Reverse mathematics and marriage problems with finitely many solutions

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Abstract

We analyze the logical strength of theorems on marriage problems with a fixed finite number of solutions via the techniques of reverse mathematics. We show that if a marriage problem has k solutions, then there is a finite set of boys such that the marriage problem restricted to this set has exactly k solutions, each of which extend uniquely to a solution of the original marriage problem. The strength of this assertion depends on whether or not the marriage problem has a bounding function. We also answer three questions from our previous work on marriage problems with unique solutions.

Our aim is to analyze some marriage theorems via the techniques of reverse mathematics. The subsystems of second order arithmetic used are RCA_0, which includes comprehension for recursive (or computable) sets of natural numbers, WKL_0, which appends a weak form of König's Lemma for trees, and ACA_0, which appends comprehension for arithmetically definable sets. We refer the reader to Simpson [5] for an extensive development of the program of reverse mathematics.

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We use the standard anthropocentric terminology for marriage theorems. A *marriage problem* consists of a set B of boys, a set G of girls, and a relation $R \subseteq B \times G$ where $(b, g) \in R$ means "b knows g." A *solution* of the marriage problem is an injection $f : B \to G$ such that for all $b \in B$, $(b, f(b)) \in R$. Informally, f matches each boy to a girl from among his acquaintances while avoiding polygamy. In general, marriage theorems provide necessary and sufficient conditions for solutions to exist. In this paper, we consider marriage problems with a fixed finite number of solutions. Formally, we say a marriage problem has k solutions if there is a sequence $f_1, f_2, \ldots f_k$ such that each f_i is a solution, $f_i = f_j$ if and only if $i = j$ and any other function g that is a solution is equal to f_i for some i. Marriage theorems are often expressed using other terminology such as transversals, systems of disctinct representatives (SDRs), and matchings in bipartite graphs.

We note that for every marriage problem considered in this paper, each boy is assumed to know only finitely many girls. Marriage problems in which boys are allowed to know infinitely many girls are considerably more complex and not considered here.

As a notational convenience, we use some set theoretic notation as abbreviations for more complicated formulas of second order arithmetic. If $b \in B$, we write $G(b)$ for $\{g \in G \mid (b, g) \in R\}$. As each boy knows at most finitely many girls, for any choice of b, RCA_0 can prove the existence of $G(b)$. Although $G(b)$ looks like function notation, it is not. We further abuse this notation by using formulas like $g \in G(B_0)$ to abbreviate $\exists b \in B_0((b, g) \in R)$. In settings that address more than one marriage problem, we write $G_M(B_0)$ to denote girls known by boys in B_0 in the marriage problem M. Cardinality notation like $|X| \leq |Y|$ abbreviates the assertion that there is an injection from X into Y. The formula $|X| < |Y|$ abbreviates the conjunction of $|X| \leq |Y|$ and $|Y| \not\leq |X|$. For finite sets, RCA_0 can prove many familiar statements about cardinality, for example, if X is finite and $y \notin X$ then $|X| < |X \cup \{y\}|$.

If a marriage problem has finitely many solutions, then the set of boys on which the solutions differ must be finite. This assertion is equivalent to WKL_0, as shown by our first theorem.

Theorem 1. (RCA_0) *The following are equivalent:*

(1) WKL_0.

(2) *Suppose $M = (B, G, R)$ is a marriage problem in which each boy knows*

finitely many girls, and M has exactly k solutions, $f_1, f_2, \dots f_k$. Then there is a finite set $B_0 \subset B$ such that for all $i < j \leq k$ and $b \in B$, if $f_i(b) \neq f_j(b)$ then $b \in B_0$.

Proof. To prove that (1) implies (2), assume WKL_0 and let $M = (B, G, R)$ be as described in (2). Suppose $B_0 = \{b \in B \mid \exists i \exists j f_i(b) \neq f_j(b)\}$ is unbounded. Our goal is to show that M has more than k solutions.

For each pair $i < j \leq k$, let $R_{ij} = \{(b, g) \mid f_i(b) = g \vee f_j(b) = g\}$ and define $M_{ij} = (B, G, R_{ij})$. Note that f_i and f_j are solutions of M_{ij} and any solution of M_{ij} is also a solution of M. Viewing M_{ij} as a bipartite graph with vertex sets B and G, the connected components of M_{ij} are among the forms represented in Figure 1. From left to right, we describe these as single links, finite cycles, and linear paths. The linear path in Figure 1 is finite and has two endpoints, but M_{ij} could contain infinite linear paths with either one endpoint or no endpoints. We will consider two cases based on the prevalence of finite cycles in M_{ij}.

