The Journal of Symbolic Logic
Volume 00, Number 0, XXX 0000

HINDMAN'S THEOREM, ULTRAFILTERS, AND REVERSE MATHEMATICS (TO APPEAR IN JSL)

JEFFRY L. HIRST
APPALACHIAN STATE UNIVERSITY

Abstract. Assuming CH, Hindman [2] showed that the existence of certain ultrafilters on the power set of the natural numbers is equivalent to Hindman's Theorem. Adapting this work to a countable setting formalized in RCA_0, this article proves the equivalence of the existence of certain ultrafilters on countable Boolean algebras and an iterated form of Hindman's Theorem, which is closely related to Milliken's Theorem. A computable restriction of Hindman's Theorem follows as a corollary.

Throughout this paper, proofs are carried out in the formal system RCA_0. For a full exposition of RCA_0 and reverse mathematics, Simpson's book [6] is the best source. For the following discussion, the salient features of RCA_0 are that it is a subsystem of second order arithmetic with induction restricted to Σ^0_1 formulas and comprehension restricted to relatively computable sets.

The language of second order arithmetic is remarkably expressive, and the following concepts are easily formalized. A *countable field of sets* is a countable sequence of subsets of $\mathbb{N}$ which is closed under intersection, union, and relative complementation. We use X^c to denote $\{x \in \mathbb{N} \mid x \notin X\}$, the complement of X relative to $\mathbb{N}$. Given a set $X \subseteq \mathbb{N}$ and an integer $n \in \mathbb{N}$, we use $X - n$ to denote the set $\{x - n \mid x \in X \wedge x \geq n\}$, the translation of X by n. A *downward translation algebra* is a countable field of sets which is closed under translation.

Given a countable collection $\{G_i \mid i \in \mathbb{N}\}$ of subsets of $\mathbb{N}$, RCA_0 suffices to prove the existence of the downward translation algebra generated by $\{G_i \mid i \in \mathbb{N}\}$, which is denoted by $\langle\{G_i \mid i \in \mathbb{N}\}\rangle$ and consists of all finite unions of finite intersections of translations of elements and complements of elements of $\{G_i \mid i \in \mathbb{N}\}$. RCA_0 can also prove that $\langle\{G_i \mid i \in \mathbb{N}\}\rangle$ is a downward translation algebra.

Note that in RCA_0, we encode $\langle\{G_i \mid i \in \mathbb{N}\}\rangle$ as a countable sequence of countable sets, and that each set in $\langle\{G_i \mid i \in \mathbb{N}\}\rangle$ may be repeated many times in the sequence. By allowing this repetition, we can organize the encoding sequence so that given the indices of any elements of the algebra we may deterministically compute indices for their unions, intersections, complements, and translations.

1991 *Mathematics Subject Classification.* 03B30, 03F35, 03D80, 05D10.
Key words and phrases. Hindman's Theorem, Milliken's Theorem, reverse mathematics, computability.

0022-4812/00/0000-0000/$00.00

Suppose that $F = \{X_i \mid i \in \mathbb{N}\}$ is a countable field of sets. In $\mathsf{RCA_0}$, we define an *ultrafilter* on F as a set $U \subseteq \mathbb{N}$ satisfying the following four properties:

- If $X_i = \emptyset$ then $i \notin U$.
- If $i, j \in U$ and $X_k = X_i \cap X_j$, then $k \in U$.
- If $i \in U$ and $X_j \supseteq X_i$, then $j \in U$.
- If $X_j = X_i^c$, then $i \in U$ or $j \in U$.

Where no confusion will arise, we will abuse notation by writing U for $\{X_i \mid i \in U\}$, so that we may write statements using the standard notation for ultrafilters. For example, consider the following definition and lemma. An ultrafilter U is an *almost downward translation invariant ultrafilter* if $\forall X \in U\ \exists x \in X\ (x \neq 0 \wedge X - x \in U)$. While the definition only requires one translating x for each set, the existence of many is shown by the following lemma.

