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ABSTRACT

Many interesting combinatorial theorems can be expressed in the language of $\mathbf{Z}_2$, formal second order arithmetic. Unlike formulas of first order arithmetic, formulas of second order arithmetic can refer to sets of integers. This thesis analyzes formalizations of many theorems of countable combinatorics, determining which set existence axioms are necessary in their proofs.

The framework of axioms used here consists primarily of the subsystems $\mathbf{RCA}_0$, $\mathbf{WKL}_0$, and $\mathbf{ACA}_0$. Much of the pioneering work done in these subsystems is due to Friedman and Simpson. In many cases it is possible to show that a theorem is equivalent to a set comprehension axiom over the weak base system $\mathbf{RCA}_0$. Results of this sort, called reverse mathematics, leave no doubt as to what set existence axioms are necessary in a proof. A surprising number of theorems of countable combinatorics yield results of reverse mathematics.

The combinatorial theorems analyzed include various infinite marriage theorems and related results concerning infinite graphs and partial orders. Several results of infinite Ramsey theory, including Hindman's theorem and Milliken's theorem, are also considered. In some cases, independence results are proven, using model theoretic techniques developed in the thesis.

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CHAPTER 1

SECOND ORDER ARITHMETIC

This thesis extends the work of Simpson and Friedman in subsystems of second order arithmetic. The primary goal here is to determine what axioms are necessary to prove various statements of combinatorics. Combinatorics is interpreted in a broad sense, including transversal theory, general graph theory, and Ramsey theory. Simpson's program of reverse mathematics includes the analysis of many theorems of ordinary mathematics within formal second order arithmetic. Ordinary mathematics is the term used by Simpson to describe non-set-theoretic mathematics, including geometry, calculus, differential equations, countable algebra, real and complex analysis, and certain topics in topology and functional analysis. Ordinary mathematics does not include abstract set theory or general topology. The combinatorial theorems presented here certainly fall well within the heading of ordinary mathematics.

Despite the emphasis placed above on ordinary mathematics and formal arithmetic, the methodology of this thesis is not restricted to these areas. For example, the model theory developed in Chapter 5 is neither purely syntactic nor ordinary. However, the set theoretic techniques are used to reveal aspects of the relative strength of combinatorial statements. Although such results are not theorems of reverse mathematics, they do contribute to the program of reverse mathematics.

1.1. $\mathbf{Z}_2$

The system $\mathbf{Z}_2$, also known as Π_1^∞–$\mathbf{CA}_0$, is a formalization of second order arithmetic. $\mathbf{Z}_2$ is a two typed first order theory with number variables and set variables. Number variables are denoted by lower case letters like x , y , and z , while set variables are assigned upper case letters like $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$. The language of $\mathbf{Z}_2$ contains quantifiers for both number and set variables. The intended domain of the number quantifiers $\forall n$ and $\exists n$ is the set of natural numbers, $\mathbf{N}$. The set quantifiers $\forall\mathbf{X}$ and $\exists\mathbf{X}$ are intended to range over the subsets of $\mathbf{N}$. Symbols are included for the constants 0 and 1, and also for the binary operations $'$ (successor), $+$ (addition), and $*$ (multiplication). Numerical terms are built up as usual from combinations of number variables, constants and the operations.

The language of $\mathbf{Z}_2$ also includes the relation symbols $=$ and $\in$. Atomic formulas are of the form $t_1 = t_2$ and $t_1 \in \mathbf{X}$ where t_1 and t_2 are numerical terms. Formulas in the language are constructed from atomic formulas using the usual propositional connectives, number quantifiers, and set quantifiers.

Formulas may be classified according to the following scheme. A formula with no quantifiers is called a Σ_0^0 (or Π_0^0) formula. A Σ_n^0 formula is of the form $\exists n Q\, \theta$ where θ is Σ_0^0 and $\exists n Q$ represents a sequence starting with an existential quantifier and containing n alternating quantifier blocks. The class of Π_n^0 formulas is defined similarly. The superscript 0 indicates the possible presence of set parameters. A formula is arithmetical if it contains no set quantifiers. The above scheme can be extended to classes such as Σ_n^1 and Π_n^1 of formulas prefixed by alternating blocks of set quantifiers.

The axioms of $\mathbf{Z}_2$ consist of twelve basic axioms plus a comprehension scheme. The basic axioms consist of the following ordered semi-ring axioms (B1-B11), and the induction axiom (B12).

B1: $n' \neq 0$.

B2: $n' = m' \rightarrow n = m$.

B3: $n + 1 = n'$.

B4: $n + 0 = n$.

B5: $n + (m') = (n + m)'$.

B6: $n * 0 = 0$.

B7: $n * (m') = (n * m) + n$.

B8: $n < m \leftrightarrow \exists r\, (r \neq 0 \wedge n + r = m)$.

B9: $n = m \rightarrow m = n$.

B10: $(k = m \wedge m = n) \rightarrow k = n$.

B11: $n = n$.

B12: $(0 \in \mathbf{X} \wedge \forall n\, (n \in \mathbf{X} \rightarrow n + 1 \in \mathbf{X})) \rightarrow \forall n\, (n \in \mathbf{X})$

The comprehension scheme for $\mathbf{Z}_2$ consists of formulas of the form

$$\exists \mathbf{X} \forall n\, (\phi(n) \leftrightarrow n \in \mathbf{X})$$

where $\phi(n)$ is any formula in the language of $\mathbf{Z}_2$ in which $\mathbf{X}$ does not occur free.

A more formal development of the material in this section can be found in [50].

1.2. Subsystems of $\mathbf{Z}_2$

Many of the proofs in this thesis are carried out in weak subsystems of $\mathbf{Z}_2$. The subsystems of interest here are called $\mathbf{RCA}_0$, $\mathbf{WKL}_0$, and $\mathbf{ACA}_0$. Each of these subsystems is defined below, and an indication of its strength is given.

The axioms of the subsystem $\mathbf{RCA}_0$ consist of the basic axioms (B1-B12), the Σ_1^0 induction scheme, and the recursive comprehension scheme. The Σ_1^0 induction scheme, denoted $\mathbf{I}\Sigma_1^0$, consists of formulas of the form

$$(\phi(0)\wedge\forall n\,(\phi(n)\rightarrow\phi(n+1)))\rightarrow\forall n\,(\phi(n))$$

where $\phi(n)$ is a Σ_1^0 formula, possibly with set parameters. The recursive comprehension scheme consists of formulas of the form

$$\forall n\,(\phi(n)\leftrightarrow\psi(n))\rightarrow\exists\mathbf{X}\forall n\,(n\in\mathbf{X}\leftrightarrow\phi(n))$$

where $\phi\in\Sigma_1^0$, $\psi\in\Pi_1^0$, and $\mathbf{X}$ occurs free in neither ϕ nor ψ. Many notions of ordinary mathematics are expressible via codes within $\mathbf{RCA}_0$. The pairing function $(x,y)_p=\frac{1}{2}(x+y)(x+y+1)+x$ can be used to code a function $f:\mathbf{N}\rightarrow\mathbf{N}$ as a set of integer codes for pairs. Exponentiation can be used to code finite sequences of integers and finite sets of integers as single integers. Sufficient amounts of elementary number theory can be proved in $\mathbf{RCA}_0$ to insure that such codes are well behaved. These codes can be used to create codes for more complicated structures. For instance, codes representing countable algebraic structures, countable sequences of reals, and continuous functions on the reals can be devised [50]. Some basic properties of these structures can be proved in $\mathbf{RCA}_0$, but many theorems require stronger axiom systems. Because of this, $\mathbf{RCA_0}$ serves ideally as a weak base system for the program of reverse mathematics described in the next section.

The next stronger subsystem is called $\mathbf{WKL_0}$. The axioms of $\mathbf{WKL_0}$ consist of the axioms of $\mathbf{RCA_0}$, together with an axiom called Weak König's Lemma. Weak König's Lemma asserts that every infinite 0–1 tree contains an infinite path. A 0–1 tree is a subset of $\mathbf{Seq_2}$, the set of all finite sequences of zeros and ones, which is closed under initial segments. An infinite path for a 0–1 tree $\mathbf{T}$ is a function, $f : \mathbf{N} \rightarrow 2$, such that for each n the sequence given by f restricted to n is an element of $\mathbf{T}$. The subsystem $\mathbf{WKL_0}$ is sufficient to develop a reasonable theory of continuous functions (see [8],[48],[49],[50]).

The subsystem $\mathbf{ACA_0}$ is strong enough to prove a large portion of the theorems of ordinary mathematics. For example, a good theory of convergence can be developed in $\mathbf{ACA_0}$ [50]. The strength of $\mathbf{ACA_0}$ can be nicely characterized in two ways. Considering only purely first order formulas, the theorems of $\mathbf{ACA_0}$ are exactly those of first order Peano Arithmetic, $\mathbf{PA}$. The full theory of $\mathbf{ACA_0}$ isolates exactly that portion of mathematical practice called "predicative analysis" by Weyl [51]. The axioms of $\mathbf{ACA_0}$ are those of $\mathbf{RCA_0}$ plus the arithmetical comprehension scheme. This comprehension scheme consists of formulas of the form

$$\exists X \forall n \, (n \in \mathbf{X} \leftrightarrow \phi(n))$$

where $\phi(n)$ is an arithmetical formula in which $\mathbf{X}$ does not occur. $\mathbf{ACA_0}$ is strictly stronger than $\mathbf{WKL_0}$ [8].

The subsystem $\mathbf{ACA_0^+}$ is referred to in Chapter 7. This system extends the axioms of $\mathbf{ACA_0}$ by requiring that the ω-jump of every infinite set exists. Although not used here, the subsystems $\mathbf{ATR}_0$ and Π^1_1–$\mathbf{CA}_0$ are also commonly used. The weaker of these systems, $\mathbf{ATR}_0$ is much stronger than $\mathbf{ACA}_0^+$. Information on these subsystems may be found in [3], [8], [9], [48], and [49].

1.3. Reverse Mathematics

Many of the results in this thesis are contributions to the program of reverse mathematics. This program, set forth and advanced by Simpson ([48], [49], [50]), was inspired by the work of Friedman [8]. The goal of the program is to determine very precisely the proof theoretic strength of statements of ordinary mathematics. The process is straightforward. First, one proves a theorem **T** in a supersystem, S, of $\mathbf{RCA}_0$. Then one proves the axioms of **S** within the system consisting of $\mathbf{RCA}_0$ and **T**. This second step, called a reversal, is possible only when the system **S** is the weakest in which **T** can be proved. In this way, reverse mathematics quickly isolates exactly those set existence axioms necessary in the proof of theorems of ordinary mathematics. We now state five examples of reverse mathematics used extensively throughout this thesis. Proofs of these results may be found in [50].

Theorem 1.1: ($\mathbf{RCA}_0$) The following are equivalent:

i) $\mathbf{WKL}_0$.

ii) If **T** is a tree and $h:\mathbf{N}\rightarrow\mathbf{N}$ is a function such that for every $\tau\in\mathbf{T}$

$$\forall n < \mathrm{lh}(\tau)(\tau(n) < h(n)),$$

then there is an infinite path for **T**. (Here $\mathrm{lh}(\tau)$ denotes the length of τ and $\tau(n)$ denotes the n^{th} element of τ.)

Theorem 1.2: ($\mathbf{RCA_0}$) The following are equivalent:

i) $\mathbf{WKL_0}$.

ii) If $f:\mathbf{N}\to\mathbf{N}$ and $g:\mathbf{N}\to\mathbf{N}$ are injections such that for all $j,k\in\mathbf{N}$ $g(j)\neq f(k)$, then there is a set $\mathbf{X}$ such that

$$\forall j\,\forall n\,((f(j)=n\to n\in\mathbf{X})\wedge(g(j)=n\to n\notin\mathbf{X}))\ .$$

Theorem 1.3: ($\mathbf{RCA_0}$) The following are equivalent:

i) $\mathbf{ACA_0}$.

ii) (König's Lemma) If $\mathbf{T}$ is a finitely branching tree, that is,

$$\forall n\,\exists k\,((\sigma\in\mathbf{T}\wedge\mathrm{lh}(\sigma)=n)\to\sigma(n-1)<k)\ ,$$

then there is an infinite path for $\mathbf{T}$.

Theorem 1.4: ($\mathbf{RCA_0}$) The following are equivalent:

i) $\mathbf{ACA_0}$.

ii) If $f:\mathbf{N}\to\mathbf{N}$ is an injection, then the set $\mathbf{Ran}(f)=\{y\in\mathbf{N}:\exists x f(x)=y\}$ exists.

Theorem 1.5: ($\mathbf{RCA_0}$) The following are equivalent:

i) $\mathbf{ACA_0}$.

ii) Ramsey's theorem for triples and two colors. (This is $\mathbf{RT}(3,2)$ in the notation of Chapter 6.)

iii) Ramsey's theorem for n-tuples, for any fixed $n\in\omega$.

1.4. ω-models

Another way of comparing subsystems of $\mathbf{Z}_2$ is to examine their ω-models. By an ω-model, we mean a model in which the integer domain is ω and the set domain is some subset of the power set of ω. The ω-models of subsystems of $\mathbf{Z}_2$ can be used to determine the recursion theoretic content of theorems of ordinary mathematics.

The minimal ω-model of $\mathbf{RCA}_0$ is $<\omega,\mathbf{REC}>$, where **REC** is the class of recursive sets. In general, the set domain of an ω-model of $\mathbf{RCA}_0$ is a Turing ideal. A Turing ideal is a subset of the power set of ω closed under relative recursiveness and join.

The set domains of ω-models of $\mathbf{WKL}_0$ are Scott systems. These classes of sets have been extensively studied in the recursion theory literature [45]. By applying the Shoenfield-Kreisel low basis theorem [46], one can easily prove the following theorem.

Theorem 1.6: There is an ω-model of $\mathbf{WKL}_0$ in which every set is of low degree, i.e. for each set $\mathbf{X}$ in the model, if $\mathbf{a}=\mathbf{deg}(\mathbf{X})$, then $\mathbf{a}' \leq 0'$.

The set domains of ω-models of $\mathbf{ACA}_0$ are called jump ideals. A jump ideal is a Turing ideal closed under the jump operation. Thus every ω-model of $\mathbf{ACA}_0$ contains every finite jump of 0.

1.5. Overview of the Thesis

Chapters 2, 3, and 4 contain many theorems of reverse mathematics. Basic versions of Hall's marriage theorem are treated in Chapter 2. Chapter 3 contains variations and applications of the material in Chapter 2. Several theorems concerning graphs, chromatic numbers, and partial orders are presented there. Chapter 4 analyzes countable Boolean rings within subsystems of $\mathbf{Z}_2$. The material in Chapter 4 is independent of that in Chapters 2 and 3.

Chapter 5 develops model theory for the formal subsystems. Notions of ultrapowers and clones are introduced. These constructions are used in the independence results of Chapter 6 and the model theoretic results of Chapter 7.

Ramsey's theorem for singletons and pairs is attacked in Chapter 6. Several independence results are presented, and a program for solving the 2-3 problem is outlined. The chapter concludes with a theorem of reverse mathematics concerning min-homogeneous sets.

Hindman's theorem and Milliken's theorem are the topic of Chapter 7. An algebraic version of Hindman's theorem based on the Boolean rings of Chapter 4 is presented. Then the model theoretic techniques of Chapter 5 are used to prove a finite combinatorial theorem. The ultrafilters in this proof are related to Milliken's theorem in the final section.

Throughout the thesis, notation is defined as it is introduced. Much of the notation for common sets of codes in $\mathbf{Z}_2$ is that used by Simpson [49], [50]. The notation used in Chapter 5 for first order models is similar to that used by Paris and Kirby [25], [26], [36].

CHAPTER 2

THE MARRIAGE THEOREM

This chapter consists of two examples of reverse mathematics. We will consider three versions of the marriage theorem and their provability in the systems of second order arithmetic outlined in Chapter 1.

A marriage problem consists of two sets, **B** and **G**, and a binary relation, **R**, such that $\mathbf{R} \subseteq \mathbf{B} \times \mathbf{G}$. In anthropomorphic terms, **B** is a set of boys, **G** is a set of girls, and $(x, y) \in \mathbf{R}$ means boy x knows girl y. A solution to the marriage problem given by **R** is a one to one function $f \subseteq \mathbf{R}$ mapping **B** into **G**. Thus $f(x) = y$ means that boy x marries girl y. Polygamy is disallowed, and every boy must have a wife. However, a solution does not guarantee that every girl will have a husband.

Certainly, not every marriage problem has a solution. The marriage theorems state necessary and sufficient conditions for marriage problems to have solutions. A condition common to all such theorems is condition **H**. We say a marriage problem satisfies condition **H** if every subset of n boys knows at least n girls. Since condition **H** is clearly necessary for a solution, only its sufficiency remains to be proved. With this terminology, we are prepared to examine some particular marriage theorems.

2.1. The Finite Marriage Theorem and $\mathbf{RCA}_0$

In this section we will consider a version of the marriage theorem which is provable in $\mathbf{RCA}_0$. We will call a marriage problem **R** finite if **B**, the set of boys, is finite. The set of girls, **G**, and hence the relation **R**, need not be finite. Philip Hall

[14] proved that condition **H** is sufficient to ensure that a finite marriage problem has a solution. The following theorem states that this proof can be carried out in $\mathbf{RCA}_0$.

Theorem 2.1 ($\mathbf{RCA}_0$) Any finite marriage problem satisfying condition **H** has a solution.

Proof: Let **R** be a finite marriage problem satisfying condition **H**. By applying $I\Sigma_1^0$ we can find a finite subset of the relation **R** satisfying condition **H** and including each boy from the original relation. Thus, we may assume that **R** is a finite relation, that is, each boy knows only finitely many girls. By $I\Sigma_0^0$, there is a subset $\mathbf{S}\subseteq\mathbf{R}$ such that **S** includes all the boys for **R**, satisfies condition **H**, and no proper subset of **S** meets these requirements.

We claim that **S** is actually a solution to the marriage problem. Since **S** satisfies condition **H**, it suffices to show that **S** contains a unique pairing for each boy. Suppose not. Fix b such that $(b,g_1),(b,g_2)\in\mathbf{S}$ where $g_1\neq g_2$. By the minimality of **S**, $\mathbf{S}_1=\mathbf{S}-\{(b,g_2)\}$ does not satisfy condition **H**. Let $\mathbf{B}_1$ be a collection of boys such that $|\mathbf{G}_1|<|\mathbf{B}_1|$ where $\mathbf{G}_1=\{t:\exists x\in\mathbf{B}_1(x,t)\in\mathbf{S}_1\}$. Since **S** satisfies condition **H**, $b\in\mathbf{B}_1$, and so $g_1\in\mathbf{G}_1$. Similarly, using $\mathbf{S}_2=\mathbf{S}-\{(b,g_1)\}$, we can find $\mathbf{B}_2$ and $\mathbf{G}_2$ such that $|\mathbf{G}_2|<|\mathbf{B}_2|$, $b\in\mathbf{B}_2$, and $g_2\in\mathbf{G}_2$. Let $\mathbf{B}_3=(\mathbf{B}_1\cap\mathbf{B}_2)-\{b\}$ and $\mathbf{G}_3=\{t:\exists x\in\mathbf{B}_3\ (x,t)\in\mathbf{S}\}$. Since **S** satisfies condition **H**, $|\mathbf{G}_3|\geq|\mathbf{B}_3|$. Let $\mathbf{B}_4=\mathbf{B}_1\cup\mathbf{B}_2$. We know that $b\in\mathbf{B}_1\cap\mathbf{B}_2$, so

$$|\mathbf{B}_4|=|\mathbf{B}_1|+|\mathbf{B}_2|-|\mathbf{B}_3|-1.$$

Consider the set $\mathbf{G}_4=\{t:\exists x\in\mathbf{B}_4\ (x,t)\in\mathbf{S}\}$. We know that $g_1\in\mathbf{G}_1$ and $g_2\in\mathbf{G}_2$, so $\mathbf{G}_4=\mathbf{G}_1\cup\mathbf{G}_2$. Thus,

$$|G_4| \leq |G_1| + |G_2| - |G_3|.$$

Summarizing, we have

$$|B_4| = |B_1| + |B_2| - |B_3| - 1 > |G_1| + |G_2| - |G_3| \geq |G_4| \quad ,$$

so $|B_4| > |G_4|$. But **S** satisfies condition **H**, so $|B_4| \leq |G_4|$, a contradiction. ■

Using this version of the finite marriage theorem, we can pass on to two infinite marriage theorems. Since neither of these will be provable in $\mathbf{RCA_0}$, each will provide an example of reverse mathematics.