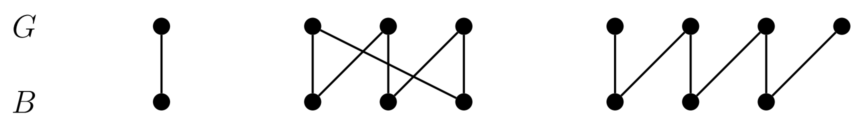

Figure 1: Typical subgraphs of M_{ij}

For the first case, suppose there is a subset B_1 of B_0 of size $k(k-1)(2k+1)$ such that no element lies in any finite cycle of any M_{ij} for any value of i and j. For any $b \in B_1$, we have $b \in B_0$, so for some i and j, $f_i(b) \neq f_j(b)$ and b lies in a linear path in M_{ij}. There are only $k(k-1)$ pairs of indices, so there must be a subset $B_2 \subset B_1$ of size $2k+1$ and a fixed pair i_0 and j_0 such that $\forall b \in B_2 (f_{i_0}(b) \neq f_{j_0}(b))$. We consider two subcases. First, suppose that there are at least $k+1$ boys b in B_2 such that $f_{i_0}(b)$ is not in the range of f_{j_0}. Then for each such b, the function matching b to $f_{i_0}(b)$ and agreeing with f_{j_0} elsewhere is a solution of $M_{i_0 j_0}$, yielding $k+1$ distinct solutions of M. For the second subcase, suppose that there are less than $k+1$ boys in B_2 such that $f_{i_0}(b)$ is not in the range of f_{j_0}. Recall that $|B_2| = 2k+1$, so there are at least $k+1$ boys b in B_2 such that $f_{i_0}(b)$ is in the range of f_{j_0}. Fix any such b and find b' such that $f_{j_0}(b') = f_{i_0}(b)$. Naïvely, the function sending b to $f_{j_0}(b)$ and b' to $f_{i_0}(b')$ can be extended to a matching of the linear path containing b, and indeed to to a solution of $M_{i_0 j_0}$ that includes every girl in the range of f_{i_0} except for $f_{i_0}(b)$. More formally, for any finite set B' of boys including our fixed b, there is a solution of $M_{i_0j_0}$ restricted to B' that does not include $f_{i_0}(b)$ but does include each girl in $\{f_{i_0}(t) \mid t \in B' \wedge t \neq b\}$. So the tree of initial segments of solutions of $M_{i_0j_0}$ avoiding (only) $f_{i_0}(b)$ is infinite, and by WKL_0 must contain an infinite path encoding a solution of $M_{i_0j_0}$ avoiding only $f_{i_0}(b)$. Because there are $k+1$ possible values we could fix for b, we can construct $k+1$ trees and apply weak König's lemma for sequences of trees (provable in WKL_0 as in Lemma 5 of Hirst [3]) to find $k+1$ distinct solutions of M. This yields a contradiction, completing our discussion of the first case.

For the second case, suppose the first case fails, so that for any collection of $k(k-1)(2k+1)$ boys from B_0, at least one must lie in some finite cycle of some M_{ij}. Using RCA_0, we can construct an infinite enumeration of finite cycles from the various M_{ij} such that no cycle is repeated. (Cycles from different M_{ij} may overlap.) Among the first $k^2(k-1)$ cycles in this enumeration, k must match indices. Suppose they all lie in $M_{i_0j_0}$. For any subset of these cycles, the function that matches f_{i_0} on the cycles in the subset and matches f_{j_0} everywhere else is a solution of $M_{i_0j_0}$ distinct from the solution corresponding to any other subset. Thus we have $2^k > k$ distinct solutions of M, completing our discussion of the second case and our proof that (1) implies (2).

To prove that (2) implies (1), suppose T is a nontrivial $0-1$ tree with no infinite paths. (In Simpson's [5] terminology, a $0-1$ tree is a binary tree in which each node is labeled with a 0 or a 1.) We will use (2) to prove that T is finite. Identify each node in T with the finite sequence of zeros and ones leading to it, and let $\rho = \langle\ \rangle$ denote the empty sequence at the root of the tree. For any sequence $\sigma \in T$, let $\sigma+$ denote the next node in the pre-order depth first traversal of T. That is, if σ has a successor, then $\sigma+ = \sigma^\frown 0$ if $\sigma^\frown 0 \in T$ and $\sigma+ = \sigma^\frown 1$ otherwise. If σ has no successor, let l be the largest number less than the length of σ such that $\sigma(l) = 0$ and $\tau = \langle \sigma(0)\dots,\sigma(l),1\rangle \in T$, and define $\sigma+ = \tau$. If σ has no successor and no such l exists (so σ is the rightmost leaf of T) let $\sigma+ = \rho$. RCA_0 can prove that $\forall\sigma(\sigma \neq \sigma+)$ and $\forall\sigma\forall\tau(\sigma \neq \tau \to \sigma+ \neq \tau+)$. Also, because T has no infinite paths, the function $\sigma \to \sigma+$ is onto. To see this, suppose τ is not equal to $\sigma+$ for any choice of σ. Then the rightmost path through T lying to the left of τ exists by recursive comprehension and must be infinite.