Lemma 1. ($\mathsf{RCA_0}$) *If U is an almost downward translation invariant ultrafilter and X is an element of U, then the set $\{x \in X \mid X - x \in U\}$ is unbounded.*

Proof. Suppose by way of contradiction that x is the largest number in X such that $X - x \in U$. Since $X - x \in U$, there is a $y \in X - x$ such that $y \neq 0$ and $(X - x) - y \in U$. Since $y \in X - x$, we have $x + y \in X$. Additionally, $(X-x)-y = X-(x+y)$, so we have $x+y \in X$, $x+y > x$, and $X-(x+y) \in U$, contradicting our choice of x. ⊣

If $X \subseteq \mathbb{N}$, then the notation $\mathsf{FS}(X)$ denotes the set of all nonrepeating sums of nonempty finite subsets of X. For example, $\mathsf{FS}(\{1, 2, 5\}) = \{1, 2, 3, 5, 6, 7, 8\}$. Using this notation, it is easy to state Hindman's Theorem.

Theorem 2 (Hindman's Theorem). *If $f : \mathbb{N} \to k$ is a function mapping $\mathbb{N}$ into the natural numbers less than k, then there is an infinite set $X \subseteq \mathbb{N}$ such that f is constant on $\mathsf{FS}(X)$.*

Proof. The original non-formalized proof appears in [3]. For a proof of Hindman's theorem in the subsystem ACA_0^+, see [1]. ⊣

The set X in the statement of Theorem 2 is called an infinite homogeneous set for the partition. Given any set $G \subseteq \mathbb{N}$, we refer to the statement of Theorem 2 with $f(x)$ as the characteristic function for G as Hindman's Theorem for G.

1. Equivalence results. We begin with the central result of the section, linking an iterated form of Hindman's Theorem to the existence of almost downward translation invariant ultrafilters.

Theorem 3. ($\mathsf{RCA_0}$) *The following are equivalent:*

1. IHT *(Iterated Hindman's Theorem) If $\{G_i \mid i \in \mathbb{N}\}$ is a collection of subsets of $\mathbb{N}$, then there is an increasing sequence $\langle x_i \rangle_{i \in \mathbb{N}} \subseteq \mathbb{N}$ such that for every $j \in \mathbb{N}$, $\{x_i \mid i > j\}$ satisfies Hindman's Theorem for G_j.*
2. *Every countable downward translation algebra has an almost downward translation invariant ultrafilter.*

Proof. To prove that (2) implies (1), suppose $\{G_i \mid i \in \mathbb{N}\}$ is a collection of subsets and let U be an almost downward translation invariant ultrafilter on $\langle\{G_i \mid i \in \mathbb{N}\}\rangle$, the downward translation algebra generated by $\{G_i \mid i \in \mathbb{N}\}$. Define a sequence of (indices for) nested sets $\langle X_i \rangle_{i \in \mathbb{N}}$ and an increasing sequence of integers $\langle x_i \rangle_{i\in\mathbb{N}}$ as follows. Let X_0 be whichever of G_0 and G_0^c is in U. Let $x_0 = 0$. Suppose that x_n has been chosen and that X_n has been chosen so that $X_n \in U$ and either $X_n \subseteq G_n$ or $X_n \subseteq G_n^c$. Since U is an almost downward translation invariant ultrafilter, by Lemma 1 we can find a least $x_{n+1} \in X_n$ such that $x_{n+1} > x_n$ and $X_n - x_{n+1} \in U$. Let $\widehat{G}_{n+1}$ denote whichever of G_{n+1} and G_{n+1}^c is in U, and define X_{n+1} by $X_{n+1} = X_n \cap (X_n - x_{n+1}) \cap \widehat{G}_{n+1}$.