2.2. The Marriage Theorem and $\mathbf{ACA_0}$

In this section we will consider an infinite version of the marriage theorem. By an infinite marriage theorem, we mean one in which the set of boys, **B**, is infinite. Marshall Hall showed that such marriage problems have solutions provided that each person knows only finitely many members of the opposite sex, and condition **H** holds [15]. This version is equivalent to $\mathbf{ACA_0}$ over the base system of $\mathbf{RCA_0}$. In proving this equivalence, the following theorem provides an example of reverse mathematics.

Theorem 2.2 ($\mathbf{RCA_0}$) The following are equivalent:

i) $\mathbf{ACA_0}$

ii) Any marriage problem in which each boy knows only finitely many girls, and in which condition **H** is satisfied, has a solution.

Proof: First, we prove that i) implies ii). By Theorem 1.3 it suffices to prove ii) using König's lemma for arbitrary finitely branching trees. Call a sequence σ of length n a partial solution to the marriage problem if

1) for all $m < n$, $(m, \sigma(m)) \in \mathbf{R}$, and

2) for all $j, k < n$, $\sigma(j) = \sigma(k)$ implies $j = k$.

Let $\mathbf{T}$ be the set of all partial solutions. $\mathbf{T}$ is Δ_1^0 in $\mathbf{R}$, so $\mathbf{T}$ exists. $\mathbf{T}$ is a finitely branching tree, since each person has finitely many acquaintances. By Theorem 2.1 and condition $\mathbf{H}$, $\mathbf{T}$ is infinite. Hence, by König's lemma, $\mathbf{T}$ has an infinite path. Such a path gives a solution to the marriage problem.

We now prove the reversal, i.e. that ii) implies i). This proof is similar to a recursion theoretic proof of McAloon [32]. Let f be a total function mapping $\mathbf{N}$ one-to-one into $\mathbf{N}$. By Theorem 1.4 it suffices to show that the range of f is defined, using ii). Define the relation $\mathbf{R}$ of the marriage problem as follows:

1) For all n, $(2n, 2n) \in \mathbf{R}$.

2) If $f(n) < n$, then $(2n+1, 2f(n)) \in \mathbf{R}$ and $(2f(n), 2n+1) \in \mathbf{R}$.

3) If $f(n) \geq n$, then $(2n+1, 2n+1) \in \mathbf{R}$.

It can easily be shown that $\mathbf{R}$ satisfies the hypotheses of ii). Let g be a solution to the marriage problem given by $\mathbf{R}$, i.e. $g(x) = y$ implies "x marries y." Then the range of f is Δ_1^0 in g. In fact, n is in the range of f precisely when $g(2n) \neq 2n$ or there is an $m \leq n$ such that $f(m) = n$. ■

Since the above theorem shows that an infinite version of the marriage theorem is equivalent to a strong version of König's lemma, it seems reasonable to expect that some version would relate similarly to $\mathbf{WKL_0}$. Indeed this is the case, as is shown in the next section.

2.3. The Marriage Theorem and $\mathbf{WKL_0}$

In this section, we consider another version of the infinite marriage problem. We will call a marriage problem bounded if there is an auxiliary function h from $\mathbf{B}$ into $\mathbf{G}$ such that $(x,y)\in\mathbf{R}$ implies $y \leq h(x)$. In anthropomorphic terms, $h(x)$ is an upper bound on the "addresses" of the girls which boy x knows. The following theorem states the connection between bounded marriage problems and $\mathbf{WKL_0}$. It is another example of reverse mathematics.

Theorem 2.3 ($\mathbf{RCA_0}$) The following are equivalent:

i) $\mathbf{WKL_0}$.

ii) Every bounded marriage problem in which each boy knows only finitely many girls, and which satisfies condition $\mathbf{H}$ has a solution.

Proof: To prove that i) implies ii), we mimic the proof of Theorem 2.2. Since the tree $\mathbf{T}$ associated with a bounded marriage problem is itself bounded, by Theorem 1.1, $\mathbf{WKL_0}$ suffices to prove ii).

The reverse implication has a proof similar to one used by Manaster and Rosenstein [29]. Let $\mathbf{T}$ be a 0-1 tree, where ρ denotes the root of $\mathbf{T}$ and σ denotes a typical node. Let the set of boys, $\mathbf{B}$, be the set of nodes of $\mathbf{T}$. The relation $\mathbf{R}$ is defined as follows:

1) If σ is a successor of ρ then $(\rho,\sigma)\in\mathbf{R}$.

2) For each $\sigma\neq\rho$, $(\sigma,\sigma)\in\mathbf{R}$.

3) For each $\sigma\neq\rho$, if τ is a successor of σ, then $(\sigma,\tau)\in\mathbf{R}$.

Since each boy knows the girl whose "name" is the same as his own, condition **H** is trivially satisfied. Furthermore, **T** is a 0-1 tree, so **R** gives a bounded marriage problem. Finally, each person knows at most three other people. Thus, by ii), there is a function g from the nodes of **T** into the nodes of **T**, giving a solution to the marriage problem defined by **R**.

It remains to show that g codes a path through **T**. Let $\sigma_0 = g(\rho)$, and $\sigma_{n+1} = g(\sigma_n)$. Note that for any σ and τ, $g(\sigma) = g(\tau)$ if and only if $\sigma = \tau$. Since $g(\rho) \neq \rho$, it follows that for every n, $\sigma_n \neq \sigma_{n+1}$. By the definition of **R**, for each n σ_{n+1} must be a successor of σ_n. In this way, g codes a path through **T**. ■

The above theorem is the final example of reverse mathematics in this chapter. Besides being interesting itself, it considerably simplifies the proofs of many of the results in the next section.

2.4. Recursion Theoretic Results

In this section, we give four results in recursion theory which can be easily proved using the previous theorems. A marriage problem is called recursive if the relation **R** is recursive. The degree of its solution is the degree of the function g, coded as a set of integers. A recursive marriage problem is recursively bounded if the auxiliary function h is recursive. For each result below, references to the original recursion theoretic proofs are given.

Porism 2.4 (McAloon [32]) There is a recursive marriage problem for which $0'$ is recursive in every solution.

Proof: Let f be a recursive function enumerating $0'$ in the proof of the reversal in Theorem 2.2. ■

Porism 2.5 (Manaster and Rosenstein [29]) There is a recursively bounded recursive marriage problem with no recursive solution.

Proof: Let **T** be a recursive 0-1 tree with no recursive paths in the proof of the reversal in Theorem 2.3. ■

Corollary 2.6 (Manaster and Rosenstein [29]) Every recursively bounded recursive marriage problem has a solution of degree **a**, where $\mathbf{a}' \leq 0'$.

Proof: By Theorem 1.6 there is an ω-model of $\mathbf{WKL_0}$ in which the degree of every set satisfies $\mathbf{a}' \leq 0'$. By Theorem 2.3, every recursively bounded recursive marriage problem has a solution in this model. ■

Porism 2.7 There is a recursively bounded recursive marriage problem which has no solution that is a finite Boolean combination of recursively enumerable sets.

Proof: Jockusch [20] proved the existence of a recursive 0-1 tree having no path that is a finite Boolean combination of recursively enumerable sets. Let **T** be such a tree in the proof of the reversal in Theorem 2.3. ■

CHAPTER 3

COMBINATORIAL VARIATIONS

This chapter consists of several variations on the basic marriage theorems of Chapter 2. Each variation is shown to be equivalent to a subsystem of second order arithmetic. The equivalences are then used to provide short proofs of some recursion theoretic results. Furthermore, all the applications in this chapter are combinatorial in nature.

The first section of this chapter is concerned with symmetric versions of the basic marriage theorems. In the second section, the bounded symmetric marriage theorem is used in developing a version of Banach's theorem. These results are then applied to problems in graph theory, including the existence of certain subgraphs and node colorings.

3.1. Symmetric Marriage Theorems

This section is concerned with solutions to marriage problems of the sort introduced in Chapter 2. Here, however, we will attempt to avoid the creation of spinsters. In more mathematical terms, we will require that the solution function, f , is a one to one matching of the set of boys, $\mathbf{B}$, to the set of girls, $\mathbf{G}$. When such a solution exists, we will call it a symmetric solution. In order to assure the existence of symmetric solutions, we will need a symmetric condition, condition $\mathbf{H}_{sym}$. We will say that a marriage problem satisfies condition $\mathbf{H}_{sym}$ if every subset of n boys knows at least n girls and every subset of n girls knows at least n boys. Using this terminology we can examine the existence of symmetric solutions to infinite marriage problems in $\mathbf{ACA}_0$ and $\mathbf{WKL}_0$.

The following theorem is the symmetric version of Theorem 2.2. The proof is very similar.

Theorem 3.1 ($\mathbf{RCA}_0$) The following are equivalent:

i) $\mathbf{ACA}_0$

ii) Any marriage problem in which each person knows only finitely many members of the opposite sex, and in which condition $\mathbf{H}_{sym}$ is satisfied, has a symmetric solution.

Proof: First we prove that i) implies ii) by proving ii) using König's lemma for arbitrary finitely branching trees. Let σ denote a partial symmetric solution to the marriage problem. Here, σ denotes a sequence of n pairs such that:

1) for all $m < n$, $(m, \sigma_1(m)) \in \mathbf{R}$,

2) for all $m < n$, $(\sigma_2(m), m) \in \mathbf{R}$, and

3) for all $j, k < n$, $\sigma_1(j) = k$ if and only if $\sigma_2(k) = j$.

Informally, the m^{th} pair in the partial solution determines the wife of the m^{th} boy and the husband of the m^{th} girl. The first two conditions insure that prospective spouses are acquainted. The third condition simultaneously prohibits polygamy and insures that the both spouses witness their marriage. Let $\mathbf{T}$ be the set of all partial solutions. $\mathbf{T}$ is Δ_1^0 in $\mathbf{R}$, so $\mathbf{T}$ exists. $\mathbf{T}$ is a finitely branching tree, since each person has finitely many acquaintances. By Theorem 2.1 and condition $\mathbf{H}_{sym}$, $\mathbf{T}$ is infinite. By König's Lemma, **T** has an infinite path. Such a path is a symmetric solution to the marriage problem.

The proof of the reversal is immediate from the proof of Theorem 2.2. Since the relation **R** of the previous proof is symmetric, condition $\mathbf{H}_{sym}$ holds. By ii), this marriage problem has a symmetric solution. The existence of any solution is sufficient to prove $\mathbf{ACA}_0$, as was previously shown. ■

Given the above proof, we can immediately convert Porism 2.4 into a symmetric result.

Porism 3.2 There is a recursive marriage problem for which $0'$ is recursive in every symmetric solution.

We now present a symmetric version of the bounded marriage problem. We will call a marriage problem symmetrically bounded if there are two functions, h_1 and h_2, such that $(x,y)\in\mathbf{R}$ implies $y < h_1(x)$ and $x < h_2(y)$. A special case of the symmetrically bounded marriage problem is a k-society. A k-society is a marriage problem in which each person knows exactly k other people. As was the case in Chapter 2, the existence of solutions to bounded marriage problems is equivalent to $\mathbf{WKL}_0$ over $\mathbf{RCA}_0$.

Theorem 3.3 ($\mathbf{RCA}_0$) The following are equivalent:

i) $\mathbf{WKL}_0$

ii) Every symmetrically bounded marriage problem in which each person knows only finitely many members of the opposite sex, and which satisfies condition $\mathbf{H}_{sym}$ has a symmetric solution.

iii) For every $k \geq 1$, every k-society has a symmetric solution.

iv) Every 2-society has a symmetric solution.

Proof: To prove that i) implies ii), we mimic the proof of Theorem 3.1, applying Theorem 1.1 in place of König's lemma for arbitrary finitely splitting trees. To prove that ii) implies iii), it suffices to note that any k-society is symmetrically bounded and satisfies condition $\mathbf{H}_{sym}$. Since iii) clearly implies iv), it remains only to show that iv) implies i).

By Theorem 1.2 it suffices to use iv) to find a set $\mathbf{X}$ which separates the disjoint ranges of two injections. Call the injections c_1 and c_2. The proof uses the pairing function $(x,y)_p = \frac{1}{2}(x+y)(x+y+1)+x$. $\mathbf{RCA}_0$ proves that this pairing function is one to one and onto [50]. Let $(z)_0$ and $(z)_1$ be inverse pairing functions; that is, $(x,y)_p = z$ implies $(z)_0 = x$ and $(z)_1 = y$.

The basic idea of the proof is to construct infinitely many marriage problems such that the symmetric solution of the n^{th} problem decides whether or not n is included in the separating set. The construction of the n^{th} problem is halted after $2m$ people if $c_1(m)=n$ or $c_2(m)=n$. At this point it is appropriately "carrier" problem, guaranteeing that its solution will be "in phase" with all the other solutions. In general, we define the 2-society $\mathbf{R}$ as follows:

1) $(0,0) \in \mathbf{R}$.

2) For all $n \in \mathbf{N}$, if $(n)_1=0$ then $(3n+2,3n+2) \in \mathbf{R}$.

3) For all $n \in \mathbf{N}$, if $c_1((n)_1)=(n)_0$ and $(n)_1$ is even, or $c_2((n)_1)=(n)_0$ and $(n)_1$ is odd then include $(3n,3n+1)$, $(3n+2,3n)$, $(3n+1,3n+3)$, $(3n+1,3n+2)$,

$(3n+3,3n+1)$, and $(3((n)_0,(n)_1+1)_p+2,3((n)_0,(n)_1+1)_p+2)$ in $\mathbf{R}$.

4) For all $n \in \mathbf{N}$, if $c_1((n)_1)=(n)_0$ and $(n)_1$ is odd, or $c_2((n)_1)=(n)_0$ and $(n)_1$ is even, then include $(3n,3n+1)$, $(3n+1,3n)$, $(3n+1,3n+3)$, $(3n+2,3n+1)$, $(3n+3,3n+2)$, and $(3((n)_0,(n)_1+1)_p+2,3((n)_0,(n)_1+1)_p+2)$ in $\mathbf{R}$.

5) For all $n \in \mathbf{N}$, if $c_1((n)_1)\neq(n)_0$ and $c_2((n)_1)\neq(n)_0$, then include the six pairs $(3n,3n+1)$, $(3n+1,3n)$, $(3n+1,3n+3)$, $(3n+3,3n+1)$, $(3n+2,3((n)_0,(n)_1+1)_p+2)$, and $(3((n)_0,(n)_1+1)_p+2)$ in $\mathbf{R}$.

Claim 1: $\mathbf{R}$ is a 2-society.

Proof : We will show that each boy knows exactly two girls. Suppose the boy's "name" is $3n$. If $n=0$, by 1) he knows exactly one girl with a name less than or equal to his. If $n>0$, exactly one of 3), 4), and 5) holds for $n-1$. In each case, boy $3n$, under his alias of $3(n-1)+3$, has a "prior acquaintance" with a girl whose name is less than or equal to his. For any n, exactly one of 3), 4), and 5) holds for n. In each case, boy $3n$ knows girl $3n+1$, but no girls with larger names. Thus boy $3n$ knows exactly two girls. Each boy $3n+1$ meets exactly two girls through 3), 4), or 5). Finally, each boy $3n+2$ meets one girl in clause 3), 4) or 5) for n, and has one prior acquaintance via 3), 4), or 5), if $(n)_1\neq 0$ or via 2) if $(n)_1=0$. By a similar argument, each girl knows exactly two boys.

Let $h:\mathbf{N}\rightarrow\mathbf{N}$ be a symmetric solution to the marriage problem given by $\mathbf{R}$. Since this is a symmetric solution to a 2-society, without loss of generality we may assume that $h(0)=0$. Let $\mathbf{X}=\{n \in \mathbf{N}: h(3(n,0)_p+2)=3(n,0)_p+2\}$. Claims 3 and 4 will show that $\mathbf{X}$ is the desired set. As an intermediate step, we prove:

Claim 2: i) If $h(j)=3n$ then $j \leq 3n$.

ii) If $h(j)=3n+1$ then $j>3n+1$.

Proof : Since $h(0)=0$, for $n=0$ the claim is obvious. Suppose that the claim holds for n. If 3) holds for n, $h(3n+2)\neq 3n$. By symmetry of the solution, $h(3n+1)=3n+3$, so i) holds for $n+1$. Since ii) holds for n, $h(3n+3)=3n+1$, so $h(3n+3)\neq 3(n+1)+1$ and so ii) must hold for $n+1$.

If 4) or 5) hold for n, then $h(3n+1)=3n$, so $h(3n+1)\neq 3n+3$ and i) holds for $n+1$. Furthermore, $h(3n+3)\neq 3n+3$, so $h(3n+3)\neq 3(n+1)+1$. Thus ii) holds for $n+1$.

Claim 3: $\mathbf{Ran}(c_1)\subseteq \mathbf{X}$.

Proof : Suppose $c_1(j)=k$ and $n=3(k,j)_p+2$. If j is even, by Claim 2.i) and part 3) of the definition of **R**, we have that $h(n)=3(k,j-1)_p+2$. Since $h(3(k,j-2)_p+2)\neq 3(k,j-1)_p+2$, $h(3(k,j-2)_p+2)=3(k,j-3)_p+2$. Continuing in this fashion, we find that $h(3(k,2)_p+2)=3(k,1)_p+2$, which implies that $h(3(k,0)_p+2)=3(k,0)_p+2$, so $k\in\mathbf{X}$.

If j is odd, by Claim 2.ii) and part 4) of the definition of **R**, $h(n)=n-1$. Since $h(n)\neq 3(k,j-1)_p+2$ and h is one to one, $h(3(k,j-2)_p+2)=3(k,j-1)_p+2$. Continuing in this fashion, we find that $h(3(k,1)_p+2)=3(k,2)_p+2$, forcing $h(3(k,0)_p+2)=3(k,0)_p+2$, i.e. $k\in\mathbf{X}$.

Claim 4: $\mathbf{Ran}(c_2)\cap\mathbf{X}=\emptyset$.

Proof : Suppose $c_2(j)=k$ and $n=3(k,j)_p+2$. If j is even and $h(3(k,0)_p+2)=3(k,0)_p+2$, then $h(n-2)=n-1$, contradicting Claim 2.ii). If j is odd and $h(3(k,0)_p+2)=3(k,0)_p+2$, then $h(n)=n-2$, contradicting Claim 2.i).

Claims 3 and 4 complete the proof that a symmetric solution to the given marriage problem yields a separating set. This concludes the proof that the existence of symmetric solutions to 2-societies implies $\mathbf{WKL}_0$. This last implication completes the proof of Theorem 3.3. ∎

Using Theorem 3.3, we can immediately convert the remaining recursion theoretic results of Chapter 2 into symmetric results. For the sake of completeness, we list these in the following corollary.

Corollary 3.4 i) (Manaster and Rosenstein [29]) There is a recursively symmetrically bounded recursive marriage problem which has a symmetric solution, but has no recursive symmetric solution.

ii) (Manaster and Rosenstein [29]) Every recursively symmetrically bounded recursive marriage problem which has a symmetric solution, has a symmetric solution of degree $\mathbf{a}$, where $\mathbf{a}' \leq 0'$.

iii) (Manaster and Rosenstein [30]) For each $k > 2$ there is a recursive k-society which has no recursive symmetric solution.

iv) For every $k \in \omega$, every recursive k-society has a symmetric solution of degree $\mathbf{a}$, where $\mathbf{a}' \leq 0'$.

3.2. Banach's Theorem

In this section, we will examine two special cases of the symmetric marriage theorem. In order for a symmetric solution for a marriage problem to exist, it is clearly necessary for each person to know at least one other person. Banach's theorem tells us that in some cases this is sufficient. This leads us to formulate the following version of Banach's theorem.