Now we can define the marriage problem associated with T. Let $M = (B, G, R)$ where $B = \{b_\sigma \mid \sigma \in T\}$, $G = \{g_\sigma \mid \sigma \in T\}$, and $R = \{(b_\sigma, g_\sigma) \mid \sigma \in T\} \cup \{(b_\sigma, g_{\sigma+}) \mid \sigma \in T\}$. Every boy knows exactly two girls. The functions f_0 and f_1 defined by $f_0(b_\sigma) = g_\sigma$ and $f_1(b_\sigma) = g_{\sigma+}$ are solutions of M, differing at every element of B. We claim that M has no other solutions. Let f be a solution that differs from f_0. Then for some σ, $f(b_\sigma) = g_{\sigma+}$. We will show that f must be f_1 in two steps.

For the first step, we will show that if $f(b_\sigma) = g_{\sigma+}$ then $f(b_\tau) = g_{\tau+}$ for all extensions τ of σ. By way of contradiction, suppose this is not the case. Let τ be the shortest extension of σ such that $f(b_\tau) = g_\tau$. Write $\tau = \sigma_0{}^\frown i$ where $\sigma_0 \supset \sigma$ and $i \in \{0, 1\}$. Because τ is shortest, $f(\sigma_0) = \sigma_0+$ and because f is injective, $\sigma_0+ \neq \tau$. Thus, $\sigma_0+ = \sigma_0{}^\frown 0$ and $\tau = \sigma_0{}^\frown 1$. Recall that $f(\sigma_0) = \sigma_0+ = \sigma_0{}^\frown 0$, so $f(\sigma_0{}^\frown 0) = \sigma_0{}^\frown 0+$, and that $\tau = \sigma_0{}^\frown 1$, so $f(\sigma_0{}^\frown 1) = \sigma_0{}^\frown 1$. Thus, we have found an extension σ_0 of σ such that $f(\sigma_0{}^\frown 0) = \sigma_0{}^\frown 0+$ and $f(\sigma_0{}^\frown 1) = \sigma_0{}^\frown 1$. If every extension of σ with this property has a proper extension with this property, then we can construct an infinite path through T. The tree T has no infinite paths, so there is an extension σ_1 of σ such that $f(\sigma_1{}^\frown 0) = \sigma_1{}^\frown 0+$, $f(\sigma_1{}^\frown 1) = \sigma_1{}^\frown 1$, and $f(\sigma_1{}^\frown 0^\frown \mu) = \sigma_1{}^\frown 0^\frown \mu+$ for every $\sigma_1{}^\frown 0^\frown \mu \in T$. Because T has no infinite paths, we can find the rightmost leaf in T extending $\sigma_1{}^\frown 0$; call it $\sigma_1{}^\frown 0^\frown \mu_0$ and note that $\sigma_1{}^\frown 0^\frown \mu_0+ = \sigma_1{}^\frown 1$. We have $f(\sigma_1{}^\frown 0^\frown \mu_0) = \sigma_1{}^\frown 0^\frown \mu_0+ = \sigma_1{}^\frown 1$ and $f(\sigma_1{}^\frown 1) = \sigma_1{}^\frown 1$, contradicting the claim that f is injective. This contradiction completes the first step, and shows that if $f(b_\sigma) = g_{\sigma+}$ then $f(b_\tau) = g_{\tau+}$ for all extensions τ of σ.

For the second step, we will show that if $f(b_\sigma) = g_{\sigma+}$ for some σ, then $f(b_\rho) = g_{\rho+}$. To see this, suppose $f(b_\sigma) = g_{\sigma+}$ for some σ. We may choose σ so that it is the shortest sequence with this property and the rightmost node of T at this level with this property. (We will eventually see that $\sigma = \rho$.) Using the fact that T has no infinite paths, let τ be the leaf of the rightmost path of T extending σ. By the preceding paragraph, $f(b_\tau) = g_{\tau+}$. Because f is injective, $f(b_{\tau+}) \neq g_{\tau+}$. By the definition of successor $(+)$ and the fact that τ is a rightmost leaf, the length of $\tau+$ must be less than or equal to the length of σ. By minimality it cannot be less, so it must be equal, and because σ is rightmost, $\tau+ = \sigma$. Thus $\sigma = \tau+$ is a proper initial segment of τ. By the definition of the successor operation, this only happens when $\sigma = \rho = \langle\ \rangle$. Thus $f(b_\rho) = g_{\rho+}$, as desired.