We will show that $\mathsf{FS}(\langle x_n \rangle_{n>t}) \subseteq X_t$. Let $x_{i_1}, \ldots, x_{i_k}$ be a finite sequence of elements of $\langle x_n \rangle_{n>t}$. Note that $i_k > 0$, so X_{i_k-1} exists, and $x_{i_k} \in X_{i_k-1}$. Treating this as the base case in an induction, we have $\sum_{m=k}^k x_{i_m} \in X_{i_k-1}$. For the induction step, suppose that $\sum_{m=j+1}^k x_{i_m} \in X_{i_{j+1}-1}$. Since $i_j \leq i_{j+1} - 1$, $\sum_{m=j+1}^k x_{i_m} \in X_{i_j}$. Since $i_j \geq 1$, by the definition of X_n, $X_{i_j} \subseteq X_{i_j-1} \cap (X_{i_j-1} - x_{i_j}) \cap \widehat{G}_{i_j}$, so $\sum_{m=j+1}^k x_{i_m} \in X_{i_j-1} - x_{i_j}$. That is, $\sum_{m=j}^k x_{i_m} \in X_{i_j-1}$, completing the induction step. By induction on quantifier-free formulas, we have shown that $\sum_{m=1}^k x_{i_m} \in X_{i_1-1}$. Since $t < i_1$, we have $t \leq i_1 - 1$, so $X_{i_1-1} \subseteq X_t$ and $\sum_{m=1}^k x_{i_m} \in X_t$. Since $x_{i_1}, \ldots, x_{i_k}$ was an arbitrary non-repeating sequence of elements of $\langle x_n \rangle_{n>t}$, this suffices to show that $\mathsf{FS}(\langle x_n \rangle_{n>t}) \subseteq X_t$.

To complete the argument, note that for each t, $X_t \subseteq G_t$ or $X_t \subseteq G_t^c$. Thus for each t, $\mathsf{FS}(\langle x_n \rangle_{n>t}) \subseteq G_t$ or $\mathsf{FS}(\langle x_n \rangle_{n>t}) \subseteq G_t^c$, so the sequence $\langle x_n \rangle_{n>0}$ satisfies the iterated version of Hindman's Theorem presented in (1).

To prove that (1) implies (2), suppose that $\{G_i \mid i \in \mathbb{N}\}$ is a countable downward translation algebra. Since $\{G_i \mid i \in \mathbb{N}\}$ is a countable collection of subsets, we may apply (1) to find an increasing sequence of integers $\langle x_i \rangle_{i\in\mathbb{N}}$ such that for all j, $\mathsf{FS}(\langle x_i \rangle_{i>j}) \subseteq G_j^*$, where either $G_j^* = G_j$ or $G_j^* = G_j^c$. We will show that $U = \{G_i^* \mid i \in \mathbb{N}\}$ is an almost downward translation invariant ultrafilter on $\{G_i \mid i \in \mathbb{N}\}$.

If $X \in \{G_i \mid i \in \mathbb{N}\}$, then for some j, $G_j^* = X$ or $G_j^* = X^c$, so either $X \in U$ or $X^c \in U$. For each j, $G_j^* \supseteq \mathsf{FS}(\langle x_i \rangle_{i>j}) \neq \emptyset$, so $\emptyset \notin U$. Suppose $X_1, X_2 \in U$. Then for some $k, j \in \mathbb{N}$, $X_1 = G_k^*$ and $X_2 = G_j^*$. Also, for some m, $G_m^* = X_1 \cap X_2$ or $G_m^* = (X_1 \cap X_2)^c$. Let $t = \max\{j, k, m\}$. then $\mathsf{FS}(\langle x_i \rangle_{i>t}) \subseteq G_m^* \cap G_k^* \cap G_j^* = G_m^* \cap X_1 \cap X_2$. Since $G_m^* \cap X_1 \cap X_2 \neq \emptyset$, $G_m^* = X_1 \cap X_2$. Thus, if $X_1, X_2 \in U$ then $X_1 \cap X_2 \in U$. Finally, if $X \in U$ and Y is an element of the downward translation algebra satisfying $X \subseteq Y$, then for some j and k, $X = G_j^*$ and $Y = G_k$. Let $t = \max\{j, k\}$ and note that $X \cap G_k^* \supseteq \mathsf{FS}(\langle x_i \rangle_{i>t}) \neq \emptyset$, so $G_k^* = Y$ and $Y \in U$. Thus U is an ultrafilter on the downward translation algebra.