Countable Bounded Banach Theorem: Let $f:\mathbf{N}\to\mathbf{N}$ and $g:\mathbf{N}\to\mathbf{N}$ be injections such that the ranges of f and g exist. Then there is a bijection $h:\mathbf{N}\to\mathbf{N}$ such that for all $m,n\in\mathbf{N}$, $h(n)=m$ implies $f(n)=m$ or $g(m)=n$.

Since this version of Banach's theorem deals only with maps from $\mathbf{N}$ to $\mathbf{N}$, it is clearly countable. The existence of the ranges of f and g is essentially a bounding condition. As will be seen in the following theorem and its proof, the Countable Bounded Banach Theorem is indeed a special case of the bounded symmetric marriage theorem. Thus, the only difficulties in the proof arise in proving the reversal.

Theorem 3.5 ($\mathbf{RCA}_0$) The following are equivalent:

i) $\mathbf{WKL}_0$

ii) Countable Bounded Banach Theorem.

Proof: We first show that i) implies ii). Let f and g be as in ii). Let $\mathbf{R}$ be the relation given by $(m,n)\in\mathbf{R}$ if and only if $f(m)=n$ or $g(n)=m$. Construct h_1 by 1) $h_1(m)=n$ if $g(n)=m$ and $n>f(m)$, and 2) $h_1(m)=f(m)$ otherwise. Construct h_2 by 1) $h_2(m)=n$ if $f(n)=m$ and $n>g(m)$, and 2) $h_2(m)=g(m)$ otherwise. Since the ranges of f and g exist, h_1 and h_2 are Δ_1^0 in f, g, and their ranges. Thus h_1 and h_2 exist.

$\mathbf{R}$ defines a marriage problem. Since f and g are injections, condition $\mathbf{H}_{sym}$ is satisfied. Furthermore, $\mathbf{R}$ is symmetrically bounded by h_1 and h_2. Thus, by Theorem 3.3, there is a symmetric solution to the marriage problem. Since the solution is a subset of $\mathbf{R}$, it gives the desired bijection, h.

We now turn to the proof of the reversal. By Theorem 1.2, it suffices to use ii) to find a set **X**, which separates the ranges of two injections, c_1 and c_2, which have disjoint ranges. Toward this end, we define two new injections, f and g, in terms of c_1 and c_2.

For each n which is not a power of a prime, let $f(n)=g(n)=n$. Let p_n denote the n^{th} prime. We define f on powers of p as follows:

1) $f(p_n)=p_n$.

2) If $i>0$, $f(p_n^{2i})=p_n^{2i+1}$.

3) If $i>0$, $f(p_n^{2i+1}) = \begin{cases} p_n^{2i+2} & \text{if } \exists j<i-1(c_1(j)=n)\vee\exists j<i(c_2(j)=n), \\ p_n^{2i} & \text{otherwise.} \end{cases}$

The function g is defined on powers of primes by:

4) If $i\geq 0$, $g(p_n^{2i+1})=p_n^{2i+2.}$

5) If $i>0$, $g(p_n^{2i}) = \begin{cases} p_n^{2i+1} & \text{if } \exists j<i-1(c_1(j)=n)\vee\exists j<i-1(c_2(j)=n), \\ p_n^{2i-1} & \text{otherwise.} \end{cases}$

We now show that f and g satisfy the hypotheses of the countable bounded Banach theorem.

Claim 1: f and g are injective.

Proof : Clearly, f is injective on integers which are not powers of primes. Also, f is clearly injective from even powers of primes to odd powers of primes. Thus, we need only show that f is injective from odd powers of primes to even powers of primes. Suppose, by way of contradiction, that $f(p_n^k)=f(p_n^l)=p_n^2$. Then, without loss of generality, $k=1$ and $l=3$. But, $p_n=f(p_n)=f(p_n^k)$, contradicting our assumption.

Now suppose that $f(p_n^k)=f(p_n^l)=p_n{}^{2i+2}$, where $i>0$. Then, without loss of generality, $k=2i+1$ and $l=2i+3$. Since $f(p_n^k)=f(p_n{}^{2i+1})=p_n{}^{2i+2}$, we have

$$\exists j<i-1(c_1(j)=n) \bigvee \exists j<i(c_2(j)=n.$$

Thus, $\exists j<i(c_1(j)=n)$ or $\exists j<i+1(c_2(j)=n)$. So, by the definition of f, we have $f(p_n^l)=f(p_n{}^{2(i+1)+1})=p_n{}^{2i+4}$, contradicting our assumption.

To show that g is injective, we proceed in a similar manner. In this case, we need only show that g is injective from even powers of primes into odd powers. Suppose, by way of contradiction, that $g(p_n^k)=g(p_n^l)=p_n{}^{2i+1}$. Without loss of generality, we may assume that $k=2i$ and $l=2(i+1)$. Since $g(p_n{}^{2i})=p_n{}^{2i+1}$, there is a $j<i-1$ such that $c_1(j)=n$ or $c_2(j)=n$. Thus $g(p_n^l)=g(p_n{}^{2(i+1)})=p_n{}^{2(i+1)+1}$, contradicting our assumption.

Claim 2: f, g, and their ranges exist.

Proof: The functions f and g are Δ_1^0 in c_1 and c_2, and so exist. To see that their ranges exist, note that:

1) $\mathbf{Ran}(f) = \{k\in\mathbf{N}:\forall i<k\,\forall n<k\,((c_1(i-1)=n \bigvee c_2(i)=n)\rightarrow k\neq p_n{}^{2i+2})\}$,

and

2) $\mathbf{Ran}(g) = \{k\in\mathbf{N}:\forall i<k\,\forall n<k\,((c_1(i-1)=n \bigvee c_2(i-1)=n)\rightarrow k\neq p_n{}^{2i+1})\}$.

These sets are also Δ_1^0 in c_1 and c_2.

So far, we have shown that f and g satisfy the hypotheses of the countable bounded Banach theorem. Applying ii) yields a bijection, h, such that $h(n)=m$ if and only if $f(n)=m$ or $g(m)=n$. Let $\mathbf{X}$ be the set of all n such that $h(p_n)=p_n$. We will now show that $\mathbf{X}$ separates the range of c_1 from the range of c_2.

Claim 3: $\mathbf{Ran}(c_1) \subseteq \mathbf{X}$.

Proof : Suppose that $c_1(m)=n$. Then p_n^{2m+3} is not in $\mathbf{Ran}(g)$. Since n is not in $\mathbf{Ran}(c_2)$, and c_1 is injective, $h(p_n^{2m+3})=f(p_n^{2m+3})=p_n^{2m+2}$. Now, h is injective, so $h(p_n^{2m+1})\neq p_n^{2m+2}= g(p_n^{2m+1})$. Thus $h(p_n^{2m+1})=f(p_n^{2m+1})$. Continuing in this fashion, we get $h(p_n)=f(p_n)=p_n$, so $n \in \mathbf{X}$.

Claim 4: $\mathbf{Ran}(c_2)$ is disjoint from $\mathbf{X}$.

Proof : Suppose $c_2(m)=n$. Then p_n^{2m+2} is not in $\mathbf{Ran}(f)$. Thus,

$$h(p_n^{2m+1})=g^{-1}(p_n^{2m+1})=p_n^{2m+2}.$$

Since h is well defined, $h(p_n^{2m+1}) \neq f(p_n^{2m+1})=p_n^{2m}$, and so, $h(p_n^{2m-1})=g^{-1}(p_n^{2m})=p_n^{2m}$. Continuing in this manner, we get $h(p_n)=g^{-1}(p_n)=p_n^2$. By the definition of $\mathbf{X}$, n is not in $\mathbf{X}$. ■

Theorem 3.5 has two recursion theoretic corollaries.

Corollary 3.6: (Remmel [41]) There are two recursive injections with recursive ranges, f and g, such that no recursive bijection can be constructed from them. That is, for no recursive bijection h is it the case that for all $m,n \in \mathbf{N}$, $h(m)=n$ implies $f(m)=n$ or $g(n)=m$.

Proof: Let c_1 and c_2 be recursive injections with recursively inseparable disjoint ranges in the proof of Theorem 3.5. The functions f and g constructed in the proof are the desired injections. ■

Corollary 3.7: Let f and g be recursive injections with recursive ranges. Then there is a bijection h such that:

i) $h(m)=n$ implies $f(m)=n$ or $g(n)=m$, and

ii) The degree of h is $\mathbf{a}$, where $\mathbf{a}' \leq 0'$.

Proof: By Theorem 1.6, there is a model of $\mathbf{WKL}_0$ in which every set is of degree at most $\mathbf{a}$, where $\mathbf{a}' \leq 0'$. By Theorem 3.5, given f and g, an appropriate h must exist in this model. ■

One might ask if the existence of the ranges of f and g is truly necessary in the proof of Theorem 3.5. Questions of this sort are easily answered in the framework of reverse mathematics, as the following theorem indicates.

Theorem 3.8: ($\mathbf{RCA}_0$) The following are equivalent:

i) $\mathbf{ACA}_0$

ii) Let $f:\mathbf{N}\to\mathbf{N}$ and $g:\mathbf{N}\to\mathbf{N}$ be injections. Then there is a bijection, $h:\mathbf{N}\to\mathbf{N}$ such that $h(n)=m$ implies $f(n)=m$ or $g(m)=n$.

iii) Let $f:\mathbf{N}\to\mathbf{N}$ and $g:\mathbf{N}\to\mathbf{N}$ be injections such that $\mathbf{Ran}(g)$ exists. Then there is a bijection, $h:\mathbf{N}\to\mathbf{N}$ such that $h(n)=m$ implies $f(n)=m$ or $g(m)=n$.

Proof: First note that ii) is a special case of the unbounded symmetric marriage theorem. Thus, by Theorem 3.1, i) implies ii). Furthermore, iii) is a restricted version of ii), so ii) implies iii). It remains to show that iii) implies i). By Theorem 1.4, it suffices to use iii) to prove that the range of an arbitrary function c exists. We define injections f and g as follows:

1) If n is not a power of a prime, $f(n)=g(n)=n$.

2) For each prime p , $f(p)=g(p)=p^2$.

$$3) \text{ If } m>1,\ f(p_n^m)=\begin{cases} p_n & \text{if } c(m-2)=n \\ p_n^m & \text{if } \exists j<m-2(c(j)=n) \\ p_n^{m+1} & \text{otherwise.} \end{cases}$$

$$4) \text{ If } m>1,\ g(p_n^m)=\begin{cases} p_n^{m+1} & \text{if } \exists j<m-2(c(j)=n) \\ p_n^m & \text{otherwise.} \end{cases}$$

The proofs that f and g are well defined injections are straightforward. Furthermore, $\mathbf{Ran}(g)=\{k\in\mathbf{N}:\forall m<k\ \forall n<k\ (c(m-2)=n \rightarrow k\neq p_n^m)\}$, so $\mathbf{Ran}(g)$ exists. Let h be the injection given by iii). Define $\mathbf{X}$ by $\mathbf{X}=\{n\in\mathbf{N}:h(p_n)\neq p_n\}$. Suppose that $c(m)=n$. Then p_n^m is not in $\mathbf{Ran}(g)$, so $h(p_n^m)=f(p_n^m)=p_n$, and $n\in\mathbf{X}$. On the other hand, if n is not in $\mathbf{Ran}(c)$, then p_n is not in $\mathbf{Ran}(f)$. Thus, $h(p_n)=g^{-1}(p_n)=p_n$, and n is not in $\mathbf{X}$. ■

Theorem 3.8 may also be converted into a recursion theory result.

Corollary 3.9: There are recursive injections f and g such that:

i) g has a recursive range, and

ii) For any bijection h constructed from f and g , $0'$ is recursive in the degree of h .

Proof: Let c be a recursive function with $0'$ recursive in its range. The injections constructed from c in the proof of Theorem 3.8 are the desired ones. ■

This proof concludes our study of Banach's theorem. We now turn to other combinatorial applications of the marriage theorems.

3.3. Graph Theory

This section contains proofs of the equivalence of several graph theoretic results to subsystems of second order arithmetic. A graph, **G**, consists of a set of vertices **V**, and a set of edges, **E**. Each edge consists of a non-ordered pair of vertices, i.e. $\mathbf{E}\subseteq[\mathbf{V}]^2$. The graph **G** is bounded if there is a function $h:\mathbf{V}\to\mathbf{V}$, such that $(v_1,v_2)\in\mathbf{E}$ implies $h(v_1)>v_2$. $\mathbf{G}=[\mathbf{X},\mathbf{Y}]$ is bipartite if and only if there are disjoint subsets **X** and **Y** of **V** such that if (x,y) is an edge of **G**, then $x\in\mathbf{X}$ and $y\in\mathbf{Y}$. For bipartite graphs we will often abuse notation by treating an edge as an ordered pair, the first component lying in **X** and the second in **Y**. A subset $\mathbf{E}'$ of **E** is said to be incident to a subset $\mathbf{X}'$ of **X** (respectively $\mathbf{Y}'$ of **Y**) if every element of $\mathbf{X}'$ ($\mathbf{Y}'$) is the endpoint of some edge in $\mathbf{E}'$. A subset $\mathbf{E}'$ of **E** is independent if no two edges of $\mathbf{E}'$ share a common endpoint. The next two theorems are slight generalizations on the symmetric marriage theorems.

Theorem 3.10 ($\mathbf{RCA}_0$) The following are equivalent:

i) $\mathbf{ACA}_0$.

ii) (Ore's Theorem) Let $\mathbf{G}=[\mathbf{X},\mathbf{Y}]$ be a bipartite graph with edge set $\mathbf{E}\subseteq\mathbf{X}\times\mathbf{Y}$. Let $\mathbf{X}'\subseteq\mathbf{X}$ and $\mathbf{Y}'\subseteq\mathbf{Y}$. If there is an independent set **S** incident to $\mathbf{X}'$ and an independent set **T** incident to $\mathbf{Y}'$, then there is an independent set **M** incident to each of $\mathbf{X}'$ and $\mathbf{Y}'$.

Proof: First, we will assume $\mathbf{ACA}_0$ and prove ii). If $\mathbf{X}'$ is finite, let $\mathbf{S}'=\{(x,y)\in\mathbf{S}:y\notin\mathbf{Y}'\}$, and take $\mathbf{M}=\mathbf{S}'\cup\mathbf{T}$. A similar argument eliminates the case where $\mathbf{Y}'$ is finite. If both $\mathbf{X}'$ and $\mathbf{Y}'$ are infinite, set $\mathbf{R}=\mathbf{S}\cup\mathbf{T}$. **R** satisfies condition $\mathbf{H}_{sym}$ and each vertex knows at most two other vertices. By Theorem 3.1,

R has a symmetric solution, g. $\mathbf{M}=\{(x,y):g(x)=y\}$ is the desired independent set.

To prove that ii) implies i), let **G** be the graph constructed as follows from the injections f and g given in the proof of Theorem 3.8. Let $\mathbf{E}=\{(x,y)\in\mathbf{N}\times\mathbf{N}: f(x)=y \vee g(y)=x\}$. (Technically, we should use coding so that **E** is the cartesian product of disjoint sets.) Set $\mathbf{T}=\{(n,f(n)):n\in\mathbf{N}\}$ and $\mathbf{S}=\{(g(n),n):n\in\mathbf{N}\}$. Any **M** as given by ii) defines the bijection needed in the proof of Theorem 3.8. ■

Theorem 3.11: ($\mathbf{RCA}_0$) The following are equivalent:

i) $\mathbf{WKL}_0$.

ii) (Bounded Ore's Theorem) Let $\mathbf{G}=[\mathbf{X},\mathbf{Y}]$ be a bounded bipartite graph with edge set $\mathbf{E}\subseteq\mathbf{X}\times\mathbf{Y}$. Let $\mathbf{X}'\subseteq\mathbf{X}$ and $\mathbf{Y}'\subseteq\mathbf{Y}$. If there is an independent set **S** incident to $\mathbf{X}'$ and an independent set **T** incident to $\mathbf{Y}'$, then there is an independent set **M** incident to each of $\mathbf{X}'$ and $\mathbf{Y}'$. (This is identical to ii in Theorem 3.10 except for the added hypothesis that **G** is bounded.)

Proof: The proof of this theorem differs from the proof of Theorem 3.10 in only two places. First, in proving that i) implies ii), Theorem 3.3 is used in place of Theorem 3.1. Secondly, in proving that ii) implies i), the proof of Theorem 3.5 is used in place of Theorem 3.8. ■

So far, we have concerned ourselves with bipartite graphs. One feature of these graphs is their lack of cycles of odd length. A cycle is a sequence of edges (l_1,r_1), (l_2,r_2), . . . , (l_n,r_n), such that for all $i<n$, $r_i=l_{i+1}$, and $l_1=r_n$. The length of the cycle is simply the number of edges in the sequence. It can be proved in $\mathbf{RCA}_0$

that every bipartite graph has no cycles of odd length. A well known theorem of ordinary mathematics states that any graph which has no cycles of odd length is bipartite. This can be proved for finite graphs in $\mathbf{RCA_0}$. However, the statement cannot be proved in $\mathbf{RCA_0}$ for arbitrary graphs, as is shown by the following theorem. Two other notions are included in the statement of the following theorem. A graph $\mathbf{G}$ is k-regular if for each $x \in \mathbf{V}$, the cardinality of the set $\{y \in \mathbf{V}:(x,y)\in \mathbf{E}\}$ is precisely k. Intuitively, $\mathbf{G}$ is k-regular if each vertex has exactly k immediate neighbors. The dual graph $\overline{\mathbf{G}}$ of a graph $\mathbf{G}$ is the graph formed by treating each edge of $\mathbf{G}$ as a vertex of $\overline{\mathbf{G}}$. Two vertices of $\overline{\mathbf{G}}$ define an edge in $\overline{\mathbf{G}}$ if and only if their corresponding edges in $\mathbf{G}$ share an endpoint.

Theorem 3.12: ($\mathbf{RCA_0}$) The following are equivalent:

i) $\mathbf{WKL_0}$.

ii) Every graph with no cycles of odd length is bipartite.

iii) The dual graph of a bipartite graph is bipartite.

iv) The dual graph of a 2-regular bipartite graph is bipartite.

Proof: First, we will prove ii) in $\mathbf{WKL_0}$. Since $\mathbf{RCA_0}$ proves ii) for finite graphs, we may restrict attention to infinite graphs. Let $\mathbf{G}$ be a graph with no odd cycles, and let $\mathbf{V} = \langle v_i : i \in \mathbf{N} \rangle$ be an enumeration of its vertices. Let $\mathbf{E}$ be the edge set of $\mathbf{G}$. Define the tree $\mathbf{T}$ of sequences from $\mathbf{Seq_2}$ as follows. Let $\sigma \in \mathbf{T}$ if and only if for all $i, j < \mathrm{lh}(\sigma)$, $\sigma(i) = \sigma(j)$ implies $(v_i, v_j) \notin \mathbf{E}$. $\mathbf{T}$ is infinite, since $\mathbf{RCA_0}$ proves ii) for finite graphs. By $\mathbf{WKL_0}$, $\mathbf{T}$ has an infinite path. This path partitions the vertices of $\mathbf{G}$ into two parts as is required.

It can easily be proved in $\mathbf{RCA}_0$ that the dual graph of a bipartite graph has no cycles of odd length. The existence of such a cycle immediately gives a cycle of odd length in the original graph, contradicting the fact that it is bipartite. From this, it is clear that ii) implies iii). Since iv) is a special case of iii), iii) implies iv). It remains to show that iv) implies i).

Let $\mathbf{G}$ be the graph corresponding to a 2-society. $\mathbf{G}$ is bipartite; the vertices consist of the set of boys and the set of girls. $\mathbf{G}$ is 2-regular since each person knows exactly two other people. By iv), the dual graph $\overline{\mathbf{G}}$ is bipartite. Let g be an appropriate partition of the vertices of $\overline{\mathbf{G}}$. Let $\mathbf{E}_0$ and $\mathbf{E}_1$ be the induced partition on the edges of $\mathbf{G}$. Both $\mathbf{E}_0$ and $\mathbf{E}_1$ give symmetric solutions to the 2-society. By Theorem 3.3, this suffices to prove $\mathbf{WKL}_0$. ■

We now turn to the existence of some node colorings of graphs. Let $\mathbf{G}$ be a graph with vertex set $\mathbf{V}$ and edge set $\mathbf{E}$. We say that $f:\mathbf{V}\rightarrow k$ is a k-coloring of $\mathbf{G}$ if f always assigns different colors to the endpoints of an edge. That is, f is a k-coloring if $(x,y)\in\mathbf{E}$ implies $f(x)\neq f(y)$. If $\mathbf{G}$ has a k-coloring, then we say that $\mathbf{G}$ is k-chromatic. Note that if $\mathbf{G}$ is k-chromatic, then $\mathbf{G}$ is j-chromatic for all $j>k$. The following theorem contains several equivalent statements involving these notions.