Combining the two steps, if $f(b_\sigma) = g_{\sigma+}$ for any $\sigma \in T$, then $f(b_\rho) = g_{\rho+}$ and $f(b_\tau) = g_{\tau+}$ for every τ extending ρ. So if f agrees with f_1 on one boy, then f is f_1. If f does not agree with f_1 on any boy, then it agrees with f_0 on every boy. Thus, M is a marriage problem with exactly two solutions and those solutions differ on every boy. Applying (2), the set of boys must be finite. Thus, the set of nodes in T is finite, completing the proof of the reversal and the theorem. □

Suppose that $M = (B, G, R)$ is a marriage problem in which B and G are subsets of $\mathbb{N}$. We say that M is *bounded* if there is a function $h : B \to G$ such that for each $b \in B$, $G(b) \subseteq \{0, 1, \ldots, h(b)\}$. The function h acts as a uniform bound on the girls that each boy knows, and also insures that each boy knows only finitely many girls. Given such a bounding function, recursive comprehension proves the existence of the function mapping each b to (the code for) the finite set $G(b)$. The next theorem is a stronger version of Lemma 3 of Hirst and Hughes [4]. Although it considers only marriage problems with unique solutions, is useful in the proofs of later results on problems with finite numbers of solutions.

Theorem 2. (RCA_0) *The following are equivalent:*

(1) WKL_0.

(2) *Suppose $M = (B, G, R)$ is a bounded marriage problem with a unique solution. Then there is an enumeration of the boys $\langle b_i \rangle_{i \geq 1}$ such that for every $n \geq 1$, we have $|G(b_1, \ldots b_n)| = n$.*

(3) *Suppose $M = (B, G, R)$ is a bounded marriage problem with a unique solution. Then for any finite $B_0 \subset B$, there is a finite set F such that $B_0 \subset F \subset B$ and $|G(F)| = |F|$.*

Proof. Theorem 6 of Hirst and Hughes [4] shows that (1) is equivalent to (2). To see that (2) implies (3), suppose $M = (B, G, R)$ is a bounded marriage problem with a unique solution, and B_0 is a finite subset of B. Apply (2) to obtain and enumeration $\langle b_i \rangle_{i \geq 1}$ of B such that $|G(b_1, \ldots b_n)| = n$ for all $n \geq 1$. The set B_0 is finite, so by the Σ^0_0 bounding principle (which is a consequence of Σ^0_1 induction), there is a $t \in \mathbb{N}$ such that $B_0 \subset \{b_1, \ldots b_t\}$. Let $F = \{b_1, \ldots b_t\}$. Then $B_0 \subset F$ and $|G(F)| = t = |F|$.

To complete the proof we will deduce WKL_0 from the special case of (3) for $|B_0| = 1$. Let T be a $0-1$ tree with no infinite paths. We will prove that T is finite. Using the sequence notation from the reversal of Theorem 1, construct the marriage problem $M = (B, G, R)$ by letting $B = \{b_\sigma \mid \sigma \in T\}$, $G = \{g_\sigma \mid \sigma \in T\}$, and

$$R = \{(b_\sigma, g_\sigma) \mid \sigma \in T\} \cup \{(b_\sigma, g_{\sigma^\frown i}) \mid \sigma \in T \wedge \sigma^\frown i \in T\}$$

where $i \in \{0,1\}$. The tree is binary, so B is a bounded marriage problem. Define $f : B \to G$ by $f(b_\sigma) = g_\sigma$. Clearly f is a solution of M; we claim that it is the only solution. To see this, suppose f_2 is a solution distinct from f. Then $f_2(b_\sigma) = g_{\sigma^\frown j}$ for some σ and some $j \in \{0,1\}$. Fix such a σ and define $\sigma_0 = \sigma$. Given σ_n, let $\sigma_{n+1} = \tau$ where $f_2(b_{\sigma_n}) = g_\tau$. Because f is injective, an induction argument shows that σ_{n+1} must always be a proper extension of σ_n. Hence $\langle \sigma_n \mid n \in \mathbb{N} \rangle$ is the tail of an infinite path through T, yielding a contradiction. Thus f is the unique solution of M.