To show that U is an almost downward translation invariant ultrafilter we must show that for each $X \in U$ we can find an $x \neq 0$ such that $X - x \in U$. Suppose $X \in U$, and find j so that $X = G_j^* \supseteq \mathsf{FS}(\langle x_i \rangle_{i>j})$. For every $y \in \mathsf{FS}(\langle x_i \rangle_{i>j+1})$, we have $x_{j+1} + y \in X$, so $X - x_{j+1} \supseteq \mathsf{FS}(\langle x_i \rangle_{i>j+1})$. Because $\{G_i \mid i \in \mathbb{N}\}$ is a downward translation algebra, there is a k such that $G_k = X - x_{j+1}$. If $t = \max\{k, j+1\}$, then $G_k^* \supseteq \mathsf{FS}(\langle x_i \rangle_{i>t})$, so $G_k^* \cap G_k \neq \emptyset$. Thus $G_k^* = G_k$, so $X - x_{j+1} \in U$ as desired. Summarizing, we have used (1) to find an almost downward translation invariant ultrafilter on a given downward translation algebra, completing the proof of the theorem. $\dashv$

The remainder of this section extends the equivalence of the preceding theorem to a version of Milliken's Theorem. This proof will use the functional variant of the iterated Hindman's Theorem which appears as statement (2) in the following result.

LEMMA 4. ($\mathsf{RCA_0}$) *The following are equivalent:*

(1) IHT *(See Theorem 3 for a full statement.)*
(2) *Given a collection of functions* $\{f_i \mid i \in \mathbb{N}\}$*, each defined on* $\mathbb{N}$ *with a bounded range, there is a sequence* $\langle x_i \rangle_{i\in\mathbb{N}}$ *such that for each* j*,* f_j *is constant on* $\mathsf{FS}(\{x_i \mid i > j\})$*.*

PROOF. To prove that (2) implies (1), just let f_j be the characteristic function for G_j. To prove the converse, if $f_j : \mathbb{N} \to k$, for each $i < k$ define $G_{j_i} = \{x \mid f_j(x) = i\}$, and add the sets $G_{j_0}, \dots, G_{j_{k-1}}$ to the collection for (1). Note that if $\mathsf{FS}(X)$ is homogeneous for $G_{j_0}, \dots, G_{j_{k-1}}$, then f_j is constant on $\mathsf{FS}(X)$. ⊣

Now we are ready to introduce Milliken's Theorem using the following terminology. Given two finite subsets A and B of $\mathbb{N}$, we write $A < B$ if $\max A < \min B$. As a convenient shorthand, we will write $\sum A$ for $\sum_{x\in A} x$. Let $\mathsf{FS}n(X)$ denote the set of all n element sets of the form $(\sum A_0, \dots, \sum A_{n-1})$, where for each $i < j < n$, we have $A_i \subseteq X$, $A_j \subseteq X$, and $A_i < A_j$.

In showing that Milliken's Theorem is equivalent to the iterated Hindman's Theorem, it is useful to have the generalized version which appears as statement (2) in the following lemma. The original proof of Milliken's theorem appears in [5].

LEMMA 5. ($\mathsf{RCA_0}$) *The following are equivalent:*

(1) $\mathsf{MT}(n)$ *(Milliken's theorem for* n*-tuples.) If* $f : [\mathbb{N}]^n \to k$ *then there is an increasing sequence* $X = \langle x_i \rangle_{i\in\mathbb{N}}$ *such that* f *is constant on* $\mathsf{FS}n(X)$*.*
(2) *Suppose* $S \subseteq \mathbb{N}$ *and* $f : [\mathbb{N}]^n \to k$*. Then there is an increasing sequence* $\langle A_i \rangle_{i\in\mathbb{N}}$ *of finite subsets of* S *such that for* $A = \{\sum A_i \mid i \in \mathbb{N}\}$*, the function* f *is constant on* $\mathsf{FS}n(A)$*.*

PROOF. The statement of (1) is a special case of (2) where S is set equal to $\mathbb{N}$. Consequently, we need only prove that (1) implies (2).

