \in T\}$. We must show that C is closed. Let U be the open set consisting of the union of

$$\{M_s - \{c_s\} \mid s \in 2^{<\mathbb{N}}\}$$

and

$$\{(a'_{s^\frown\langle\epsilon\rangle}, b'_{s^\frown\langle\epsilon\rangle}) \mid s \in T \wedge \epsilon \in \{0,1\} \wedge s^\frown\langle\epsilon\rangle \notin T\}.$$

We can show that C is the complement of U by showing that for each $x \in [0,1]$, x is in either C or U, but not both. First suppose that x is not in the Cantor middle third set. Then for some s, $x \in M_s$. Thus $x = c_s \in C$ or $x \in M_s - c_s \subset U$. Since $M_s \cap C = \{c_s\}$, only one of these possibilities can hold. Now suppose that x is in the Cantor middle third set. Then for some $f : \mathbb{N} \to \{0,1\}$, we have

$$x = \sum_{i=0}^{\infty} \frac{2 \cdot f(i)}{3^{i+1}} \notin C.$$

Since T contains no infinite paths, there is an $m \in \mathbb{N}$ such that $f[m] \in T$ but $f[m+1] \notin T$. Thus $x \in (a'_{f[m+1]}, b'_{f[m+1]}) \subseteq U$, completing the proof that C is the complement of U.

We have shown that C is a closed set of rationals in $[0,1]$. Consider the open cover of C given by $V = \{M_s \mid s \in T\}$. By 2), C is covered by a finite subset of V. Each element of V contains exactly one element of C, so C is finite. Each sequence in T corresponds to exactly one element of C, so T is finite. $\dashv$

In the preceding theorem, a closed set was viewed as the complement of an open set. We say that $\bar{X}$ is *separably closed* if there is a countable set of points X such that $\bar{X}$ is the set of limit points of X. If the closed sets of the preceding result are replaced by separably closed sets, the logical strength of the Heine/Borel theorem changes, yielding the following equivalence.

Theorem 2 ($\mathsf{RCA_0}$). *The following are equivalent:*

1. $\mathsf{ACA_0}$
2. *Every separably closed set of rationals in* $[0,1]$ *is Heine/Borel compact.*

Proof. To prove that $\mathsf{ACA_0}$ proves 2), note that $\mathsf{ACA_0}$ proves that every separably closed set is closed [1]. To prove the reversal, let $f : \mathbb{N} \to \mathbb{N}$ be an injection, and define $S = \langle r_i \mid i \in \mathbb{N}\rangle$ by

$$r_i = \sum_{j=0}^{i} 2^{-f(j)-1}.$$

Consider the set of open intervals $\{(0, r_i) \mid i \in \mathbb{N}\}$. Any element of $\bar{S}$ which is irrational or not contained in the intervals encodes the range of f. $\dashv$

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