Theorem 3.13 ($\mathbf{RCA}_0$) The following are equivalent:

i) $\mathbf{WKL}_0$.

ii) If every finite subset of a graph $\mathbf{G}$ is k-chromatic, then $\mathbf{G}$ is k-chromatic.

iii) If every finite subset of a bounded graph $\mathbf{G}$ is 2-chromatic, then $\mathbf{G}$ is 2-chromatic.

iv) Every 2-regular graph with no cycles of odd length is 2-chromatic.

Proof: First we will prove that ii) is provable in $\mathbf{WKL}_0$. Let $\mathbf{G}$ be a graph such that every finite subgraph of $\mathbf{G}$ is k-chromatic. Let $<v_i : i \in \mathbf{N}>$ be an enumeration of the vertices of $\mathbf{G}$. Define the tree $\mathbf{T}$ of sequences from $\mathbf{Seq}_k$ as follows. Let $\sigma \in \mathbf{T}$ if and only if $f(v_i) = \sigma(i)$ defines a k-coloring on the subgraph of $\mathbf{G}$ with vertex set $\{v_i : i < \mathrm{lh}(\sigma)\}$. Since every finite subgraph of $\mathbf{G}$ is k-chromatic, $\mathbf{T}$ is an infinite tree. By Theorem 1.1, $\mathbf{T}$ has an infinite path. Such a path codes a k-coloring of $\mathbf{T}$.

Since iii) is a special case of ii), ii) implies iii). Every 2-regular graph is bounded, and a graph with no cycles of odd length has 2-chromatic finite subgraphs. Thus iii) implies iv). It remains only to show that iv) implies i).

Let $\overline{\mathbf{G}}$ be the dual graph constructed in the proof of Theorem 3.12. $\overline{\mathbf{G}}$ is 2-regular and has no odd cycles. By iv), $\overline{\mathbf{G}}$ is 2-chromatic. Let $f : \mathbf{V} \to 2$ be a 2-coloring of $\overline{\mathbf{G}}$. Then $\mathbf{V}_0 = \{v \in \mathbf{V} : f(v) = 0\}$ and $\mathbf{V}_1 = \{v \in \mathbf{V} : f(v) = 1\}$ define a partition of the vertices of $\overline{\mathbf{G}}$ which proves that $\overline{\mathbf{G}}$ is bipartite. By Theorem 3.12, this implies $\mathbf{WKL}_0$. ■

In the previous theorem the conclusion that $\mathbf{G}$ has a 2-coloring is necessary to obtain the reversal. Schmerl [43] has shown that $\mathbf{RCA}_0$ proves that if every finite subset of a bounded graph $\mathbf{G}$ is 2-chromatic, then $\mathbf{G}$ is 3-chromatic. His result is actually phrased as a recursion theoretic theorem. The above theorem yields the following recursion theoretic corollaries. An edge k-coloring of $\mathbf{G}$ is simply a k-coloring of the dual graph of $\mathbf{G}$.

Corollary 3.14: Every k-chromatic recursive graph has a k-coloring of degree $\mathbf{a}$ where $\mathbf{a}' \leq 0'$.

Proof: Let **G** be a k-chromatic recursive graph. By Theorem 3.13 ii), **G** has a k-coloring in any model of $\mathbf{WKL_0}$. By Theorem 1.6, there is an ω-model of $\mathbf{WKL_0}$ in which every set is of degree **a** where $\mathbf{a}' \leq 0'$. ■

Porism 3.15: There is a 2-regular recursive graph with a recursive 2-coloring but no recursive edge 2-coloring.

Proof: Let c_1 and c_2 be recursive functions with recursively inseparable ranges. Let **G** be the graph corresponding to the 2-society constructed from c_1 and c_2 as in the proof of Theorem 3.3. Since **G** is bipartite, it has a recursive 2-coloring; the vertices can be separated by gender. However, **G** has no recursive edge 2-coloring, since such a coloring yields a symmetric solution to the 2-society. ■

Corollary 3.16: There is a 2-regular recursive graph with a recursive edge 2-coloring but no recursive 2-coloring.

Proof: Use the dual of the graph in the proof of Porism 3.15. ■

Corollary 3.17: There is a 2-regular recursive graph with no recursive 2-coloring and no recursive edge 2-coloring.

Proof: Let **G** be the graph in the proof of Porism 3.15 and let $\overline{\mathrm{G}}$ be its dual graph. Form a graph **H** by letting the subgraph of **H** consisting of only the odd numbered vertices be **G**, and that consisting of only the even numbered vertices be $\mathbf{G}'$. **H** is the desired graph. ■

3.4. Partial Orders

This section contains several theorems relating $\mathbf{WKL_0}$ to statements about partial orders. A partial order consists of a set **P**, together with a set of ordered pairs

$\mathbf{R} \subseteq \mathbf{P} \times \mathbf{P}$. If $(a, b) \in \mathbf{R}$, then we write $a \leq^{\mathbf{P}} b$. As a relation, **R** must be reflexive, transitive, and antisymmetric. That is, for any a, b, and c in **P**, i) $a \leq^{\mathbf{P}} a$, ii) $a \leq^{\mathbf{P}} b$ and $b \leq^{\mathbf{P}} c$ imply $a \leq^{\mathbf{P}} c$, and iii) $a \leq^{\mathbf{P}} b$ and $b \leq^{\mathbf{P}} a$ imply $a = b$. A partial order is often simply called **P**, when the relation **R** is clear.

Many of the results in this section concern comparability graphs. A graph **G** is comparability graph if there is a partial order **P** such that (a, b) is an edge of **G** if and only if $a <^{\mathbf{P}} b$ or $b <^{\mathbf{P}} a$. In this case we say that **P** satisfies **G**. It is easy to show that $\mathbf{RCA}_0$ proves that every partial order satisfies some comparability graph. It is less easy, however, to decide whether or not a particular graph is a comparability graph, as is shown by the following theorem.

Theorem 3.18: ($\mathbf{RCA}_0$) The following are equivalent:

i) $\mathbf{WKL}_0$

ii) If every finite subgraph of **G** is a comparability graph, then **G** is a comparability graph.

Proof: The proof of ii) for finite graphs is trivial. Assume i) and let **G** be an infinite graph such that each of its finite subgraphs is a comparability graph. Let $\mathbf{V} = \langle v_i : i \in \mathbf{N} \rangle$ be an enumeration of the vertices of **G**. Let **T** be a tree of finite partial orders on subsets of **V**, such that σ is in the n^{th} level of **T** if and only if σ codes a partial order on $v_1, v_2, \ldots, v_n$ satisfying **G** restricted to $v_1, v_2, \ldots, v_n$. **T** is ordered by inclusion. Since **G** is infinite, **T** is also infinite. Furthermore, it is easy to arrange the coding used in **T** to insure that **T** is bounded. By Theorem 1.1, **T** has an infinite path. Such a path clearly gives the desired partial order.

The proof that ii) implies i) is even more simple. Let **G** be a 2-regular graph with no cycles of odd length. Each finite subgraph of **G** is a comparability graph. To see this, separate a given finite subgraph into its connected components. Within each component, arbitrarily fix the ordering on one pair of connected vertices. Since **G** is 2-regular, the rest of the ordering is completely determined by this choice. By ii), **G** is a comparability graph, so there is some partial order **P** corresponding to **G**. Let f be the function on the vertices of **G** given by:

i) $f(v)=0$ if $v <^{\mathbf{P}} w$ where (v, w) is an edge of **G**,

ii) $f(v)=1$ otherwise.

Since **G** is 2-regular, f is Δ_1^0 in **P**. Since **P** contains no chains of length three, f is a 2-coloring of **G**. By Theorem 3.13, this is equivalent to $\mathbf{WKL}_0$. ∎

A useful characterization of comparability graphs was given by Ghouilà-Houri [11], Gilmore, and Hoffman [12]. A g-cycle of a graph **G** is a finite sequence of vertices $v_1, v_2, \ldots, v_n$ such that (v_1, v_n) and (v_i, v_{i+1}) for $i<n$ are edges of **G**. In addition, for no vertices a and b and integers $i, j<n$ is it the case that both $a=v_i=v_j$ and $b=v_{i+1}=v_{j+1}$ or that both $a=v_{n-1}=v_1$ and $b=v_n=v_2$. A g-cycle is called odd if n is odd, and even if n is even. Finally, an edge is a triangular chord for a g-cycle if it is of the form (v_i, v_{i+2}) where $1 \leq i \leq n-2$, or (v_{n-1}, v_1), or (v_n, v_2). With this vocabulary we can characterize all finite comparability graphs.

Theorem 3.19: $(\mathbf{RCA}_0)$ Let **G** be a finite graph. If **G** has no odd g-cycles without triangular chords, then **G** is a comparability graph.

Proof: The proof of this theorem in $\mathbf{RCA}_0$ amounts to a formalization of Gilmore and Hoffman's proof [12], up to the application of Zorn's Lemma. As is noted after the proof of Lemma 6 in [12], this portion of the proof for infinite graphs suffices to prove the statement for finite graphs. ∎

Although Gilmore and Hoffman's proof make use of Zorn's Lemma to extend Theorem 3.19 to infinite graphs, the theorem can be extended to countable graphs within $\mathbf{WKL}_0$. Not surprisingly, this result also has a reversal, as is noted in the following theorem.

Theorem 3.20: ($\mathbf{RCA}_0$) The following are equivalent:

i) $\mathbf{WKL}_0$

ii) If $\mathbf{G}$ is a graph such that every odd g-cycle has a triangular chord, then $\mathbf{G}$ is a comparability graph.

Proof: The proof that i) implies ii) follows immediately from Theorem 3.18 and Theorem 3.19. To prove that ii) implies i), it suffices to show that the graph used in the proof of Theorem 3.18 has no odd g-cycles. Since every odd g-cycle contains an odd cycle, this is clearly the case. ∎

The next three theorems relate versions of Dilworth's theorem to $\mathbf{RCA}_0$ and $\mathbf{WKL}_0$. For the statements of these theorems, it is useful to introduce the concepts of height and width of partial orders. A partial order has width at most n if it contains no antichains of length $n+1$. It has height at most n if it contains no chains of length $n+1$. Throughout the following, a chain or antichain may be empty.

Theorem 3.21: ($\mathbf{RCA}_0$) Every finite partial order of width at most n can be partitioned into n disjoint chains. Dually, every finite partial order of height at most n can be partitioned into n disjoint antichains.

Proof: The proof of the first statement is a straightforward formalization of the usual proof of the finite Dilworth theorem (see [5] or [34]). The dual statement is proved by induction on the cardinality of the partial order. If the cardinality is 1, the statement is trivial. Suppose that the statement is true for partial orders of size less than n. Let $\mathbf{P}$ be a partial order of size n and height at most k. Let $\mathbf{A}_1=\{x \in \mathbf{P} : \forall y \in \mathbf{P}\, y \not< x\}$, that is, $\mathbf{A}_1$ is the set of minimal elements of $\mathbf{P}$. Since $\mathbf{P}$ is finite, the existence of this set is proved by $\mathbf{RCA}_0$. Clearly, $\mathbf{A}_1$ is an antichain of cardinality at least 1. Thus the partial order $\mathbf{P}-\mathbf{A}_1$ has size strictly less than n. Furthermore, since every chain in $\mathbf{P}-\mathbf{A}_1$ has a proper extension in $\mathbf{P}$, the height of $\mathbf{P}-\mathbf{A}_1$ is at most $k-1$. By the induction hypothesis, $\mathbf{P}-\mathbf{A}_1$ can be partitioned into $k-1$ disjoint antichains. Call these $\mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_k$. Then $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_k$ is a partition of $\mathbf{P}$ into k antichains. ■

Theorem 3.21 also has a graph theoretic version. A graph is called complete if every pair of vertices is connected by an edge. Complete subgraphs of comparability graphs correspond to chains in any partial order satisfying them. Independent sets of vertices correspond similarly to antichains.

Corollary 3.22: ($\mathbf{RCA}_0$) Let $\mathbf{G}$ be a finite comparability graph. If $\mathbf{G}$ contains no set of $k+1$ independent vertices, then the vertices of $\mathbf{G}$ can be partitioned into k sets, $\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_k$, so that for each $i \leq k$, $\mathbf{G}$ restricted to $\mathbf{V}_i$ is complete. Dually, if $\mathbf{G}$ contains no complete subgraphs on $k+1$ vertices, then $\mathbf{G}$ is k-chromatic.

Proof: Let $\mathbf{G}$ be a finite comparability graph. Let $\mathbf{P}$ be a partial order satisfying $\mathbf{G}$. If $\mathbf{G}$ has no set of $k+1$ independent vertices, then $\mathbf{P}$ has width at most k. By

Theorem 3.21, **P** can be partitioned into k disjoint chains. The subgraph of **G** associated with each chain is complete. Similarly, if **G** contains no complete subgraphs on $k+1$ vertices, then **P** has height at most k. By Theorem 3.21, **P** can be partitioned into k disjoint antichains. The corresponding partition of the vertices of **G** is a k-coloring. ■

We will now examine Dilworth's theorem for countable partial orders. Theorem 3.23 analyzes the basic Dilworth theorem, while Theorem 3.24 analyzes the dual statement. To simplify the graph theoretic statements involved we will use the following abbreviation. A graph has property **C** if each of its finite subgraphs is a comparability graph. By Theorem 3.19, a graph has property **C** if and only if it has no odd g-cycles without triangular chords.

Theorem 3.23: ($\mathbf{RCA_0}$) The following are equivalent:

i) $\mathbf{WKL_0}$

ii) If **G** is a graph with property **C** and contains no set of $k+1$ independent vertices, then the vertices of **G** can be partitioned into k sets, $\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_k$, such that for each $i \leq k$, **G** restricted to $\mathbf{V}_i$ is complete.

iii) If **G** is a graph with property **C** and contains no set of three independent vertices, then the vertices of **G** can be partitioned into two sets, $\mathbf{V}_1$ and $\mathbf{V}_2$, such that for each $i \leq 2$, **G** restricted to $\mathbf{V}_i$ is complete.

iv) If **P** is a partial order of width at most n, then **P** can be partitioned into n chains.

v) If **P** is a partial order of width 2, then **P** can be partitioned into two chains.

Proof: Clearly, ii) implies iii) and iv) implies v). Since $\mathbf{RCA}_0$ proves the existence of comparability graphs for given partial orders, ii) implies iv) and iii) implies v). It remains to show that i) implies ii) and v) implies i).

To prove that i) implies ii), assume $\mathbf{WKL}_0$ and let $\mathbf{G}$ be a graph with property $\mathbf{C}$ having no set of $k+1$ independent vertices. Let $\mathbf{V} = < v_i : i \in \mathbf{N} >$ be an enumeration of the vertices of $\mathbf{G}$. Let $\mathbf{T}$ be the tree of appropriate partial partitions of $\mathbf{V}$, ordered by inclusion. That is, σ is in the n^{th} level of $\mathbf{T}$ if and only if σ codes a partition of $\{v_i : i \leq n\}$ into at most k sets, $\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_k$, so that $\mathbf{G}$ restricted to $\mathbf{V}_i$ is complete for each $i \leq k$. Choosing appropriate codings for $\mathbf{T}$ insures that $\mathbf{T}$ is bounded. By Corollary 3.22, $\mathbf{T}$ is infinite. By Theorem 1.1, $\mathbf{T}$ has an infinite path, which codes the desired partition of $\mathbf{V}$.

By Theorem 1.2, to prove that v) implies i) it suffices to use v) to separate the ranges of two functions. Let f and g be injections with disjoint ranges. We will partially order the set $\mathbf{P}$, where $\mathbf{P}$ consists of (codes for the elements of) the sets $\{n_k : n \in \mathbf{N} \text{ and } 1 \leq k \leq 4\}$, $\{\mathbf{I}_{j,k} : j \in \{1,2\} \text{ and } k \in \mathbf{N}\}$, and $\{\mathbf{O}_k : k \in \mathbf{N}\}$. The relation on $\mathbf{P}$ is given by:

1) $m_j <^{\mathbf{P}} n_k$ if $m < n$ or both $m = n$ and $j < k$.

2) $m_i <^{\mathbf{P}} \mathbf{I}_{j,k}$ if $m < f(k)$.

3) $\mathbf{I}_{j,k} <^{\mathbf{P}} n_l$ if $f(k) < n$.

4) $n_1 <^{\mathbf{P}} \mathbf{I}_{1,m} <^{\mathbf{P}} n_3$ and $n_2 <^{\mathbf{P}} \mathbf{I}_{2,m} <^{\mathbf{P}} n_4$ if $f(m) = n$.

5) $m_i <^{\mathbf{P}} \mathbf{O}_k$ if $m < g(k)$.

6) $\mathbf{O}_i <^{\mathbf{P}} m_k$ if $g(i) < m$.

7) $m_1 <^{\mathbf{P}} \mathbf{O}_k <^{\mathbf{P}} m_4$ if $g(k) = m$.

It is straightforward to verify that $\mathbf{P}$ together with the ordering $<^{\mathbf{P}}$ is a partial order of width 2. Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be a partition of $\mathbf{P}$ into two chains. For all $n \in \mathbf{N}$, if $n \in \mathbf{Ran}(f)$, then exactly one of $n_2 \in \mathbf{C}_1$ and $n_3 \in \mathbf{C}_1$ hold. If $n \in \mathbf{Ran}(g)$, then $\{n_2, n_3\} \subseteq \mathbf{C}_1$ or $\{n_2, n_3\} \cap \mathbf{C}_1 = \emptyset$. Thus, $\{n \in \mathbf{N}$: exactly one of n_2 and n_3 is in $\mathbf{C}_1\}$ contains $\mathbf{Ran}(f)$ and is disjoint from $\mathbf{Ran}(g)$. ∎

Theorem 3.24: ($\mathbf{RCA}_0$) The following are equivalent:

i) $\mathbf{WKL}_0$

ii) If $\mathbf{G}$ is a graph with property $\mathbf{C}$ and contains no complete subgraph on $k+1$ vertices then $\mathbf{G}$ is k-chromatic.

iii) If $\mathbf{G}$ is a graph with property $\mathbf{C}$ and contains no triangles, then $\mathbf{G}$ is 2-chromatic.

iv) If $\mathbf{P}$ is a partial order of height at most n, then $\mathbf{P}$ can be partitioned into n antichains.

v) If $\mathbf{P}$ is partial order of height 2, then $\mathbf{P}$ can be partitioned into two antichains.

Proof: Clearly ii) implies iii) and iv) implies v). Since $\mathbf{RCA}_0$ proves the existence of comparability graphs for given partial orders, ii) implies iv) and iii) implies v). The proof of i) implies ii) follows immediately from Corollary 3.22 and Theorem 3.13.

It remains to show that v) implies i). By Theorem 1.2, it suffices to use v) to prove the existence of a set separating the ranges of two functions. Let f and g be injections with disjoint ranges. We will partially order the set $\mathbf{P}$, where $\mathbf{P}$ consists of (codes for) the elements of the sets $\{\mathbf{I}_n : n \in \mathbf{N}\}$, $\{\mathbf{O}_n : n \in \mathbf{N}\}$, and $\mathbf{N}$. The partial order on $\mathbf{P}$ is defined by:

1) $\mathbf{O}_n <^{\mathbf{P}} \mathbf{I}_m$ for all n, m.

2) $\mathbf{O}_n <^{\mathbf{P}} j$ if $\exists t \leq n \;\; g(t) = j$.

3) $j <^{\mathbf{P}} \mathbf{I}_n$ if $\exists t \leq n \;\; f(t) = j$.