Using ρ to denote the empty sequence at the root of T, apply (3) to find a finite $|F|$ such that $\{b_\rho\} \subset F$ and $|G(F)| = |F|$. The unique solution f restricted to F is a solution of the restricted problem (F, G, R) and because $|G(F)| = |F|$, it must be a bijection between F and $G(F)$. Consequently, if $g_\sigma \in G(F)$ then $b_\sigma \in F$. Thus, if $b_\sigma \in F$ and $\sigma^\frown i \in T$, then $g_{\sigma^\frown i} \in G(F)$, and so $b_{\sigma^\frown i} \in F$. So $F = B$ and T is finite. □

Item (3) of the preceding theorem is a generalization of Lemma 3 of Hirst and Hughes [4]. The proof that (3) implies (1) shows that Lemma 3 of [4] is equivalent to WKL_0, answering the question in that paper about the exact strength of the lemma.

The next theorem can be viewed as an analysis of characterization theorems for bounded marriage problems with finitely many solutions. Item (2) states that if a bounded marriage problem has finitely many solutions, then there must be a set F with the three given properties. This implication is equivalent to WKL_0. RCA_0 suffices to prove the converse, that is if such an F exists, then the marriage theorem must have finitely many solutions. Similarly, WKL_0 is equivalent to the implication in item (3), and its converse is provable in RCA_0. Thus, item (2) and item (3) could be stated as biconditionals without changing the strength of the theorem.

Theorem 3. (RCA_0) *The following are equivalent:*

(1) WKL_0.

(2) *Suppose $M = (B, G, R)$ is a bounded marriage problem with k solutions $f_1, f_2, \dots f_k$. Then there is a finite set F satisfying the following three properties:*

  (P1) *The solutions of (F, G, R) are precisely the restrictions of the solutions $f_1, f_2, \dots f_k$ to F.*

(P2) *For all* $i < j \leq k$ *and* $b \in B$*, if* $f_i(b) \neq f_j(b)$ *then* $b \in F$*.*

(P3) *Let* $G_0 = \{g \mid \forall i \leq k \, \exists b \in F \, f_i(b) = g\}$*. Then there is an enumeration* $\langle b_i \rangle_{i \geq 1}$ *of* $B{-}F$ *such that for all* n*,* $|G(b_1, \ldots b_n){-}G_0| = n$*.*

(3) *Suppose* $M = (B, G, R)$ *is a a bounded marriage problem with* k *solutions* $f_1, f_2, \ldots f_k$*. Then there is a finite set* $F \subset B$ *such that* M *restricted to* F *has exactly* k *solutions, each of which extends uniquely to a solution of* M*.*

Proof. To see that (1) implies (2), assume WKL_0 and let $M = (B, G, R)$ be a bounded marriage problem with k solutions $f_1, f_2, \ldots f_k$. Our goal is to construct the finite set F. Begin by applying Theorem 1 to find $B_0 \subset B$ such that for all $i < j \leq k$ and $b \in B$, if $f_i(b) \neq f_j(b)$ then $b \in B_0$. Because M is bounded, the set $G(B_0)$ exists by recursive comprehension. Consider the finite marriage problem $M_0 = (B_0, G(B_0), R)$. List all the solutions of M_0, beginning with the restrictions of $f_1, f_2, \ldots f_k$ to B_0, so the entire list is $f_1, f_2, \ldots f_k, f_{k+1}, \ldots f_n$. For each j with $k + 1 \leq j \leq n$, the marriage problem $(B{-}B_0, G{-}\{f_j(b) \mid b \in B_0\}, R)$ has no solution, so by Theorem 2.3 of Hirst [2], there is a finite set $B_j \subset B{-}B_0$ such that $|G(B_j) - \{f_j(b) \mid b \in B_0\}| < |B_j|$. Thus, f_j cannot be extended to a solution of any restriction of M that includes $B_0 \cup B_j$ among the boys. Because we know that for each j (a code for) such a finite B_j exists, recursive comprehension suffices to prove the existence of a sequence of finite sets $\{B_j \mid k + 1 \leq j \leq n\}$ blocking extensions of $f_{k+1}, \ldots f_n$. Let $B_1 = \cup\{B_j \mid k + 1 \leq j \leq n\}$ and note that every solution of $(B_0 \cup B_1, G(B_0 \cup B_1), R)$ must match one of the solutions f_1 through f_k on the set B_0. Let $G_0 = \cap_{i \leq k}\{f_i(b) \mid b \in B_0\}$ and consider the marriage problem $M_1 = (B{-}B_0, G{-}G_0, R)$. For $1 \leq i \leq k$, f_i restricted to $B{-}B_0$ is a solution of M_1. Because f_1 through f_k agree outside B_0, these constitute a single solution of M_1. We claim this is the unique solution. By way of contradiction, suppose f is a solution of M_1 that differs from f_1 for some some fixed $b_0 \in B{-}B_0$. If there is an $i \leq k$ such that $f(b_0) \notin \{f_i(b) \mid b \in B_0\}$, then the function f' defined by $f'(b) = f_i(b)$ if $b \in B_0$ and $f'(b) = f(b)$ for $b \in B{-}B_0$ is a solution of M differing from f_1 on a boy in $B{-}B_0$, contradicting the construction of B_0. Thus $f(b_0) \in \cap_{i \leq k}\{f_i(b) \mid b \in B_0\} = G_0$, contradicting the claim that f maps $B{-}B_0$ into $G{-}G_0$. Summarizing, $M_1 = (B{-}B_0, G{-}G_0, F)$ is a bounded marriage problem with a unique solution. Using the fact that $B_1 \subset B{-}B_0$, we can apply Theorem 2 to find a finite set F_1 such that $B_1 \subset F_1 \subset B{-}B_0$ and