Fix S, f, n, and k as in the statement of (2). Given any infinite sequence of integers and any finite collection of moduli, we can find a finite subsequence which sums to a value which is congruent to 0 for each of the given moduli. Consequently, without loss of generality, we may assume that $S = \langle s_i \rangle_{i\in\mathbb{N}}$ satisfies the congruence conditions

$$i < j \text{ implies } \forall t \le s_i (s_j \equiv 0 \mod t).$$

Define $h : \mathsf{FS}(S) \to \mathbb{N}$ by setting $h(\sum_{i\in F} s_i) = \sum_{i\in F} 2^i$ for each finite subset F of $\mathbb{N}$. Because of the congruence condition on S, h is bijective. Thus h^{-1} is well-defined, and we may define $g : [\mathbb{N}]^n \to k$ by

$$g(m_1, \dots, m_n) = f(h^{-1}(m_1), \dots, h^{-1}(m_n)).$$

Apply (1) to g to find a homogenous set for g and denote it by $\langle b_i \rangle_{i\in\mathbb{N}}$. As with S, we may assume that for $i < j$, we have $\forall t \le b_i + 1 (b_j \equiv 0 \mod t)$. For each

j, find the unique F such that $b_j = \sum_{i \in F} 2^i$, and let $A_j = \{s_i \mid i \in F\}$. The congruence condition on $\langle b_i \rangle_{i \in \mathbb{N}}$ insures that the sequence $\langle A_j \rangle_{j \in \mathbb{N}}$ is increasing. The choice of g guarantees that if $A = \{\sum A_j \mid j \in \mathbb{N}\}$, then f is constant on $\mathsf{FS}n(A)$, as desired.

As an alternative proof, one could utilize a version of Milliken's Theorem for unions, emulating the version of Hindman's Theorem for unions presented in [1]. An argument of this sort appears in Chapter 7 of [4]. $\dashv$

We conclude the section with the proof that Milliken's Theorem is equivalent to the iterated version of Hindman's Theorem appearing in Theorem 3. Consequently, for natural numbers $n \geq 3$, $\mathsf{MT}(n)$ can be appended to the list of equivalent statements in Theorem 3.

THEOREM 6. *For each standard natural number $n \geq 3$, RCA_0 can prove that the following are equivalent:*

(1) IHT *(Iterated Hindman's Theorem.) See Theorem 3 for a full statement.*
(2) $\mathsf{MT}(n)$ *(Milliken's Theorem for n-tuples.) See Lemma 5 for a full statement.*

PROOF. To prove that (2) implies (1), we assume RCA_0 and Milliken's Theorem for triples. Let $\{G_i \mid i \in \mathbb{N}\}$ be a collection of subsets of $\mathbb{N}$ as in the statement of (1). For $k < m < n$, let $f(k, m, n) = 1$ if for every $j \leq k$, $m \in G_j$ if and only if $n \in G_j$. Let $f(k, m, n) = 0$ otherwise. Apply (2) to f to find a homogeneous set $X = \langle x_i \rangle_{i \in \mathbb{N}}$ for f. A pigeonhole argument shows that $f(\mathsf{FS}3(X)) = 1$. Since $j \leq x_j$ for all j, $\{x_i \mid i > j\}$ satisfies Hindman's theorem for G_j.