It is straightforward to verify that $\mathbf{P}$ together with $<^{\mathbf{P}}$ is a partial order of height 2. Let $\mathbf{A}_1$ and $\mathbf{A}_2$ partition $\mathbf{P}$ into two antichains. Without loss of generality, let $\mathbf{O}_0 \in \mathbf{A}_1$. Then $\{\mathbf{I}_n : n \in \mathbf{N}\} \subseteq \mathbf{A}_2$, so $\{\mathbf{O}_n : n \in \mathbf{N}\} \subseteq \mathbf{A}_1$. From this we can see that if $n \in \mathbf{Ran}(f)$, then $n \in \mathbf{A}_1$, and if $n \in \mathbf{Ran}(g)$, then $n \in \mathbf{A}_2$. Thus, $\mathbf{N} \cap \mathbf{A}_1$ is the desired separating set. ■

This section concludes with an assortment of recursion theoretic porisms and corollaries following from the preceding theorems.

Porism 3.25: There is a highly recursive comparability graph which is not the comparability graph for any recursive partial order.

Proof: Use the graph from Porism 3.16. ■

Corollary 3.26: Every recursive comparability graph is the comparability graph of a partial order of degree $\mathbf{a}$, where $\mathbf{a}' \leq 0'$.

Proof: Let $\mathbf{G}$ be a recursive comparability graph. By Theorem 3.18, $\mathbf{G}$ is a comparability graph in any ω-model of $\mathbf{WKL}_0$. By Theorem 1.6, there is an ω-model of $\mathbf{WKL}_0$ whose set universe consists only of sets of low degree. ■

Porism 3.27: There is a recursive partial order of width 2 which cannot be partitioned into two recursive chains.

Proof: In the proof of Theorem 3.23, let f and g be recursive functions with recursively inseparable ranges. ∎

Kierstead [23] has constructed a recursive partial order of width 2 which cannot be partitioned into four recursive chains. This recursive counterexample is considerably more involved than the construction of Theorem 3.23, and may or may not generalize to a reversal, as increasing the number of allowable chains can reduce their complexity. For example, Kierstead has shown that any recursive partial order of width 2 can be partitioned into six recursive chains.

Corollary 3.28: Every recursive partial order of width at most n can be partitioned into n chains of degree $\mathbf{a}_i$ for $1 \leq i \leq n$, such that for each i, $\mathbf{a}_i' \leq 0'$.

Proof: Let **P** be a recursive partial order of width at most n. By Theorem 3.23, **P** can be partitioned into n chains in any ω-model of $\mathbf{WKL_0}$. By Theorem 1.6, there is an ω-model of $\mathbf{WKL_0}$ whose set universe consists only of sets of low degree. ∎

Porism 3.29: There is a recursive partial order of height 2 which cannot be partitioned into two recursive antichains. Furthermore, any recursive partial order of height n can be partitioned into n antichains of degree $\mathbf{a}_i$ for $1 \leq i \leq n$, such that for each i, $\mathbf{a}_i' \leq 0'$.

Proof: Similar to the proofs of Corollary 3.27 and Porism 3.28 except for the use of Theorem 3.24. ∎

3.5. Rado's Selection Principle

In this section, we will consider several versions of Rado's selection principle. This theorem resembles a marriage theorem in that it asserts the existence of a choice function for a family of finite subsets. A function $f : \mathbf{I} \to \mathbf{E}$ is a choice function for a family of non-empty subsets of a base set $\mathbf{E}$ indexed by $\mathbf{I}$ if for each $i \in \mathbf{I}$, $f(i)$ is an element of the set with index i. For countable families of finite sets, we can further specify the nature of the choice function. Since all the results of this section concern such countable families, we will always use $\mathbf{N}$ as both the index set and the base set.

The variants of the selection principle listed below are the result of two factors. First, although the usual statement of the selection principle involves arbitrary finite subsets of the index set, it is sufficient to consider only initial segments of $\mathbf{N}$. The second factor is the availability of two reasonable codings for families of finite sets. We say that a set of ordered pairs $\mathbf{A}$ codes a family of finite subsets of $\mathbf{N}$ if for each i, the set of j's such that $(i,j) \in \mathbf{A}$ is finite. The intuitive meaning of $(i,j) \in \mathbf{A}$ is that j is an element of the i^{th} subset. On the other hand, we may code a family of finite subsets of $\mathbf{N}$ by a sequence of integers, $\mathbf{C} = \langle c_i : i \in \mathbf{N} \rangle$, where c_i is the code for the i^{th} finite set. Thus each c_i is an element of $\mathbf{P}_{<\mathbf{N}}(\mathbf{N})$, the set of all codes for finite subsets of $\mathbf{N}$. When n is an element of the set coded by c_i, we will write $n \in \mathbf{X}_{c_i}$. This statement is actually a Σ^0_0 formula without set parameters. Although the ordered pair representation of a family of set is Δ^0_1 in the finite set representation, the converse relation does not hold. Thus, in $\mathbf{RCA}_0$, we must explicitly state which representation is used. Oddly enough, in this section the choice of representation makes no difference, but this takes some proof.

The following theorem was first proved by Q. Feng and S. Simpson.

Theorem 3.30: $(\mathbf{RCA}_0)$ The following are equivalent:

i) $\mathbf{ACA}_0$

ii) Let $\mathbf{A}$ be a set of ordered pairs coding a family of non-empty finite subsets of $\mathbf{N}$. Let $\mathbf{F}=<f_i : i \in \mathbf{N}>$ be a sequence of finite functions such that $\mathbf{Dom}(f_i)=\{j \in \mathbf{N} : j < i\}$, and for $j < i \in \mathbf{N}$, $(j, f_i(j)) \in \mathbf{A}$. Then there is a choice function f, such that for each $m \in \mathbf{N}$, there is an $n > m$ such that for all $t < m$, $f(t)=f_n(t)$.

iii) Let $\mathbf{C}=<c_i : i \in \mathbf{N}>$ be a sequence of codes for non-empty finite subsets of $\mathbf{N}$. Let $\mathbf{F}=<f_i : i \in \mathbf{N}>$ be a sequence of finite functions such that $\mathbf{Dom}(f_i)=\{j \in \mathbf{N} : j < i\}$, and for $j < i \in \mathbf{N}$, $f_i(j) \in \mathbf{X}_{c_i}$. Then there is a choice function f such that for each $m \in \mathbf{N}$, there is an $n > m$ such that for all $t < m$, $f(t)=f_n(t)$.

iv) (Rado's Selection Principle) Let $\mathbf{A}$ be a set of ordered pairs coding a family of non-empty finite subsets of $\mathbf{N}$. Let $\mathbf{F}=\{f_t : t \in \mathbf{P}_{<\mathbf{N}}(\mathbf{N})\}$ be a set of finite functions such that $\mathbf{Dom}(f_t)=\mathbf{X}_t$, and for all $t \in \mathbf{P}_{<\mathbf{N}}(\mathbf{N})$ and $i \in \mathbf{X}_t$, $(i, f_t(i)) \in \mathbf{A}$. Then there is a choice function f such that for each $s \in \mathbf{P}_{<\mathbf{N}}(\mathbf{N})$, there is a $t \in \mathbf{P}_{<\mathbf{N}}(\mathbf{N})$ such that $\mathbf{X}_s \subseteq \mathbf{X}_t$ and for all $i \in \mathbf{X}_s$, $f_t(i)=f(i)$.

v) Let $\mathbf{C}=<c_i : i \in \mathbf{N}>$ be a sequence of codes for non-empty finite subsets of $\mathbf{N}$. Let $\mathbf{F}=\{f_t : t \in \mathbf{P}_{<\mathbf{N}}(\mathbf{N})\}$ be a set of finite functions such that for all

$t \in \mathbf{P}_{<\mathrm{N}}(\mathbf{N})$, $\mathbf{Dom}(f_t)=\mathbf{X}_t$, and for all $i \in \mathbf{X}_t$, $f_t(i) \in \mathbf{X}_{c_i}$. Then there is a choice function f such that for each $s \in \mathbf{P}_{<\mathrm{N}}(\mathbf{N})$ there is a $t \in \mathbf{P}_{<\mathrm{N}}(\mathbf{N})$ such that $\mathbf{X}_s \subseteq \mathbf{X}_t$ and for all $i \in \mathbf{X}_s$, $f_t(i)=f(i)$.

Proof: By the previous comments concerning coding of families of finite sets, it is clear that ii) implies iii), and iv) implies v). Since every finite subset of $\mathbf{N}$ is contained in some initial segment, it is easy to show that ii) implies iv), and iii) implies v). We will complete the proof by showing that i) implies ii), and v) implies i).

To show that i) implies ii), assume $\mathbf{ACA}_0$ and let $\mathbf{A}$ and $\mathbf{F}$ be as in ii). We define a tree $\mathbf{T} \subseteq \mathbf{Seq}_{<\mathrm{N}}$ by:

$$\sigma \in \mathbf{T} \text{ if and only if } \forall i < \mathrm{lh}(\sigma)\ \exists j\ \forall t < i\ (\sigma(t)=f_j(t)).$$

$\mathbf{T}$ is Σ^0_1 in $\mathbf{F}$, so $\mathbf{ACA}_0$ proves the existence of $\mathbf{T}$. Since $\mathbf{F}$ is infinite, so is $\mathbf{T}$. Furthermore, since $\{j \in \mathbf{N}:(i,j) \in \mathbf{A}\}$ is finite for each i, and $f_n(i)=j$ implies $(i,j) \in \mathbf{A}_j$, $\mathbf{T}$ is finitely branching. By Theorem 1.3, we may apply König's lemma to $\mathbf{T}$. The resulting infinite path is the desired choice function.

By Theorem 1.4, to show that v) implies i), it suffices to use v) to define the range of an injection g. Let $\mathbf{C}=<c_i : i \in \mathbf{N}>$ be the sequence given by $\mathbf{X}_{c_i}=\{0,1\}$ for all $i \in \mathbf{N}$. For $t \in \mathbf{P}_{<\mathrm{N}}(\mathbf{N})$, we define f_t for $i \in \mathbf{X}_t$ by:

$$f_t(i)=1 \text{ if and only if } \exists k < \max(\mathbf{X}_t)\ g(k)=i.$$

Thus $\mathbf{F}=\{f_t : t \in \mathbf{P}_{<\mathrm{N}}(\mathbf{N})\}$ is Δ^0_1 in g, so $\mathbf{F}$ exists. Let f be the choice function guaranteed by v). If $g(t)=i$, then for some k such that $\{t,i\} \subseteq \mathbf{X}_k$, $f(i)=f_k(i)=1$. If $i \notin \mathbf{Ran}(g)$, then for every $k \in \mathbf{P}_{<\mathrm{N}}(\mathbf{N})$, if $i \in \mathbf{X}_k$, then $f_k(i)=0$, and so $f(i)=0$. Thus $\mathbf{Ran}(g)=\{n \in \mathbf{N}: f(n)=1\}$. ■

The preceding theorem can be modified by introducing a bounding function. The bounding function weakens the selection principle by reducing the variation among the local choice functions. The resulting statements are equivalent to $\mathbf{WKL}_0$.

Theorem 3.31: $(\mathbf{RCA}_0)$ The following are equivalent:

i) $\mathbf{WKL}_0$

ii) Let $\mathbf{A}$ be a set of ordered pairs coding a family of non-empty finite subsets of $\mathbf{N}$. Let $\mathbf{F}=<f_i:i\in\mathbf{N}>$ be a sequence of finite functions such that $\mathbf{Dom}(f_i)=\{j\in\mathbf{N}:j<i\}$, and for $j<i\in\mathbf{N}$, $(j,f_i(j))\in\mathbf{A}$. Let h be a bounding function such that for all $i,j\in\mathbf{N}$, there is a $k<h(i)$ such that for all $t<i$, $f_j(t)=f_k(t)$. Then there is a choice function f, such that for each $m\in\mathbf{N}$, there is an $n>m$ such that for all $t<m$, $f(t)=f_n(t)$.

iii) Let $\mathbf{C}=<c_i:i\in\mathbf{N}>$ be a sequence of codes for non-empty finite subsets of $\mathbf{N}$. Let $\mathbf{F}=<f_i:i\in\mathbf{N}>$ be a sequence of finite functions such that $\mathbf{Dom}(f_i)=\{j\in\mathbf{N}:j<i\}$, and for $j<i\in\mathbf{N}$, $f_i(j)\in\mathbf{X}_{c_j}$. Let h be a bounding function such that for all $i,j\in\mathbf{N}$, there is a $k<h(i)$ such that for all $t<i$, $f_j(t)=f_k(t)$. Then there is a choice function f such that for each $m\in\mathbf{N}$, there is an $n>m$ such that for all $t<m$, $f(t)=f_n(t)$.

iv) (Weak Rado's Selection Principle) Let $\mathbf{A}$ be a set of ordered pairs coding a family of non-empty finite subsets of $\mathbf{N}$. Let $\mathbf{F}=\{f_t:t\in\mathbf{P}_{<\mathbf{N}}(\mathbf{N})\}$ be a set of finite functions such that $\mathbf{Dom}(f_t)=\mathbf{X}_t$, and for all $t\in\mathbf{P}_{<\mathbf{N}}(\mathbf{N})$ and $i\in\mathbf{X}_t$, $(i,f_t(i))\in\mathbf{A}$. Let h be a bounding function such that for all $s,t\in\mathbf{P}_{<\mathbf{N}}(\mathbf{N})$ with $\mathbf{X}_s\subseteq\mathbf{X}_t$, there is a $u<h(s)$ such that $\mathbf{X}_s\subseteq\mathbf{X}_u$ and for all $i\in\mathbf{X}_s$, $f_t(i)=f_u(i)$. Then there is a choice function f such that for each

$s \in \mathbf{P}_{<\mathbf{N}}(\mathbf{N})$, there is a $t \in \mathbf{P}_{<\mathbf{N}}(\mathbf{N})$ such that $\mathbf{X}_s \subseteq \mathbf{X}_t$ and for all $i \in \mathbf{X}_s$, $f_t(i)=f(i)$.

v) Let $\mathbf{C}=<c_i : i \in \mathbf{N}>$ be a sequence of codes for non-empty finite subsets of $\mathbf{N}$. Let $\mathbf{F}=\{f_t : t \in \mathbf{P}_{<\mathbf{N}}(\mathbf{N})\}$ be a set of finite functions such that for all $t \in \mathbf{P}_{<\mathbf{N}}(\mathbf{N})$, $\mathbf{Dom}(f_t)=\mathbf{X}_t$, and for all $i \in \mathbf{X}_t$, $f_t(i) \in \mathbf{X}_{c_i}$. Let h be a bounding function such that for all $s, t \in \mathbf{P}_{<\mathbf{N}}(\mathbf{N})$ with $\mathbf{X}_s \subseteq \mathbf{X}_t$, there is a $u < h(s)$ such that $\mathbf{X}_s \subseteq \mathbf{X}_u$ and for all $i \in \mathbf{X}_s$, $f_t(i)=f_u(i)$. Then there is a choice function f such that for each $s \in \mathbf{P}_{<\mathbf{N}}(\mathbf{N})$ there is a $t \in \mathbf{P}_{<\mathbf{N}}(\mathbf{N})$ such that $\mathbf{X}_s \subseteq \mathbf{X}_t$ and for all $i \in \mathbf{X}_s$, $f_t(i)=f(i)$.

Proof: As in the proof of Theorem 3.30, ii) implies iii), iv) implies v), ii) implies iv), and iii) implies v) are straightforward.

The proof that i) implies ii) is essentially the same as in Theorem 3.30. We will point out where the function h is of importance. First, h bounds the existential quantifier in the definition of $\mathbf{T}$, so $\mathbf{T}$ is Σ_0^0 in $\mathbf{F}$. Thus $\mathbf{RCA}_0$ suffices to prove the existence of $\mathbf{T}$. Next, there is a function g which is Δ_1^0 in h such that for $\sigma \in \mathbf{T}$, $\sigma(i)=j$ implies $j < g(i)$. It is noteworthy that $(i,j) \in \mathbf{A}$ does not necessarily imply that $j < g(i)$, so although g provides the needed bound on $\mathbf{T}$, it says very little about $\mathbf{A}$. The existence of g allows the use of Theorem 1.1 to find the desired path through $\mathbf{T}$.

The proof that v) implies i) is necessarily different from that of Theorem 3.30. This part of the proof also serves to link the selection principle directly to a marriage theorem. Let $\mathbf{R}$ be an infinite marriage problem with bounding function h, set of boys $\mathbf{B}$ (identified with $\mathbf{N}$), and set of girls $\mathbf{G}$. Suppose that every boy knows finitely many girls and that **R** satisfies condition **H**. We will use v) to find a solution to this marriage problem. Let $\mathbf{C}=<c_i : i \in \mathbf{N}>$ be the sequence given by $\mathbf{X}_{c_b}=\{g : (b, g) \in \mathbf{R}\}$. **C** is Δ_1^0 in **R** and h. Let $t \in \mathbf{P}_{<\mathrm{N}}(\mathbf{N})$ and $<b_i>_{i \leq k}$ be an increasing enumeration of the elements of $\mathbf{X}_t$. We define f_t by the following:

1) If b_k is the j^{th} boy who knows some girl whose address is greater than $h(b_i)$ for all $i<k$, and the marriage problem restricted to $\{b_i : i<k\}$ has at least j solutions, then $<f_t(b_i)>_{i<k}$ is the j^{th} solution (in the lexicographical ordering) and $f_t(b_k)$ is the least n such that $n>h(b_i)$ for all $i<k$ and $(b_k, n) \in \mathbf{R}$.

2) In any other case, $<f_t(b_i)>_{i<k}$ is the (lexicographically) least solution to the marriage problem restricted to $\{b_i : i \leq k\}$.

The above definition, convoluted as it is, serves two purposes. First, it insures that $\mathbf{F}=<f_t : t \in \mathbf{P}_{<\mathrm{N}}(\mathbf{N})>$ is Δ_1^0 in **R** and h. Secondly, it makes sure that every solution to a subset of the marriage problem is included in **F** by some easily discernible point. In particular, there is a function $\bar{h}$ which is Δ_1^0 in h and **R** such that for any s and t in $\mathbf{P}^{<\mathrm{N}}(\mathbf{N})$, if $\mathbf{X}_s \subseteq \mathbf{X}_t$, then it is possible to find a $k<\bar{h}(t)$ such that $\forall i \in \mathbf{X}_s \; f_t(i)=f_k(i)$.

Applying v) to **C**, **F**, and $\bar{h}$ yields a choice function f. Since every local choice function f_t is an injection, so is f. Thus f is a solution to the marriage problem. By Theorem 2.3, this suffices to prove i). ■

The previous theorems have many recursion theoretic implications. The following three results are characteristic.

Porism 3.32: There are recursive sets **A** and **F** satisfying the hypotheses of Theorem 3.30 v) such that $0'$ is recursive in any choice function satisfying the conclusion of Theorem 3.30 v).

Proof: In the proof of v) implies i) for Theorem 3.30, take g to be any recursive injection with $0'$ recursive in its range. ■

Porism 3.33: There are recursive sets **F** and **A** and a recursive function h satisfying the hypotheses of Theorem 3.31 v), such that no function f satisfying the conclusion of Theorem 3.31 v) can be expressed as a finite Boolean combination of recursively enumerable sets. In particular, no such f is recursive.

Proof: In the proof of v) implies i) for Theorem 3.31, use the recursively bounded recursive marriage problem of Porism 2.7. ■

Corollary 3.34: For any recursive sets **F** and **A** and recursive function h satisfying the hypotheses of Theorem 3.31 ii), there is a function f satisfying the conclusion of Theorem 3.31 ii) which has degree $\mathbf{a}$, where $\mathbf{a}' \leq 0'$.

Proof: By Theorem 3.31, such an f exists in any ω-model of $\mathbf{WKL}_0$. By Theorem 1.6, there is an ω-model of $\mathbf{WKL}_0$ whose set universe consists only of sets of low degree. ■

CHAPTER 4

BOOLEAN RINGS

This chapter explores various aspects of countable Boolean rings. Although this topic appears to be purely algebraic, it is linked to combinatorics. For instance, Boolean rings are used in the algebraic version of Hindman's theorem presented in Chapter 7.