$|G_{M_1}(F_1)| = |F_1|$. Let $F = B_0 \cup F_1$.

We claim that F has the three properties listed in (2). Let f be any solution of $(F, G(F), R)$, the restriction of M to the boys in F. Then f is a solution of $(B_0 \cup B_1, G(B_0 \cup B_1), R)$ and so must match one of f_1 through f_k on B_0. Fix $1 \le i_0 \le k$ such that f matches f_{i_0} on B_0. Because $f_{i_0}(B_0) \supset G_0$, f must map $F_1 = F - B_0$ into $G - G_0 = G_{M_1}(F_1)$. Recall that $|G_{M_1}(F_1)| = |F_1|$, so f must be a bijection on F_1 with precisely the same range on F_1 as f_{i_0}. If f differs from f_{i_0}, then the function that matches f on F and f_{i_0} on $B - F$ is a solution of M differing from $f_1, f_2, \ldots f_k$. Thus f is the restriction of f_{i_0} to F and $(\mathsf{P1})$ holds. Recall that $B_0 \subset F$ and B_0 includes all b such that $f_i(b) \ne f_j(b)$ for $i < j \le k$. Thus $(\mathsf{P2})$ holds. Finally, for $G_0 = \{g \mid \forall i \le k\, \exists b \in F\, f_i(b) = g\}$, consider the marriage problem $M_2 = (B - F, G - G_0, R)$. Any solution of M_2 can be extended to a solution of $M_1 = (B - B_0, G - G_0, R)$, so M_2 has a unique solution. By Theorem 6 of Hirst and Hughes [4], there is an enumeration $\langle b_i \rangle_{i \ge 1}$ of $B - F$ such that for all n, $|G(b_1, \ldots b_n) - G_0| = n$, satisfying property $(\mathsf{P3})$ and completing the proof of (2) from WKL_0.

Clearly, any F satisfying (2) also satisfies (3), so we need only show that (3) implies (1). This can be done with the following modification of the reversal from Theorem 2. Given a $0-1$ tree T with no infinite paths, construct the marriage problem $M = (B, G, R)$ by letting $B = \{b_\sigma \mid \sigma \in T\}$, $G = \{g_\sigma \mid \sigma \in T\} \cup \{g\}$, and

$$R = \{(b_\sigma, g_\sigma) \mid \sigma \in T\} \cup \{(b_\sigma, g_{\sigma^\frown i}) \mid \sigma \in T \wedge \sigma^\frown i \in T\} \cup \{(b_\rho, g)\}$$

where $i \in \{0, 1\}$. The tree is binary, so B is a bounded marriage problem. Define $f_1 : B \to G$ by $f_1(b_\sigma) = g_\sigma$ for all σ, and $f_2 : B \to G$ by $f_2(b_\rho) = g$ and $f_2(b_\sigma) = g_\sigma$ for all σ other than ρ. Then f_1 and f_2 are solutions of M, and as in the proof of Theorem 2, the existence of a third solution implies the existence of an infinite path. Apply (3) to find F such that M restricted to F has exactly two solutions, each of which extends uniquely to a solution of M. Then F must contain b_ρ and, as in the proof of Theorem 2, be closed under sequence extensions. Consequently, T has finitely many nodes. $\square$

We can extend the preceding results to marriage problems without bounding functions, but the resulting statements are stronger. The next theorem is the version of Theorem 2 for this broader class of problems.