To prove that (1) implies (2), we begin by showing that RCA_0 plus (1) proves that $\mathsf{MT}(n)$ implies $\mathsf{MT}(n{+}1)$. Assume RCA_0, IHT, and $\mathsf{MT}(n)$ and fix a function $f : [\mathbb{N}]^{n+1} \to k$. Let $\langle \vec{y_i} \rangle_{i \in \mathbb{N}}$ be an enumeration of $[\mathbb{N}]^n$ and define a sequence of functions $f_j : \mathbb{N} \to k$ by setting $f_j(m) = f(\vec{y_j}, m)$. Using IHT, apply statement (2) of Lemma 4 to $\{f_j \mid j \in \mathbb{N}\}$ to find a homogeneous set $X = \langle x_i \rangle_{i \in \mathbb{N}}$ such that for each j, f_j is constant on $\mathsf{FS}(\{x_k \mid k > j\}$. Define $g : \mathsf{FS}n(X) \to k$ by setting $g(y_1, y_2, \ldots, y_n) = f(y_1, y_2, \ldots y_n, x)$ where x is the least element of X greater than y_n. Applying $\mathsf{MT}(n)$ in the guise of statement (2) from Lemma 5 to the function g yields a sequence $Y = \langle y_i \rangle_{i \in \mathbb{N}}$ such that f is constant on $\mathsf{FS}(n{+}1)(Y)$, as desired.

To complete the proof that (1) implies (2), we point out that IHT implies Hindman's Theorem, which is identical to Milliken's Theorem for singletons. Using this as a base case and the preceding paragraph as an induction step, we conclude that for each standard natural number k, RCA_0 proves that IHT implies $\mathsf{MT}(k)$. $\dashv$

Because the induction in the preceding proof is external rather than within the formal system, the proof does not show that $\mathsf{RCA}_0 + \mathsf{IHT}$ implies $\forall n\ \mathsf{MT}(n)$.

2. A computable restriction. The main result of this section is that RCA_0 with some additional induction proves Hindman's Theorem for those partitions G such that $\langle G \rangle$ does not contain all the singleton sets. The notation $\langle G \rangle$ is an abbreviation for $\langle \{G\} \rangle$, the downward translation algebra generated by a single set G. We begin with a restriction of one of the implications of Theorem 3.

THEOREM 7. (RCA_0) *Fix $G \subseteq N$. If there is an almost downward translation invariant ultrafilter on the downward translation algebra $\langle G\rangle$, then Hindman's Theorem holds for G.*

PROOF. Assume RCA_0, suppose that $G \subseteq N$, and let U be an almost downward translation invariant ultrafilter on $\langle G\rangle$. Define a sequence (of indices) of nested sets $\langle X_i\rangle$ and an increasing sequence of integers $\langle x_i\rangle$ as follows. Set X_0 to be whichever of G and G^c is in U, and let $x_0 = 0$. Suppose that $X_n \in U$ and x_n have been chosen. Since U is an almost downward translation invariant ultrafilter, applying Lemma 1 we can find a least $x_{n+1} \in X_n$ such that $x_{n+1} > x_n$ and $X_n - x_{n+1} \in U$. Let $X_{n+1} = X_n \cap (X_n - x_{n+1})$. Emulating the induction argument from the proof that (2) implies (1) in Theorem 3, show that $\mathsf{FS}(\langle x_n\rangle_{n>0}) \subseteq X_0$, thereby proving that Hindman's Theorem holds for G. ⊣

LEMMA 8. (RCA_0) *If the downward translation algebra $\langle G\rangle$ contains no singletons, then Hindman's Theorem holds for G.*

PROOF. Suppose $\langle G\rangle$ contains no singletons. Let U be the principal ultrafilter generated by 0, so $U = \{X \in \langle G\rangle \mid 0 \in X\}$. Pick $X \in U$. Since X is not a singleton, there is an $x \neq 0$ such that $x \in X$. Since $x \in X$, we have $0 \in X - x$, so $X - x \in U$. Thus U is an almost downward translation invariant ultrafilter. By Theorem 7, Hindman's theorem holds for G. ⊣

The preceding argument may be extended to a broader class of algebras, but the modifications use the induction scheme on Σ^0_2 formulas, which is denoted by $\Sigma^0_2\mathsf{-IND}$.

LEMMA 9. ($\mathsf{RCA}_0 + \Sigma^0_2\mathsf{-IND}$) *If the downward translation algebra $\langle G\rangle$ contains finitely many singletons, then Hindman's Theorem holds for G.*

PROOF. Suppose that $\langle G\rangle$ contains finitely many singletons. We will show that there is a set H differing from G only on a finite initial interval, such that $\langle H\rangle$ contains no singletons.