Although every Boolean algebra is a Boolean ring, the development here is not a frivolous generalization. As will be seen in Chapter 7, the most natural algebraic reformulation of Hindman's theorem is in terms of rings, not algebras. Furthermore, certain Boolean rings without identities can be represented as rings of finite sets. Such rings have a particularly revealing representation, as is shown in Theorem 4.10.

4.1. Basic Facts

In this section, we will define Boolean rings and prove a few basic facts. Throughout this chapter, we will use the term Boolean ring to refer to a countable Boolean ring. Note that the following definition does not require the ring to have a multiplicative identity. Indeed, it is easy to show in $\mathbf{RCA}_0$ that any Boolean ring with a multiplicative identity is a Boolean algebra. (By Boolean algebra, we mean a structure where addition corresponds to symmetric difference, not to union. Both axiomatizations are common in the literature.)

Definition 4.1: A (countable) Boolean ring consists of a set of elements, $\mathbf{R}$, and two operations (coded as sets of triples) denoted by $+$ and $*$, such that

i) $<\mathbf{R},+>$ is an abelian group with designated identity, 0.

ii) $<\mathbf{R},*>$ is a semigroup.

iii) The distributive laws, $a*(b+c)=(a*b)+(a*c)$, and $(b+c)*a=(b*a)+(c*a)$, hold for all $a,b,c\in\mathbf{R}$.

iv) Every element is a multiplicative idempotent, i.e. $\forall r\in\mathbf{R}(r*r=r)$.

Some interesting properties of Boolean rings follow immediately from the definition.

Lemma 4.2: $(\mathbf{RCA_0})$ Let $<\mathbf{R},+,*>$ be a Boolean ring. Then:

i) $\mathbf{R}$ has characteristic 2, i.e. $\forall r\,(r+r=0)$, and

ii) $\mathbf{R}$ is commutative, i.e. $\forall r\,\forall s\,(r*s=s*r)$.

Proof: To prove i), fix $r\in\mathbf{R}$. Then

$$\begin{aligned} r+r &= (r+r)*(r+r) \\ &= r^2+r^2+r^2+r^2 \\ &= r+r+r+r. \end{aligned}$$

Since $<\mathbf{R},+>$ is a group, $0=r+r$.

To prove ii), fix $r,s\in\mathbf{R}$. Then

$$\begin{aligned} r+s &= (r+s)*(r+s) \\ &= r^2+(s*r)+(r*s)+s^2 \\ &= r+(s*r)+(r*s)+s \end{aligned}$$

So, $0=(s*r)+(r*s)$. Adding $r*s$ to both sides and applying i) yields the desired result. ■

4.2. Atoms

An element r of a Boolean ring **R** is called an atom if for every $s \in \mathbf{R}$, $r * s \in \{0, r\}$. Much of the structure of a Boolean ring can be determined by looking at its atoms. Under certain circumstances, it is comparatively easy to find the atoms of a Boolean ring. The following definition gives a representation of some Boolean rings with very simple sets of atoms.

Definition 4.3: ($\mathbf{RCA_0}$) Let **R** be a set of (codes for) finite sets which is closed under intersection and symmetric difference. **R** together with the operations $+$ (symmetric difference) and $*$ (intersection) is called a Boolean ring of finite sets.

Given a Boolean ring of finite sets, it is a simple matter to find all the atoms. The following lemma and its corollary indicate this.

Lemma 4.4: ($\mathbf{RCA_0}$) Let **R** be a Boolean ring of finite sets. The set of all atoms of **R** exists.

Proof: Let **R** be a Boolean ring of finite sets. For a code $t \in \mathbf{R}$, let $\mathbf{X}_t$ denote the set coded by t. We assume a coding where $\mathbf{X}_t \subseteq \mathbf{X}_s$ implies $t \leq s$. The set of atoms of **R** is defined by

$$\mathbf{A} = \{t \in \mathbf{R} : \forall s < t\, (\mathbf{X}_s \subseteq \mathbf{X}_t \rightarrow s \notin \mathbf{R})\}.$$

Since **A** is Δ_1^0 in **R**, **A** exists. ■

Corollary 4.5: If **R** is a recursive Boolean ring of finite sets, then the set of atoms of **R** is recursive.

Proof: Let $<\omega, \mathbf{Rec}>$ denote the standard minimal model of $\mathbf{RCA_0}$. If **R** is recursive, then **R** is in $<\omega, \mathbf{Rec}>$, so by Lemma 4.4, the set of atoms of **R** is in $<\omega, \mathbf{Rec}>$. Thus the set of atoms is recursive. ■

For the general case, it takes $\mathbf{ACA}_0$ to prove the existence of the set of atoms. The following theorem of reverse mathematics confirms this.

Theorem 4.6: ($\mathbf{RCA}_0$) The following are equivalent.

i) $\mathbf{ACA}_0$.

ii) If $\mathbf{R}$ is a Boolean ring, then the set of all atoms of $\mathbf{R}$ exists.

iii) If $\mathbf{R}$ is a Boolean ring with infinitely many atoms, then an infinite subset of the atoms exists.

Proof: To prove that i) implies ii), let $<r_i>_{i\in\mathrm{N}}$ be an enumeration of the Boolean ring $\mathbf{R}$. The set of atoms of $\mathbf{R}$ is defined by

$$\mathbf{A}=\{r_i\in\mathbf{R}:\forall j\,(r_i * r_j\in\{r_i,0\})\}.$$

Since $\mathbf{A}$ is Π_1^0 in $\mathbf{R}$, $\mathbf{ACA}_0$ proves the existence of $\mathbf{A}$.

Since iii) follows immediately from ii), it remains only to show that iii) implies i). Let $f:\mathbf{N}\rightarrow\mathbf{N}$ be a total 1–1 function. By Theorem 1.4, it suffices to use iii) to show that the range of f exists. This can be done in three steps. First, we must construct the appropriate Boolean ring $\mathbf{R}$. Secondly, we must prove that $\mathbf{R}$ is indeed a Boolean ring with infinitely many atoms. Thirdly, given an arbitrary infinite subset of the atoms of $\mathbf{R}$, we must somehow decode the range of f.

The construction of $\mathbf{R}$ is necessarily somewhat messy. Intuitively, $\mathbf{R}$ is a Tychonoff product of infinitely many finite Boolean algebras. The structure of the n^{th} Boolean algebra codes all the information needed to retrieve $\mathbf{Ran}(f)\cap(n+1)$. Thus each element of the n^{th} Boolean algebra is a triple giving the number of the

Boolean algebra, information about f , and a set representing the element. Let S denote the elements of the finite Boolean algebras. Formally, $\mathbf{S}\subseteq\mathbf{N}\times\mathbf{N}\times\mathbf{P}_{<\mathbf{N}}(\mathbf{N})$ and $(n,k,\mathbf{X})\in\mathbf{S}$ if and only if the following four conditions hold.

1) $f(k)\leq n$.

2) $\emptyset\neq\mathbf{X}\subseteq\{0,1,\ldots,2^n-1\}$.

3) If $c=|\{y\leq n:\exists x\leq k f(x)=y\}|$, then for some $\mathbf{J}\subseteq\{0,1,\ldots,2^{c-1}-1\}$,

$$\mathbf{X}=\{m<2^n:\exists j\in\mathbf{J}\quad m=j \mod 2^{c-1}\}.$$

4) If c is as in 3) and $c>1$, then for all d such that $1\leq d<c$, there is no $\mathbf{J}\subseteq\{0,1,\ldots,2^{d-1}-1\}$ such that

$$\mathbf{X}=\{m<2^n:\exists j\in\mathbf{J}\quad m=j \mod 2^{d-1}\}.$$

Intuitively, for each n and k, $\{\emptyset\}\cup\{\mathbf{X}\subseteq 2^n:\exists j\leq k\,(n,j,\mathbf{X})\in\mathbf{S}\}$ defines a subalgebra of 2^n elements. Condition 1) insures proper coding of the information about f. Conditions 2) and 3) guarantee that at each stage k, a subalgebra is constructed. Condition 4) prevents any element of the n^{th} algebra from being associated with several values for k.

The underlying set for the Boolean ring **R** is easily defined in terms of S. For all $\mathbf{Y}\in\mathbf{P}_{<\mathbf{N}}(\mathbf{S})$, $\mathbf{Y}\in\mathbf{R}$ if and only if for all distinct $(n_1,k_1,\mathbf{X}_1)$ and $(n_2,k_2,\mathbf{X}_2)$ in **Y**, $n_1\neq n_2$. **R** can be viewed as the set of functions with finite support mapping **N** into the direct product of the finite algebras.

The operations on **R** are carried out component-wise in terms of the finite algebras. If $(n,k_1,\mathbf{X}_1)$ and $(n,k_2,\mathbf{X}_2)$ are elements of **S** with identical first components such that $\mathbf{X}_1\cap\mathbf{X}_2\neq\emptyset$, then

$$(n, k_1, \mathbf{X}_1) * (n, k_2, \mathbf{X}_2) = (n, j, \mathbf{X}_1 \cap \mathbf{X}_2)$$

where j is given by

$$j = \mu t \leq \max(k_1, k_2)((n, j, \mathbf{X}_1 \cap \mathbf{X}_2) \in \mathbf{S}).$$

A close examination of the definition of $\mathbf{S}$ reveals that such a j always exists. Addition is defined using symmetric difference, that is, if $\mathbf{X}_1 \triangledown \mathbf{X}_2 \neq \emptyset$, then

$$(n, k_1, \mathbf{X}_1) + (n, k_2, \mathbf{X}_2) = (n, j, \mathbf{X}_1 \triangledown \mathbf{X}_2),$$

where j is given by

$$j = \mu t \leq \max(k_1, k_2)((n, j, \mathbf{X}_1 \triangledown \mathbf{X}_2) \in \mathbf{S}).$$

Again, such a j always exists. If $\mathbf{X}_1 \cap \mathbf{X}_2 = \emptyset$ (respectively $\mathbf{X}_1 \triangledown \mathbf{X}_2 = \emptyset$) then the product (respectively sum) is undefined in the sense that $\{(n, k_1, \mathbf{X}_1) * (n, k_2, \mathbf{X}_2)\} = \emptyset$ (respectively $\{(n, k_1, \mathbf{X}_1) + (n, k_2, \mathbf{X}_2)\} = \emptyset$).

Now we can define the operations on $\mathbf{R}$. Let $\mathbf{Y}, \mathbf{Z} \in \mathbf{R}$, and let

$$m = \max \{n \in \mathbf{N} : \exists t \, \exists \mathbf{X} \ (n, t, \mathbf{X}) \in \mathbf{Y} \cup \mathbf{Z}\}.$$

For each $n \leq m$, let $\mathbf{Y}_n$ (respectively $\mathbf{Z}_n$) denote the singleton set containing the element of $\mathbf{Y}$ (respectively $\mathbf{Z}$) with first component n, if such an element exists, and $\emptyset$ otherwise. If $\mathbf{Y}_n \neq \emptyset$, write $\mathbf{Y}_n = \{y\}$ and similarly, write $\mathbf{Z}_n = \{z\}$ if $\mathbf{Z}_n \neq \emptyset$. Define addition in $\mathbf{R}$ by

$$\mathbf{Y} + \mathbf{Z} = \bigcup_{n \leq m} (\mathbf{Y}_n \dotplus \mathbf{Z}_n),$$

where $\dotplus$ is defined by

$$\mathbf{Y}_n \hat{+} \mathbf{Z}_n = \begin{cases} \{y+z\} \text{ if } \mathbf{Y} \neq \emptyset \wedge \mathbf{Z} \neq \emptyset, \\ \mathbf{Y}_n \cup \mathbf{Z}_n \text{ otherwise.} \end{cases}$$

Define multiplication in $\mathbf{R}$ by

$$\mathbf{Y} * \mathbf{Z} = \bigcup_{n \leq m} (\mathbf{Y}_n \hat{*} \mathbf{Z}_n),$$

where $\hat{*}$ is defined by

$$\mathbf{Y}_n \hat{*} \mathbf{Z}_n = \begin{cases} \{y * z\} \text{ if } \mathbf{Y} \neq \emptyset \wedge \mathbf{Z} \neq \emptyset, \\ \emptyset \text{ otherwise.} \end{cases}$$

Since $y, z \in \mathbf{S}$, these functions are well defined. It is straightforward but tedious to show that $\mathbf{R}$ and its operations are Δ_1^0 definable in f.

Next we need to show that $\mathbf{R}$ together with $+$ and $*$ is a Boolean ring with infinitely many atoms. For the most part, this is left to the ambitious reader. It is clear that it is sufficient to prove that the operations act appropriately on each component. Once one accepts that the bookkeeping components in each triple behave nicely, this is simply a verification that the finite subalgebras of the form $\{\mathbf{X} \subseteq 2^n : \exists t\,(n, t, \mathbf{X}) \in \mathbf{S}\} \cup \{\emptyset\}$ are Boolean rings. Furthermore, it is clear that any atom consists of exactly one triple from $\mathbf{S}$. An element $(n, k, \mathbf{X}) \in \mathbf{S}$ is (the sole element of) an atom if and only if $\forall t > k f(t) > n$, and for some $i < 2^{c-1}$, $\mathbf{X} = \{j < 2^n : j = i \bmod 2^{c-1}\}$ where $c = |\{y \leq n : \exists x f(x) = y\}|$.

The simple nature of the atoms of $\mathbf{R}$ makes the final step of the proof easy. Let $\mathbf{A} \subseteq \mathbf{R}$ be an infinite subset of the atoms of $\mathbf{R}$. Since there are only a finite number of atoms with any fixed first component, we can find a sequence $< a_i >_{i \in \mathrm{N}} \subseteq \mathbf{A}$ such that for each i, $a_i = (n_i, k_i, \mathbf{X}_i)$ and $n_i \geq i$. Since each a_i is an atom, $\forall t > k_i (f(t) > n_i)$ for each $i \in \mathbf{N}$. Thus for each $i \in \mathbf{N}$, $i \in \mathbf{Ran}(f)$ if and only if $\exists t \leq k_i (f(t)=i)$. By Δ_1^0 comprehension, if an infinite subset of the atoms of $\mathbf{R}$ exists, then so does $\mathbf{Ran}(f)$, as desired. ■

The following Porism points out a facet of the proof of Theorem 4.6 which will be used in the next section.

Porism 4.7: ($\mathbf{RCA_0}$) The following are equivalent.

i) $\mathbf{ACA_0}$.

ii) If $\mathbf{R}$ is a Boolean ring with infinitely many atoms such that each element can be expressed as a finite sum of atoms, then the set of atoms of $\mathbf{R}$ exists.

Proof: The Boolean ring constructed in the proof of Theorem 4.6 satisfies the hypothesis of ii). ■

4.3. Versions of Stone's Theorem

In this section we will examine two versions of Stone's representation theorem. Although both statements are easily proven in $\mathbf{ACA_0}$, no reversals have been found. The statement of the first version uses the following definition.

Definition 4.8: ($\mathbf{RCA_0}$) A Boolean ring of sets consists of a sequence $<f_i>_{i \in N}$ of 0-1 functions which is closed under $*$ and $+$, where these operations are defined as follows:

1. $(f_i + f_j)(k) = \begin{cases} 1 \text{ if } f_i(k) \neq f_j(k) \\ 0 \text{ otherwise} \end{cases}$

2. $(f_i * f_j)(k) = f_i(k) * f_j(k)$.

Intuitively, the functions are characteristic functions of sets in the Boolean ring. The operations $+$ and $*$ correspond to symmetric difference and intersection, respectively. Ideally, we would now prove that every Boolean ring is isomorphic to a Boolean ring of sets. Unfortunately, no proof of this statement in a subsystem of $\mathbf{Z}_2$ is known. We can, however, prove the result for atomic Boolean rings. An atomic Boolean ring is one in which every element contains an atom.

Theorem 4.9: ($\mathbf{ACA}_0$) Every atomic Boolean ring is isomorphic to a Boolean ring of sets.

Proof: The proof consists of mapping each element of the Boolean ring to its set of atoms. Let $< r_i >_{i \in \mathrm{N}}$ be an enumeration of the atomic Boolean ring. By Theorem 4.6, we can find an enumeration $< a_i >_{i \in \mathrm{N}}$ of the atoms of $\mathbf{R}$. The sequence of functions $< f_i >_{i \in \mathrm{N}}$ is defined by

$$f_i(j) = \begin{cases} 1 \text{ if } r_i * a_j = a_j \\ 0 \text{ otherwise.} \end{cases}$$

It is easy to verify that $\phi : r_i \rightarrow f_i$ is an isomorphism of $\mathbf{R}$ onto the Boolean ring of sets $< f_i >_{i \in \mathrm{N}}$. ■

As was observed in the previous section, some Boolean rings have a more tractable representation. In general, it is much easier to work with Boolean rings of finite sets than with Boolean rings of sets. The following theorem characterizes those Boolean rings which are isomorphic to Boolean rings of finite sets. Furthermore, it gives a weak version of Stone's theorem which is equivalent to $\mathbf{ACA}_0$.

Theorem 4.10: ($\mathbf{RCA}_0$) The following are equivalent:

i) $\mathbf{ACA}_0$.

ii) A Boolean ring $\mathbf{R}$ is isomorphic to a Boolean ring of finite sets if and only if every element of $\mathbf{R}$ can be expressed as a finite sum of atoms.

Proof: To prove that i) implies ii), first let $\mathbf{R}=<r_i>_{i\in N}$ be an enumeration of a Boolean ring in which every element can be expressed as a finite sum of atoms. By Theorem 4.6, we can find an enumeration $<a_i>_{i\in N}$ of the atoms of **R**. Define $\mathbf{S}=<s_i>_{i\in N}$ by letting s_i be the code for the set $\{j\in \mathbf{N}: r_i * a_j = a_j\}$. Clearly, **S** is a Boolean ring of finite sets, and $\phi(r_i)=s_i$ is an isomorphism of **R** onto **S**.

To complete the proof that i) implies ii), suppose that ϕ is an isomorphism of **R** onto a Boolean ring of finite sets **S**. Fix $r\in\mathbf{R}$. Let $s\in\mathbf{S}$ be the image of r under ϕ, and let $\phi(a_1), \ldots, \phi(a_n)$ be the atoms of **S** contained in (the set coded by) s. Since s is finite and **S** is closed under symmetric difference, $s=\phi(a_1)+\cdots+\phi(a_n)$. Since ϕ is an isomorphism, $r=a_i+\cdots+a_n$ is a decomposition of r into finitely many atoms.

To prove that ii) implies i), let **R** be a Boolean ring with infinitely many atoms, such that every element can be expressed as a finite sum of atoms. By ii), **R** is isomorphic to a Boolean ring of finite sets. By Lemma 4.4, the set of all atoms of **R** exists. By Porism 4.7, this implies $\mathbf{ACA_0}$. ■

4.4. Zero Divisors

The following three lemmas are needed to prove the results on Hindman's theorem presented in Chapter 7. They all concern the existence of sequences of pairwise zero divisors in Boolean rings.

Lemma 4.11: $(\mathbf{RCA_0})$ Let **R** be an infinite Boolean ring of finite sets. Then there is a sequence $<z_i>_{i\in N}$ of elements of **R** which are distinct pairwise zero divisors.

Proof: Let $<r_i>_{i\in \mathrm{N}}$ be an enumeration of a subset of **R** such that $i<j$ implies $\max(r_i) < \max(r_j)$. Since **R** is an infinite collection of finite sets, by $\mathrm{I}\Sigma_1^0$, such a sequence exists. Define z_i for $i \in \mathbf{N}$ by

$$z_i = r_i + \sum_{k<i} r_i * z_k.$$

Since $k<i$ implies $\max(r_k) < \max(r_i)$, $r_i \neq \sum_{k<i} r_i * z_k$, so for each i, $z_i \neq 0$. Furthermore, if $z_i * z_j = 0$ for all $i, j < k$ such that $i \neq j$, then

$$\begin{aligned} z_j * z_k &= z_j * (r_k + \sum_{i<k} r_k * z_i) \\ &= z_j * r_k + z_j^2 * r_k + \sum_{i<k, i\neq j} r_k * z_i * z_j \\ &= z_j * r_k + z_j^2 * r_k \\ &= z_j * r_k + z_j * r_k \\ &= 0. \end{aligned}$$

So by $\mathrm{I}\Sigma_1^0$, $<z_i>_{i\in \mathrm{N}}$ is a sequence of distinct zero divisors. ■

Lemma 4.12: (**RCA$_0$**) Let **R** be an infinite Boolean ring with finitely many atoms $<a_i>_{i\leq n}$. Then **R** contains an infinite sequence of distinct pairwise zero divisors.