Theorem 4. (RCA_0) *The following are equivalent:*

(1) ACA_0.

(2) *Suppose* $M = (B, G, R)$ *is a marriage problem, each boy knows finitely many girls, and* M *has a unique solution. Then there is an enumeration of the boys* $\langle b_i \rangle_{i \geq 1}$ *such that for every* $n \geq 1$, *we have* $|G(b_1, \ldots b_n)| = n$.

(3) *Suppose* $M = (B, G, R)$ *is a marriage problem, each boy knows finitely many girls, and* M *has a unique solution. Then for any finite* $B_0 \subset B$, *there is a finite set* F *such that* $B_0 \subset F \subset B$ *and* $|G(F)| = |F|$.

Proof. Theorem 4 of Hirst and Hughes [4] shows that (1) is equivalent to (2). To see that (2) implies (3), assume (2) and note that we may use ACA_0. Given any marriage problem as in (3), and viewing B and G as subsets of $\mathbb{N}$, the bounding function given by $h(b) = \mu t(\forall n((b, g) \in R \to g < t))$ is arithmetically definable. ACA_0 implies WKL_0, so by Theorem 2, (3) follows. To prove that (3) implies (1), imitate the reversal from Theorem 2, letting T be a tree with no infinite paths in which each node has finitely many successors. The resulting marriage problem satisfies the hypothesis of (3), and an application of (3) shows that T is finite. This yields a proof of full König's Lemma, which is equivalent to ACA_0, as noted by Friedman [1] and presented in detail as Theorem III.7.2 of Simpson [5]. □

The preceding reversal holds in the special case where B_0 is a singleton, showing that Lemma 2 of Hirst and Hughes [4] is equivalent to ACA_0, and answering the question posed in that article about the exact strength of Lemma 2. Next, we present the version of Theorem 3 for marriage problems without bounding functions.

Theorem 5. (RCA_0) *The following are equivalent:*

(1) ACA_0.

(2) *Suppose* $M = (B, G, R)$ *is a marriage problem, each boy knows finitely many girls, and* $f_1, f_2, \ldots f_k$ *are the* k *solutions of* M. *Then there is a finite set* F *satisfying the following three properties:*

- (P1) *The solutions of* (F, G, R) *are precisely the restrictions of the solutions* $f_1, f_2, \ldots f_k$ *to* F.
- (P2) *For all* $i < j \leq k$ *and* $b \in B$, *if* $f_i(b) \neq f_j(b)$ *then* $b \in F$.
- (P3) *Let* $G_0 = \{g \mid \forall i \leq k\, \exists b \in F\, f_i(b) = g\}$. *Then there is an enumeration* $\langle b_i \rangle_{i \geq 1}$ *of* $B - F$ *such that for all* n, $|G(b_1, \ldots b_n) - G_0| = n$.

(3) *Suppose $M = (B, G, R)$ is a marriage problem, each boy knows finitely many girls, and $f_1, f_2, \ldots f_k$ are the k solutions of M. Then there is a finite set $F \subset B$ such that M restricted to F has exactly k solutions, each of which extends uniquely to a solution of M.*

Proof. To see that (1) implies (2), assume ACA_0. Given a marriage problem as in (2), ACA_0 proves the existence of a bounding function. Apply (2) from Theorem 3 to complete the proof. As in the proof of Theorem 3, (2) immediately implies (3). For the reversal, let T be a finitely splitting tree with no infinite paths and imitate the construction from the reversal of Theorem 3. Apply (3) to show that T is finite, proving full König's Lemma. □

We close by answering one more question from our earlier article. Lemma 1 of Hirst and Hughes [4] shows that RCA_0 can prove that any finite marriage problem with a unique solution must contain some boy who knows exactly one girl. Theorem 7 of that article proves the infinite version of this statement in WKL_0, leaving open the question of whether WKL_0 is actually necessary. Our final theorem shows that RCA_0 suffices, so WKL_0 is not needed.