If $\langle G\rangle$ contains no singletons, let $H = G$. Otherwise, let $\{m\}$ be the largest singleton in $\langle G\rangle$. Let $G_0, \dots G_{2^{m+1}-1}$ be an enumeration of all subsets of $\mathbb{N}$ differing from G only at or below m. Note that by applying translations and Boolean operations to G and $\{m\}$, we may construct each G_i. Thus for each i, $\langle G_i\rangle \subseteq \langle G\rangle$ and in particular, the singletons of each $\langle G_i\rangle$ are a subset of the singletons of $\langle G\rangle$. Furthermore, the assertion "$\{j\}$ is an element of $\langle G_i\rangle$" holds if and only if there is a finite collection of translations of G_i and G_i^c such that for every n, n is in the intersection of the translations if and only if $n = j$. This assertion is expressible by a Σ^0_2 formula. By bounded Σ^0_2 comprehension (which is provable in RCA_0 from Σ^0_2 induction [6]), there is a set X such that for every $i < 2^{m+1} - 1$ and $j \leq m$, $(i, j) \in X$ if and only if $\{j\} \in G_i$. If there is an i such that G_i contains no singletons, pick the first such i and let $H = G_i$. Otherwise use X to define the set $Y = \{m_i \mid i < 2^{m+1} - 1\}$, the collection containing the maximum singleton from each G_i. Let p be the minimum of Y, and let H be the first G_i for which p is the largest singleton.

We will now show that $\langle H\rangle$ contains no singletons. If $\langle H\rangle$ contains a singleton, then it contains $\{p\}$ as constructed above. Since $\langle H\rangle = \langle H^c\rangle$ we may relabel as needed to insure $p \in H$.

Because $\langle H\rangle$ is a boolean algebra generated by downward translations of H and H^c, $\{p\}$ is expressible as a union of intersections of translations of H and H^c. Furthermore, because $\{p\}$ is a singleton, we may discard all but the first nonempty element of the union, and write $\{p\}$ as an intersection of translations of H and H^c. By way of contradiction, suppose $\{p\}$ is expressed as the intersection of a collection of such translations, and H itself is not in the collection. Since $p \notin H^c$, the collection must consist entirely of non-zero translations of H and H^c. Adding the value of the smallest translation to each of these sets yields a collection of elements of $\langle H\rangle$ whose intersection is a singleton which is greater than p, contradicting the fact that $\{p\}$ is the largest singleton in $\langle H\rangle$. Thus, H must be included in any intersection defining $\{p\}$, and we may write

$$\{p\} = H \cap \bigcap_{k\in F} (H^{d_k} - k)$$

where F is a finite set of positive natural numbers and d_k indicates whether or not to take the complement of H. Note that $\cap_{k\in F}(H^{d_k} - k) \subseteq H^c \cup \{p\}$.

Now consider $\langle H'\rangle$ where $H' = H - \{p\}$. Suppose by way of contradiction that $\{p\} \in \langle H'\rangle$. As argued above, we must be able to express $\{p\}$ as the intersection of a collection of translations of H' and H'^c. Since $p \notin H'$, we know H' is not in the collection. If H'^c is not in the collection, then $\langle H'\rangle$ contains a singleton larger than p. However, $H' = H \cap \{p\}^c \in \langle H\rangle$, so $\langle H'\rangle \subseteq \langle H\rangle$, implying that $\langle H'\rangle$ can contain no singletons larger than p. Thus any intersection defining $\{p\}$ in $\langle H'\rangle$ includes H'^c, and we may write

$$\{p\} = H'^c \cap \bigcap_{k\in W} (H'^{d_k} - k)$$

where W is a finite collection of positive integers. Note that $\cap_{k\in W}(H'^{d_k} - k) \subseteq (H'^c)^c \cup \{p\} = H$.