Proof: Let $<r_i>_{i\in \mathrm{N}}$ be an enumeration without repetitions of the elements of **R**. Let $t_i = r_i + \sum_{j\leq n} r_i * a_j$ for all $i \in \mathbf{N}$. Then for any $j \leq n$ and any $i \in \mathbf{N}$, $t_i * a_j = 0$. Let $<u_i>_{i\in \mathrm{N}}$ be an enumeration without repetitions of the t_i such that $t_i \neq 0$. There are infinitely many u_i. Let $v_1 = u_1$. Suppose that v_j is defined for all $j<k$. Define v_{k+1} by $v_{k+1} = v_k * u_n$, where $n = \mu j (v_k * u_j \notin \{v_k, 0\})$. Such a j always exists by definition of u_i and the fact that there are only finitely many atoms. Note

that if $j \leq k$, $v_j * v_k = v_k$. Finally, define $<z_i>_{i \in N}$ by $z_j = v_j + v_{j+1}$. Note that for $j < k$,

$$\begin{aligned} z_j * z_k &= (v_j + v_{j+1}) * (v_k + v_{k+1}) \\ &= v_j * v_k + v_{j+1} * v_k + v_j * v_{k+1} + v_{j+1} * v_{k+1} \\ &= v_k + v_k + v_{k+1} + v_{k+1} \\ &= 0. \end{aligned}$$

Since $*$ is commutative in **R**, this suffices to show that $z_j * z_k = 0$ for all $j, k \in N$ such that $j \neq k$. ■

Corollary 4.13: **(ACA_0)** Let **R** be an infinite Boolean ring. Then there is an infinite sequence of pairwise zero divisors in **R**.

Proof: If **R** has finitely many atoms, then apply Lemma 4.12. If **R** has infinitely many atoms, then by Theorem 4.6, the set of atoms exists. Such a set of atoms is a set of pairwise zero divisors. ■

CHAPTER 5

MODELS OF SUBSYSTEMS OF Z_2

Several interesting combinatorial results can be proved using model theoretic techniques. This chapter develops the basic tools necessary to carry out these proofs. The proofs and definitions of this chapter differ from those in other chapters in that they are not formulated in the language of $\mathbf{Z}_2$. The approach taken here is wildly different from that of constructive or recursive mathematics. The fact that some set theory is necessary to prove results does not inhibit our use of them.

The main techniques developed in this chapter involve the careful exploitation of the interplay between models of first and second order arithmetic. Several notational conventions simplify the presentation. The language of second order arithmetic is denoted by $\mathbf{L}_2$, while the restriction of this language to first order arithmetic is denoted by $\mathbf{L}_1$. Thus, $\mathbf{L}_1$ is $\mathbf{L}_2$ stripped of $\in$, set variables, and set quantifiers. Models of second order arithmetic are denoted by upper case Greek letters, while upper case Roman letters denote models of first order theories. Models and their domains are generally denoted by the same symbol.

It is important to note that fragments of $\mathbf{Z}_2$ can be viewed as two sorted first order theories. Consequently, standard model theoretic definitions and results apply to models of fragments of $\mathbf{Z}_2$. In particular, the definitions of isomorphism and elementary equivalence of models is as usual, and the downward Lowenheim Skolem theorem can be applied.

A model of a fragment of $\mathbf{Z}_2$ consists of a pair $\Gamma=<\mathbf{N}_\Gamma,\mathbf{S}_\Gamma>$, together with the interpretations of the relation, constant, and function symbols. Here $\mathbf{N}_\Gamma$ is the domain for the number variables, and $\mathbf{S}_\Gamma \subseteq \mathbf{P}(\mathbf{N}_\Gamma)$ is the domain of the set variables. We uniformly use $\mathbf{N}_\Gamma$ and $\mathbf{S}_\Gamma$ to denote the domains of a model Γ, and omit specific mention of the interpretations of relation, function, and constant symbols when they are clear. The model of first order arithmetic consisting of $\mathbf{N}_\Gamma$ together with the interpretations of the first order constants, relations, and functions is called the first order part of Γ. For example, the first order part of any model of $\mathbf{ACA}_0$ is a model of the first order Peano axioms.

The first order Peano axioms, denoted **PA**, consist of the first order basic axioms, denoted $\mathbf{P}^-$, together with the induction scheme for all purely first order formulas. We often consider fragments of this theory created by restricting the induction scheme to formulas of fixed complexity, for example Σ_1 formulas of $\mathbf{L}_1$. The following terminology is essentially that of Kirby and Paris [26]. Let $\mathbf{M}$ be a model of a fragment of **PA**. We write $\mathbf{I} \subseteq_e \mathbf{M}$ if $\mathbf{I}$ is a proper initial segment of $\mathbf{M}$. If $a \in \mathbf{M}$ and $a \notin \mathbf{I}$, we write $a > \mathbf{I}$. If $\mathbf{B}$ is a subset of $\mathbf{I}$, an element $g \in \mathbf{M}$ codes $\mathbf{B}$ if for any $a \in \mathbf{I}$, $a \in \mathbf{B}$ if and only if $\mathbf{M}$ models the formula stating that the a^{th} prime divides g. The set of all subsets of $\mathbf{I}$ coded by elements of $\mathbf{M}$ is denoted by $\mathbf{R}_\mathbf{M}\mathbf{I}$.

We will use the following hierarchies of collection schemas outlined by Kirby and Paris [27]. The scheme $\mathbf{I}\Sigma_n$ (Σ_n induction) consists of the universal closures of

$$(\theta(0) \wedge \forall x\, (\theta(x) \rightarrow \theta(x+1))) \rightarrow \forall x\, \theta(x)$$

where θ is a Σ_n formula of $\mathbf{L}_1$. The scheme $\mathbf{B}\Sigma_n$ (Σ_n collection) consists of the universal closures of

$$\forall x < y \exists z \, \theta(x, z) \rightarrow \exists t \, \forall x < y \exists z < t \, \theta(x, z)$$

where θ is a Σ_n formula of $\mathbf{L}_1$. The scheme $\mathbf{L}\Sigma_n$ (Σ_n least number principle) consists of the universal closures of

$$\exists x \, \theta(x) \rightarrow \exists x \, (\theta(x) \wedge \forall y < x \, \neg\theta(y))$$

for all Σ_n formulas of $\mathbf{L}_1$. The schemas $\mathbf{I}\Pi_n$, $\mathbf{B}\Pi_n$, and $\mathbf{L}\Pi_n$ are defined similarly. We will denote expansions of the schemas to include Σ_n^0 (respectively Π_n^0) formulas of $\mathbf{L}_2$ by $\mathbf{I}\Sigma_n^0$, $\mathbf{B}\Sigma_n^0$, and $\mathbf{L}\Sigma_n^0$ (respectively $\mathbf{I}\Pi_n^0$, $\mathbf{B}\Pi_n^0$, and $\mathbf{L}\Pi_n^0$). The following theorem concerning these schemas was proved by Kirby and Paris [27].

Theorem 5.1: Let $n \geq 0$. Suppose that $\mathbf{M} \models \mathbf{P}^- + \mathbf{I}\Sigma_0$. Then the following hold:

i) $\mathbf{M} \models \mathbf{I}\Sigma_n$ iff $\mathbf{M} \models \mathbf{I}\Pi_n$ iff $\mathbf{M} \models \mathbf{L}\Sigma_n$ iff $\mathbf{M} \models \mathbf{L}\Pi_n$.

ii) $\mathbf{M} \models \mathbf{B}\Sigma_{n+1}$ iff $\mathbf{M} \models \mathbf{B}\Pi_n$.

iii) $\mathbf{M} \models \mathbf{I}\Sigma_n$ implies $\mathbf{M} \models \mathbf{B}\Sigma_n$.

iv) $\mathbf{M} \models \mathbf{B}\Sigma_{n+1}$ implies $\mathbf{M} \models \mathbf{I}\Sigma_n$.

The proof of the preceding theorem is easily modified to prove the corresponding result for second order models and collection schemas.

The following terminology for initial segments of models of $\mathbf{L}_1$ is widely used. An initial segment $\mathbf{I}$ is semi-regular in $\mathbf{M}$ if $\langle \mathbf{I}, \mathbf{R}_\mathbf{M}\mathbf{I} \rangle \models \mathbf{I}\Sigma_1^0$. An initial segment $\mathbf{I}$ is regular in $\mathbf{M}$ if $\langle \mathbf{I}, \mathbf{R}_\mathbf{M}\mathbf{I} \rangle \models \mathbf{B}\Sigma_2^0$. An initial segment $\mathbf{I}$ is strong in $\mathbf{M}$ if $\mathbf{I}$ is semi-regular, and for every coded coloring of triples with two colors, there is a coded set cofinal in $\mathbf{I}$ which is monochromatic in the sense of Ramsey's theorem.

With this terminology and background in hand, we are prepared to begin our study of model theory.

5.1. Clones

The cornerstone of the development of model theory in this chapter is the notion of a clone. Two facts capture the essence of clones. First, a clone is simply a model of some second order theory. Secondly, the interesting aspect of a clone is the way in which it is built. A clone is always grown from a cut in a model of a first order theory. The following definition makes these ideas precise.

Definition 5.2: Let $\mathbf{M}$ be a model of a fragment of $\mathbf{PA}$ and let $\mathbf{I} \subseteq_e \mathbf{M}$. The second order model $<\mathbf{I},\mathbf{R_M I}>$ is called a clone. If $<\mathbf{I},\mathbf{R_M I}>$ is isomorphic to a model Γ, we say $<\mathbf{I},\mathbf{R_M I}>$ is a Γ-clone. If $<\mathbf{I},\mathbf{R_M I}>$ models a theory $\mathbf{T}$, we say $<\mathbf{I},\mathbf{R_M I}>$ is a $\mathbf{T}$-clone.

The definition of clone is so general that their existence is obvious. It is less obvious that interesting clones exist. The next two theorems show how to grow $\mathbf{WKL_0}$-clones and $\mathbf{ACA_0}$-clones from initial segments described by Kirby and Paris.

Theorem 5.3: Let $\mathbf{M}$ be a countable model of $\mathbf{P}^-$ and $\mathbf{I}\Sigma_0$. Let $\mathbf{I} \subseteq_e \mathbf{M}$ be a semi-regular initial segment of $\mathbf{M}$. Then $<\mathbf{I},\mathbf{R_M I}>$ is a $\mathbf{WKL_0}$-clone.

Proof: Let $\mathbf{I}$ and $\mathbf{M}$ be as in the statement of the theorem. Let $\Gamma = <\mathbf{I},\mathbf{R_M I}>$ be the resulting clone. Since $\mathbf{I}$ is semi-regular in $\mathbf{M}$, Γ models the basic axioms and induction for Σ_1^0 formulas. It remains only to show that Γ satisfies the desired comprehension axioms.

First, we show that if $\exists k \ \phi(k,n)$ is a Σ_1^0 formula with set parameters in $\mathbf{R_M I}$, then $\mathbf{R_M I}$ contains the code of a function f such that

$$\Gamma \models (\exists k \ \ \phi(k,n)) \equiv (\exists t \ \ (f\ (t)=n)).$$

Since $\mathbf{I}$ models $I\Sigma_1^0$, it is easy to show that for each $m \in \mathbf{I}$, there is a finite function f_c coded by $c \in \mathbf{I}$ such that

$$\mathbf{I} \models \forall n < m \ ((\exists k < m \ \ \phi(k,n)) \leftrightarrow (\exists t < n \ \ (f_c(t) = n))).$$

Pick $b_1 \in \mathbf{M}$ such that $b_1 > \mathbf{I}$. Then (for an appropriate first order translation of the formula)

$$\forall m \in \mathbf{I} \ \ \mathbf{M} \models \exists c < b_1 \forall n < m \ ((\exists k < m \ \ \phi(k,n)) \leftrightarrow (\exists t < n \ \ (f_c(t) = n))).$$

By Σ_0 induction in $\mathbf{M}$, there is a $b_2 > \mathbf{I}$ and a $c_1 < b_1$ such that

$$\mathbf{M} \models \forall m < b_2 \forall n < m \ (\exists k < m \ \ \phi(k,n)) \equiv (\exists t < m \ \ (f_{c_1}(t) = n)).$$

Thus c_1 codes a function f in $\mathbf{R_M I}$ mapping $\mathbf{I}$ to $\mathbf{I}$ so that

$$\Gamma \models \forall n \ (\exists k \ \ \phi(k,n)) \equiv (\exists t \ \ (f(t) = n)).$$

If $\Gamma \models \forall b \ \ \exists t \ \ (t > b \ \wedge \exists k \ \ \phi(k,t))$ then the above construction can be modified to insure that f is injective.

We now show that given two injections in $\mathbf{R_M I}$ with disjoint ranges, there is a coded set $\mathbf{X}$ which separates the ranges of the functions. Applying this to functions constructed as above shows that Γ models Δ_1^0-comprehension. Applying it to arbitrary injections yields that $\Gamma \models \mathbf{WKL}_0$. Let $f, g \in \mathbf{R_M I}$ be injections with disjoint ranges. Let c and d be the codes of f and g in $\mathbf{M}$. By $I\Sigma_0$ in $\mathbf{M}$, we can find some $b_1 > \mathbf{I}$ such that c and d code disjoint injections on $\{x : x < b_1\}$. For every $m \in \mathbf{I}$, $\mathbf{I}$ models the existence of a set $\mathbf{X}_k$ coded by $k \in \mathbf{I}$, such that

$$\mathbf{I} \models \forall n < m \ (((\exists t < m \ \ f(t) = n) \rightarrow n \in \mathbf{X}_k) \wedge ((\exists t < m \ \ g(t) = n) \rightarrow n \notin \mathbf{X}_k)).$$

By $I\Sigma_0$ in $\mathbf{M}$, there are elements $k, b_2 > \mathbf{I}$ such that (for an appropriate translation of the formula),

$$\mathbf{M} \models \forall n < b_2 \, (((\exists t < b_2 \, f(t) = n) \rightarrow n \in \mathbf{X}_k) \bigwedge ((\exists t < b_2 \, g(t) = n) \rightarrow n \notin \mathbf{X}_k)).$$

Since $\mathbf{X}_k \cap \mathbf{I} \in \mathbf{R_M I}$, Γ models that there is a set separating the ranges of f and g. ■

Theorem 5.4: Let **M** be a countable model of $\mathbf{P}^-$ and $\mathbf{I}\Sigma_0$. Let $\mathbf{I} \subseteq_e \mathbf{M}$ be a strong initial segment. Then $<\mathbf{I},\mathbf{R_M I}>$ is an $\mathbf{ACA}_0$-clone.

Proof: Let **M** and **I** be as stated. Let $\Gamma = <\mathbf{I},\mathbf{R_M I}>$. By Theorem 5.3, $\Gamma \models \mathbf{WKL}_0$. By the definition of strong initial segment, Γ models Ramsey's theorem for triples and two colors. By Theorem 1.5, this suffices to prove that $\Gamma \models \mathbf{ACA}_0$.

Theorems 5.3 and 5.4 do not actually prove the existence of $\mathbf{WKL}_0$-clones and $\mathbf{ACA}_0$-clones. However, existence of these clones follows immediately from the existence of semi-regular and strong initial segments. Using the results of Kirby and Paris [26] we can prove more than simple existence. The following corollary indicates that $\mathbf{ACA}_0$-clones not only exist, but are rather profuse.

Corollary 5.5: Let **M** be a countable model of **PA** and let $j \in \mathbf{M}$. Then there is an initial segment $\mathbf{I} \subseteq_e \mathbf{M}$ such that $j \in \mathbf{I}$ and $<\mathbf{I},\mathbf{R_M I}>$ is an $\mathbf{ACA}_0$-clone. Thus **M** generates countably many $\mathbf{ACA}_0$-clones. The same statements hold for $\mathbf{WKL}_0$-clones.

Proof: Let **M** and j be as in the corollary. Kirby and Paris [26] showed that there is a strong initial segment **I** of **M** such that $j \in \mathbf{I}$. By Theorem 5.4, $<\mathbf{I},\mathbf{R_M I}>$ is an $\mathbf{ACA}_0$-clone. Given any finite collection of such segments, it is possible to pick an element of **M** greater than any of them. In this way, countably many $\mathbf{ACA}_0$-clones can be generated. Since an $\mathbf{ACA}_0$-clone is automatically a $\mathbf{WKL}_0$-clone, the same statements hold for $\mathbf{WKL}_0$-clones. ■

We now have an abundance of clones. To gain more control over the nature of the clones, it is necessary to devise methods of building models of first order arithmetic with specific initial segments.

5.2. Γ-ultrapowers

In this section, we reverse the process of cloning. The method described here, using Γ-ultrapowers, builds first order models from second order models. Through this technique, great control can be gained over the structure of the first order models, resulting in the ability to grow specific clones. The details of the method parallel those of ultrapowers of models of **PA**. Indeed, many of the results in this section are best viewed as independent confirmation of theorems of Kirby [25]. This is particularly true of the analogs of Łoś's Theorem. The first step toward these results is to find the appropriate ultrafilter analog.

Definition 5.6: Let Γ be a model of $\mathbf{RCA_0}$. A collection of sets $\mathbf{U} \subseteq \mathbf{S}_\Gamma$ is a restricted ultrafilter (on $\mathbf{N}_\Gamma$ relative to Γ) if for some ultrafilter **V** on $\mathbf{N}_\Gamma$, $\mathbf{U} = \mathbf{V} \cap \mathbf{S}_\Gamma$.

There is a similar notion of ultrafilter on initial segments of models of fragments of **PA**, replacing $\mathbf{N}_\Gamma$ by **I** and $\mathbf{S}_\Gamma$ by $\mathbf{R_M I}$. Although this construction is generally called a definable ultrafilter, we will call it a restricted ultrafilter on **I** relative to **M**. The justification for abandoning the usual terminology is simple. A definable ultrafilter is generally not definable (as a sequence of sets) in $\mathbf{R_M I}$. Similarly, a restricted ultrafilter on $\mathbf{N}_\Gamma$ is generally not definable in Γ. Regardless of definability, a restricted ultrafilter does retain many of an ultrafilter's properties, as shown by the following lemma.

Lemma 5.7: Let $\Gamma \models \mathbf{RCA}_0$ and let $\mathbf{U}$ be a restricted ultrafilter on $\mathbf{N}_\Gamma$ relative to Γ. Then $\mathbf{U}$ acts like an ultrafilter. That is:

i) $\mathbf{U}$ has the finite intersection property.

ii) If $\mathbf{X} \in \mathbf{U}$ and $\mathbf{Y} \in \mathbf{S}_\Gamma$ such that $\mathbf{X} \subseteq \mathbf{Y}$, then $\mathbf{Y} \in \mathbf{U}$.

iii) If $\mathbf{X} \in \mathbf{S}_\Gamma$ then $\mathbf{X} \in \mathbf{U}$ or $\mathbf{X}^c \in \mathbf{U}$.