Theorem 6. (RCA_0) *Suppose $M = \langle B, G, R\rangle$ is a marriage problem in which each boy knows only finitely many girls and M has a unique solution. Then some boy knows exactly one girl.*

Proof. Suppose $M = \langle B, G, R\rangle$ is a marriage problem with a unique solution, f. Suppose by way of contradiction that every boy knows at least two girls. Define a function $h_0 : B \to G$ by letting $h_0(b)$ be the first girl other than $f(b)$ that b knows. Formally, $h_0(b) = \mu g((b, g) \in R \wedge f(b) \neq g)$. Define $h_1 : B \to G$ by $h_1(b) = \max\{h_0(b), f(b)\}$ and let $R' = \{(b, g) \mid f(b) = g \vee h_0(b) = g\}$. Recursive comprehension proves the existence of h_0, h_1, and R'. The society $M' = (B, G, R')$ is bounded by h_1 and has f as a solution. Every boy in M' knows exactly two girls. If there is a finite set $F \subset B$ such that $|G(F)| = |F|$, then by Lemma 1 of Hirst and Hughes [4], the marriage problem $(F, G(F), R)$ has at least two distinct solutions. Call them g_1 and g_2. The functions

$$f_i(b) = \begin{cases} g_i(b) & \text{if } b \in F \\ f(b) & \text{if } b \notin F \end{cases}$$

are distinct solutions of M, contradicting the uniqueness of f. Thus for every finite $F \subset B$, $|G(F)| > |F|$.

For each finite set F, let $\mu_F = |G(F)| - |F|$. The problem M is bounded, so μ_F is computable in RCA_0. By Σ^1_0 least element principle (a consequence of Σ^1_0 induction) there is a smallest n such that there is a finite set F with $\mu_F = n$. Denote this n by μ_0. By Σ^1_0 least element principle there is a smallest k such that $\exists F(|F| = k \wedge \mu_F = \mu_0)$. (The existential quantifier ranges over integer codes for finite sets so this formula is Σ^1_0.) Denote this value by k_0 and choose a witness F_0 such that $|F_0| = k_0$ and $\mu_{F_0} = k_0$. Thus $|G(F_0)| = k_0 + \mu_0 > k_0$. The unique solution f maps F_0 injectively into $G(F_0)$. Because $|G(F_0)| = |F_0| + \mu_0$, there are exactly μ_0 girls in $G(F_0)$ that are not in the range of f restricted to F_0. Because f is unique, every girl in $G(F_0)$ is in the range of f (as a function of B), so there are μ_0 boys not in F_0 that are mapped by f into $G(F_0)$. Let F_1 denote these boys. Thus f is a bijection between $F_0 \cup F_1$ and $G(F_0)$. We may view the inverse of f as a solution to the marriage problem $M' = (G(F_0), F_0 \cup F_1, R^{-1})$ where $R^{-1} \subseteq G(F_0) \times (F_0 \cup F_1)$ such that $(g, b) \in R^{-1}$ if and only if $(b, g) \in R$, $b \in F_0 \cup F_1$ and $g \in G(F_0)$.

We claim that every girl in $G(F_0)$ knows at least two boys in $F_0 \cup F_1$. Consider two cases. First, suppose $g \in f(F_1)$. Then there is a boy $b_1 \in F_1$ that knows g and, because $g \in G(F_0)$, there is a boy $b_0 \in F_0$ that knows g. F_0 and F_1 are disjoint, so g knows at least two boys. Second, suppose $g \in f(F_0)$. By way of contradiction, suppose g knows exactly one boy, $b_0 \in F_0$. Then $|G(F_0 - \{b_0\})| \leq |G(F_0)| - 1$. We know $|F_0| = k_0$ and $|G(F_0)| = k_0 + \mu_0$ so $|G(F_0 - \{b_0\})| \leq (k_0 - 1) + \mu_0$. By the minimality of k_0, we know that $\mu_{F - \{b_0\}} > \mu_0$. Thus $|G(F_0 - \{b_0\})| = (k_0 - 1) + \mu_{F - \{b_0\}} > (k_0 - 1) + \mu_0$, yielding the desired contradiction. Summarizing, because every girl in $G(F_0)$ is either in $f(F_1)$ or in $f(F_0)$, every girl in $G(F_0)$ knows at least two boys in $F_0 \cup F_1$.

Every girl in the problem $M' = (G(F_0), F_0 \cup F_1, R^{-1})$ knows at least two boys, so by Lemma 1 of [4], M' must have at least two distinct solutions. Because these are distinct bijections between $G(F_0)$ and $F_0 \cup F_1$, we may invert them to obtain distinct bijections between $F_0 \cup F_1$ and $G(F_0)$. Call these maps f_1 and f_2. Because $f(F_0 \cup F_1) = G(F_0)$, we can patch f_1 and f_2 into f, defining

$$g_i(b) = \begin{cases} f_i(b) & \text{if } b \in F_0 \cup F_1 \\ f(b) & \text{if } b \notin F_0 \cup F_1 \end{cases}$$

for $i \in \{1, 2\}$. Then g_1 and g_2 are distinct solutions of M, contradicting the assumption that f is unique, and completing the proof. □

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