Combining the preceding two centered equations, we have

$$\{p\} \subseteq \bigcap_{k\in F}(H^{d_k} - k) \cap \bigcap_{k\in W}(H'^{d_k} - k) \subseteq (H^c \cup \{p\}) \cap H = \{p\}.$$

Since $H' \in \langle G\rangle$, this shows that $\{p\}$ is expressible as an intersection consisting entirely of nonzero translations of H and H^c, yielding a contradiction. Thus, $\{p\} \notin \langle H'\rangle$. Since $\langle H'\rangle$ is closed under downward translation, this shows that any singletons in $\langle H'\rangle$ must be strictly less than p. However, H' differs from G only at or below $m \geq p$, so $\langle H'\rangle$ must contain a singleton greater than or equal to p, yielding a final contradiction and proving that $\langle H\rangle$ contains no singletons.

We have shown that H differs from G only at or below m, and H has no singletons. By Lemma 8, Hindman's Theorem holds for H. Given an infinite homogeneous set Z for H, the set $\{n \in Z \mid n > m\}$ is an infinite homogeneous set for G. Thus Hindman's Theorem holds for G. ⊣

THEOREM 10. ($\mathsf{RCA}_0 + \Sigma^0_2\text{–}\mathsf{IND}$) *If the downward translation algebra $\langle G\rangle$ doesn't contain all the singletons, then Hindman's Theorem holds for G.*

PROOF. By closure under downward translation, if $\langle G\rangle$ contains a singleton $\{p\}$, it contains all singletons $\{n\}$ such that $n < p$. Thus, if $\langle G\rangle$ doesn't contain all the singletons, it must contain at most finitely many singletons. By Lemma 9, Hindman's Theorem holds for G. ⊣

Corollary 11. *If G is computable and $\langle G\rangle$ doesn't contain all the singletons, then there is a computable set satisfying Hindman's Theorem for G.*

Proof. The standard natural numbers and the computable sets form a model for $\mathsf{RCA}_0 + \Sigma^0_2 - \mathsf{IND}$. If G is in this model and $\langle G\rangle$ doesn't contain all the singletons, then by Theorem 10, a homogeneous set for G must lie in the model. Since all the sets in the model are computable, the homogeneous set is computable. ⊣

The contrapositive of the preceding statement is particularly interesting in that it draws a purely algebraic conclusion from a computability theoretic hypothesis. Here is the contrapositive:

Corollary 12. *If G is computable and no computable set satisfies Hindman's Theorem for G, then $\langle G\rangle$ must include all the singletons.*

A partition satisfying the hypothesis of Corollary 12 is constructed in Theorem 2.2 of [1].

REFERENCES

[1] Andreas R. Blass, Jeffry L. Hirst, and Stephen G. Simpson, *Logical analysis of some theorems of combinatorics and topological dynamics*, ***Logic and combinatorics (Arcata, Calif., 1985)***, Contemp. Math., vol. 65, Amer. Math. Soc., Providence, RI, 1987, pp. 125–156.

[2] Neil Hindman, *The existence of certain ultra-filters on N and a conjecture of Graham and Rothschild*, ***Proc. Amer. Math. Soc.***, vol. 36 (1972), pp. 341–346.

[3] ———, *Finite sums from sequences within cells of a partition of N*, ***J. Combinatorial Theory Ser. A***, vol. 17 (1974), pp. 1–11.

[4] Jeffry L. Hirst, *Combinatorics in subsystems of second order arithmetic*, ***Ph.D. thesis***, The Pennsylvania State University, 1987, pp. vii+146 pages.

[5] Keith R. Milliken, *Ramsey's theorem with sums or unions*, ***J. Combinatorial Theory Ser. A***, vol. 18 (1975), pp. 276–290.

[6] Stephen G. Simpson, ***Subsystems of second order arithmetic***, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1999.

DEPARTMENT OF MATHEMATICAL SCIENCES
APPALACHIAN STATE UNIVERSITY
BOONE, NC 28608
USA
E-mail: jlh@math.appstate.edu
URL: www.mathsci.appstate.edu/~jlh