Proof: Let Γ and $\mathbf{U}$ be as in the lemma. Let $\mathbf{V}$ be the ultrafilter on $\mathbf{N}_\Gamma$ such that $\mathbf{U} = \mathbf{V} \cap \mathbf{S}_\Gamma$. To show i), suppose $\mathbf{X}_1, \ldots, \mathbf{X}_n \in \mathbf{U}$. Then $\bigcap_{i \leq n} \mathbf{X}_i \in \mathbf{V}$ and, since $\Gamma \models \mathbf{RCA}_0$, $\bigcap_{i \leq n} \mathbf{X}_i \in \mathbf{S}_\Gamma$. Thus $\bigcap_{i \leq n} \mathbf{X}_i \in \mathbf{U}$. To show ii), suppose $\mathbf{X} \in \mathbf{U}$, $\mathbf{Y} \in \mathbf{S}_\Gamma$, and $\mathbf{X} \subseteq \mathbf{Y}$. Then $\mathbf{X} \in \mathbf{V}$, so $\mathbf{Y} \in \mathbf{V}$, and since $\mathbf{Y} \in \mathbf{S}_\Gamma$, $\mathbf{Y} \in \mathbf{U}$. To show iii), let $\mathbf{X} \in \mathbf{S}_\Gamma$. Since $\Gamma \models \mathbf{RCA}_0$, $\mathbf{X}^c \in \mathbf{S}_\Gamma$. Since $\mathbf{V}$ is an ultrafilter, either $\mathbf{X} \in \mathbf{V}$, or $\mathbf{X}^c \in \mathbf{V}$. Thus either $\mathbf{X} \in \mathbf{U}$ or $\mathbf{X}^c \in \mathbf{U}$. ■

We now use a restricted ultrafilter on $\mathbf{N}_\Gamma$ to define equivalence classes of functions on $\mathbf{N}_\Gamma$ and a reduced product.

Definition 5.8: Let $\Gamma \models \mathbf{RCA}_0$ and let $\mathbf{U}$ be a restricted ultrafilter on $\mathbf{N}_\Gamma$ relative to Γ. Let $\mathbf{F}$ denote the collection of functions from $\mathbf{N}_\Gamma$ into $\mathbf{N}_\Gamma$ which are (coded) in $\mathbf{S}_\Gamma$. For $f, g \in \mathbf{F}$, we say f and g are U-equivalent, written $f =_{\mathbf{U}} g$, if and only if $\{i \in \mathbf{N}_\Gamma : \Gamma \models f(i) = g(i)\} \in \mathbf{U}$. The class of functions which are U-equivalent to f is denoted $[f]$, i.e. $[f] = \{g \in \mathbf{F} : f =_{\mathbf{U}} g\}$.

Lemma 5.9: Let Γ, U, and $\mathbf{F}$ be as in Definition 5.8. Then U-equivalence is an equivalence relation.

Proof: Reflexivity and symmetry are immediate. Transitivity follows from the fact that $\mathbf{U}$ has the finite intersection property, Lemma 5.7i).

Definition 5.10: Let Γ, $\mathbf{U}$, and $\mathbf{F}$ be as in Definition 5.8. The restricted reduced product of $\mathbf{N}_\Gamma$ denoted by $\prod_\mathbf{U}\mathbf{N}_\Gamma$ is defined by $\prod_\mathbf{U}\mathbf{N}_\Gamma=\{[f]: f\in\mathbf{F}\}$.

Definition 5.11 Let $\Gamma\models\mathbf{RCA}_0$ and let $\mathbf{U}$ be a restricted ultrafilter on $\mathbf{N}_\Gamma$ relative to Γ. The Γ-ultrapower (of $\mathbf{N}_\Gamma$ modulo $\mathbf{U}$) is the model $\mathbf{M}$ of $\mathbf{L}_1$ defined by:

i) The universe of $\mathbf{M}$ is $\prod_\mathbf{U}\mathbf{N}_\Gamma$.

ii) $\mathbf{M}\models[f]=[g]$ if and only if $\{i\in\mathbf{N}_\Gamma:\Gamma\models f(i)=g(i)\}\in\mathbf{U}$.

iii) $\mathbf{M}\models[f]<[g]$ if and only if $\{i\in\mathbf{N}_\Gamma:\Gamma\models f(i)<g(i)\}\in\mathbf{U}$.

iv) The interpretation of $[f]+[g]$ in $\mathbf{M}$ is $[f+g]$.

v) The interpretation of $[f]\cdot[g]$ in $\mathbf{M}$ is $[f\cdot g]$.

vi) The interpretation of 0 in $\mathbf{M}$ is $[f^0]$ where f^0 is the function (coded) in $\mathbf{S}_\Gamma$ such that $\Gamma\models\forall i\,(f^0(i)=0)$.

vii) The interpretation of 1 in $\mathbf{M}$ is $[f^1]$ where f^1 is the function (coded) in $\mathbf{S}_\Gamma$ such that $\Gamma\models\forall i\,(f^1(i)=1)$.

As per usual, we will use $\prod_\mathbf{U}\mathbf{N}_\Gamma$ to denote both $\mathbf{M}$ and the domain of $\mathbf{M}$.

The next step is to prove an analog of Łoś's theorem. Due to the use of the restricted reduced product, the result is not as strong as Łoś's theorem. However, as the following theorem and its corollary show, the Γ-ultrapower has reasonably nice properties.

Theorem 5.12: Let $\Gamma\models\mathbf{RCA}_0$ and let $\mathbf{U}$ be a restricted ultrafilter on $\mathbf{N}_\Gamma$. Let $\prod_\mathbf{U}\mathbf{N}_\Gamma$ be the Γ-ultrapower of $\mathbf{N}_\Gamma$ modulo $\mathbf{U}$. Then:

i) For any term $t(x_1,\ldots,x_n)$ of $\mathbf{L}_1$ and elements $[f_1],\ldots,[f_n]\in\prod_\mathbf{U}\mathbf{N}_\Gamma$, we have

$$t^{\prod_U N_\Gamma}([f_1], \ldots, [f_n]) = [<t(f_1(i), \ldots, f_n(i)) : i \in \mathbf{N}_\Gamma>].$$

ii) Given any Σ_0 formula $\phi(x_1, \ldots, x_n)$ of $\mathbf{L}_1$ and $[f_1], \ldots, [f_n] \in \prod_U N_\Gamma$, we have

$$\prod_U N_\Gamma \models \phi([f_1], \ldots, [f_n]) \text{ iff } \{i \in \mathbf{N}_\Gamma : \Gamma \models \phi(f_1(i), \ldots, f_n(i))\} \in \mathbf{U}.$$

iii) Given any Σ_1 formula $\phi(x_1, \ldots, x_n)$ of $\mathbf{L}_1$ and $[f_1], \ldots, [f_n] \in \prod_U N_\Gamma$, we have

$$\prod_U N_\Gamma \models \phi([f_1], \ldots, [f_n]) \text{ iff } \{i \in \mathbf{N}_\Gamma : \Gamma \models \phi(f_1(i), \ldots, f_n(i))\} \supseteq \mathbf{X} \in \mathbf{U}.$$

iv) Given any Π_2 sentence ϕ of $\mathbf{L}_1$, we have

$$\Gamma \models \phi \text{ iff } \prod_U N_\Gamma \models \phi.$$

Proof: To prove part i), we proceed by induction on the complexity of the term. The cases for variables and constant symbols are trivial. The cases for the production clauses with $+$ and $\cdot$ follow immediately from parts iv) and v) of Definition 5.11 and Δ_1^0 comprehension in Γ.

Part ii) is also proved by induction on formula complexity. To simplify the notation, we will consider formulas involving only one free variable. The proof for formulas with several free variables is essentially identical.

For atomic formulas, apply Theorem 5.12i) and Definition 5.11, part iii) or iv). For example, given terms $t_1(x)$ and $t_2(x)$ in $\mathbf{L}_1$ and $f \in \mathbf{S}_\Gamma$,

$$\prod_U N_\Gamma \models t_1^{\prod_U N_\Gamma}([f]) = t_2^{\prod_U N_\Gamma}([f])$$

iff (Theorem 5.12i))

$$\prod_U \mathbf{N}_\Gamma \models [<t_1(f\ (i)):i \in \mathbf{N}_\Gamma>] = [<t_2(f\ (i)):i \in \mathbf{N}_\Gamma>]$$

iff (Definition 5.11ii))

$$\{i \in \mathbf{N}_\Gamma : \Gamma \models t_1(f\ (i)) = t_2(f\ (i))\} \in \mathbf{U}.$$

The case for $<$ is similar.

There is a little twist in the proof of the clause for negation. Suppose that part ii) holds for $\phi([f\ ])$ where ϕ is a Σ_0 formula of $\mathbf{L}_1$. Then:

$$\prod_U \mathbf{N}_\Gamma \models \neg\phi([f\ ])$$

iff (definition of $\models$)

$$\prod_U \mathbf{N}_\Gamma \text{ does not } \models \phi([f\ ])$$

iff (induction hypothesis)

$$\mathbf{X} = \{i \in \mathbf{N}_\Gamma : \Gamma \models \phi(f\ (i))\} \notin \mathbf{U}.$$

Since ϕ is Σ_0, by Δ_1^0 comprehension in Γ, $\mathbf{X} \in \mathbf{S}_\Gamma$. Since $\mathbf{U}$ is a restricted ultrafilter on $\mathbf{N}_\Gamma$, we have:

$$\mathbf{X} \notin \mathbf{U} \text{ and } \mathbf{X} \in \mathbf{S}_\Gamma$$

iff (Lemma 5.7iii))

$$\mathbf{X}^c = \{i \in \mathbf{N}_\Gamma : \Gamma \models \phi(f\ (i))\}^c \in \mathbf{U}$$

iff (definition of $\models$)

$$\{i \in \mathbf{N}_\Gamma : \Gamma \models \neg\phi(f\ (i\ ))\} \in \mathbf{U}.$$

The clauses for conjunction and disjunction follow immediately from the induction hypothesis and Lemma 5.7. The clauses for implication and biconditional are also straightforward.

To prove the clauses for bounded quantifiers, assume that ϕ is a Σ_0 formula of $\mathbf{L}_1$, t is a term of $\mathbf{L}_1$, and $f \in \mathbf{S}_\Gamma$. Then:

$$\prod_{\mathbf{U}} \mathbf{N}_\Gamma \models \exists x < t\,([f\ ])\ \phi(x\ ,[f\ ])$$

iff (definition of $\exists x < t$)

$$\prod_{\mathbf{U}} \mathbf{N}_\Gamma \models \exists x\ \ (x < t\,([f\ ]) \wedge \phi(x\ ,[f\ ]))$$

iff (interpretation of $\exists x$)

for some $g \in \mathbf{S}_\Gamma$, $\prod_{\mathbf{U}} \mathbf{N}_\Gamma \models [g\ ] < t\,([f\ ]) \wedge \phi([g\ ],[f\ ])$

iff (induction hypothesis)

for some $g \in \mathbf{S}_\Gamma$, $\{i \in \mathbf{N}_\Gamma : \Gamma \models g\,(i\ ) < t\,(f\ \ (i\ )) \wedge \phi(g\,(i\ ), f\ \ (i\ ))\} \in \mathbf{U}$

iff (interpretation of $\exists x$ or, conversely, Δ_1^0 comprehension in Γ)

$$\{i \in \mathbf{N}_\Gamma : \Gamma \models \exists x\ \ (x < t\,(f\ \ (i\ )) \wedge \phi(x\ , f\ \ (i\ )))\} \in \mathbf{U}$$

iff (definition of $\exists x < t$)

$$\{i \in \mathbf{N}_\Gamma : \Gamma \models \exists x < t(f(i))\ \phi(x, f(i))\} \in \mathbf{U}.$$

The proof for bounded universal quantifiers follows immediately by rewriting $\forall x < t\ \ \phi$ as $\neg \exists x < t\ \ \neg\phi$. This concludes the proof of part ii) for Σ_0 formulas of $\mathbf{L}_1$.

To prove part iii), suppose that ϕ is a Σ_0 formula of $\mathbf{L}_1$, and $f \in \mathbf{S}_\Gamma$. Then:

$$\prod_{\mathbf{U}} \mathbf{N}_\Gamma \models \exists x\ \ \phi(x, [f])$$

iff (definition of $\models$)

for some $g \in \mathbf{S}_\Gamma$, $\prod_{\mathbf{U}} \mathbf{N}_\Gamma \models \phi([g], [f])$

iff (Theorem 5.12ii))

for some $g \in \mathbf{S}_\Gamma$, $\{i \in \mathbf{N}_\Gamma : \Gamma \models \phi(g(i), f(i))\} \in \mathbf{U}$

iff (interpretation of $\exists x$ or, conversely, Δ_1^0 comprehension in Γ)

$$\{i \in \mathbf{N}_\Gamma : \Gamma \models \exists x\ \ \phi(x, f(i))\} \supseteq \mathbf{X} \in \mathbf{U}.$$

The absence of parameters makes the proof of part iv) easy. Let $\phi(x, y)$ be a Σ_0 formula of $\mathbf{L}_1$. Suppose that $\prod_{\mathbf{U}} \mathbf{N}_\Gamma \models \forall x \exists y\ \ \phi(x, y)$ and $\Gamma \models \exists x \forall y\ \ \neg\phi(x, y)$. Then for some $c \in \mathbf{N}_\Gamma$, $\Gamma \models \forall y\ \ \neg\phi(c, y)$. Let $f^c \in \mathbf{S}_\Gamma$ be the constant function for c. Then $\prod_{\mathbf{U}} \mathbf{N}_\Gamma \models \exists y\ \ \phi([f^c], y)$, so for some $g \in \mathbf{S}_\Gamma$, $\prod_{\mathbf{U}} \mathbf{N}_\Gamma \models \phi([f^c], [g])$. By Theorem 5.12ii), $\{i \in \mathbf{N}_\Gamma : \Gamma \models \phi(f(i), g(i))\} \in \mathbf{U}$. Since $f(i) = c$, $\{i \in \mathbf{N}_\Gamma : \Gamma \models \phi(c, g(i))\} \in \mathbf{U}$, so $\Gamma \models \exists y\ \phi(c, y)$, contradicting the choice of c.

Conversely, suppose that $\Gamma \models \forall x \exists y\ \phi(x, y)$. Fix $f \in \mathbf{S}_\Gamma$. Then by Δ_1^0 comprehension in Γ, there is a $g \in \mathbf{S}_\Gamma$ such that $\Gamma \models \forall i\ \phi(f(i), g(i))$. Hence, $\{i \in \mathbf{N}_\Gamma : \Gamma \models \phi(f(i), g(i))\} = \mathbf{N}_\Gamma \in \mathbf{U}$. By Theorem 5.12ii), $\prod_\mathbf{U} \mathbf{N}_\Gamma \models \phi([f], [g])$, so $\prod_\mathbf{U} \mathbf{N}_\Gamma \models \exists y\ \phi([f], y)$. Since the choice of f was arbitrary, we have $\prod_\mathbf{U} \mathbf{N}_\Gamma \models \forall x \exists y\ \phi(x, y)$. ■

Corollary 5.13: Let $\Gamma \models \mathbf{RCA}_0$ and let $\mathbf{U}$ be a restricted ultrafilter on $\mathbf{N}_\Gamma$. Let $\prod_\mathbf{U} \mathbf{N}_\Gamma$ be the Γ-ultrapower of $\mathbf{N}_\Gamma$ modulo $\mathbf{U}$. Then $\prod_\mathbf{U} \mathbf{N}_\Gamma$ is a model of $\mathbf{P}^-$ plus $\mathrm{I}\Sigma_0$.

Proof: Let Γ, $\mathbf{U}$, and $\prod_\mathbf{U} \mathbf{N}_\Gamma$ be as stated. Applications of Theorem 5.12iv) show that $\prod_\mathbf{U} \mathbf{N}_\Gamma$ models $\mathbf{P}^-$. Let θ be a Σ_0 formula of $\mathbf{L}_1$. To simplify notation, we will consider the case where θ has one parameter, $[f] \in \prod_\mathbf{U} \mathbf{N}_\Gamma$. Suppose, by way of contradiction, that

$$\prod_\mathbf{U} \mathbf{N}_\Gamma \models \theta([f], 0) \wedge \forall y\, (\theta([f], y) \rightarrow \theta([f], y+1)) \wedge \neg \forall y\, (\theta([f], y))$$

Since $\prod_\mathbf{U} \mathbf{N}_\Gamma \models \exists y\, (\neg \theta([f], y))$, for some $g \in \mathbf{S}_\Gamma$, $\prod_\mathbf{U} \mathbf{N}_\Gamma \models \neg \theta([f], [g])$. By Theorem 5.12ii), $\mathbf{X} = \{i \in \mathbf{N}_\Gamma : \Gamma \models \neg \theta(f(i), g(i))\} \in \mathbf{U}$. Since f, g, and $\mathbf{X}$ are in $\mathbf{S}_\Gamma$, by Δ_1^0 comprehension in Γ, for some $h \in \mathbf{S}_\Gamma$,

$$\{i \in \mathbf{N}_\Gamma : \Gamma \models \theta(f(i), h(i)) \wedge \neg \theta(f(i), h(i)+1)\} \in \mathbf{U}.$$

By Theorem 5.12ii), $\prod_\mathbf{U} \mathbf{N}_\Gamma \models \theta([f], [h]) \wedge \neg \theta([f], [h]+1)$, contradicting the assumption that $\prod_\mathbf{U} \mathbf{N}_\Gamma \models \forall y\, (\theta([f], y) \rightarrow \theta([f], y+1))$. ■

In the proof of Theorem 5.12, we were severely hindered by the lack of set comprehension in Γ. Increasing the available comprehension to $\mathbf{WKL}_0$ does not improve the situation.

Theorem 5.14: There is an ω-model Γ of $\mathbf{WKL_0}$, and restricted ultrafilters $\mathbf{U}$ and $\mathbf{V}$ on $\omega = \mathbf{N}_\Gamma$ relative to Γ such that:

i) There is a Π_3 sentence ϕ of $\mathbf{L}_1$ such that $\Gamma \models \phi$ and $\prod_{\mathbf{U}} \mathbf{N}_\Gamma \models \neg\phi$.

ii) There is a Σ_0 formula θ of $\mathbf{L}_1$ and an $f \in \mathbf{S}_\Gamma$ such that

$\prod_{\mathbf{V}} \mathbf{N}_\Gamma \models \forall t \, \neg\theta([f\,], t)$ but for any $\mathbf{Z} \in \mathbf{V}$, there is an $i \in \mathbf{Z}$ such that

$\Gamma \models \exists t \, \theta(f\,(i), t)$.

Proof: Let Γ be a countable ω model of $\mathbf{WKL_0}$ such that $\mathbf{S}_\Gamma$ consists entirely of sets of low degree. Such a model exists by Theorem 1.6. Let $< g_i >_{i \in \omega}$ be an enumeration of the total functions in $\mathbf{S}_\Gamma$. We specify that g_0 is the function which is constantly 0. Let $\mathbf{X}$ be a set not in $\mathbf{S}_\Gamma$ such that for some Σ_0 formula θ of $\mathbf{L}_1$, we have $\mathbf{X} = \{x \in \mathbf{N}_\Gamma : \Gamma \models \exists t \, \theta(x, t)\}$, and $\Gamma \models \forall t \, \forall x \, \forall y \, ((\theta(x, t) \wedge \theta(y, t)) \rightarrow x = y)$. Note that $\mathbf{X}$ is essentially the range of a recursive injection. We will now construct $\mathbf{U}$.

We will define a filter base $< \mathbf{F}_n >_{n \in \omega}$ such that for each n the following five properties hold.

1) $\min(\mathbf{F}_n) > n$.

2) $\mathbf{F}_n$ is infinite.

3) $\mathbf{F}_{n+1} \subseteq \mathbf{F}_n$.

4) $\mathbf{F}_n \in \mathbf{S}_\Gamma$.

5) There is an $f \in \mathbf{S}_\Gamma$, such that for every $k \in \mathbf{F}_n$,

$$\Gamma \models \exists x \leq k \, ((\exists y < f\,(k) \theta(x, y)) \wedge (\forall z < g_n(k) \neg\theta(x, z))).$$

Let $t_0 = \mu n(\exists x \le n\, \theta(x,n))$. Such a t_0 exists, since otherwise, we would have $\mathbf{X} = \{x : \exists n < x\, \theta(x,n)\}$, contradicting $\mathbf{X} \notin \mathbf{S}_\Gamma$. Set $\mathbf{F}_0 = \{k \in \mathbf{N}_\Gamma : k > t_0\}$. Then $\min(\mathbf{F}_0) > 0$, $\mathbf{F}_0$ is infinite, and $\mathbf{F}_0 \in \mathbf{S}_\Gamma$. Let f^{id} be the identity function on ω. For any $k > t_0$, we have $\Gamma \models \exists x \le k(\exists y < k\, \theta(x,y) \wedge \forall z < 0 \neg\theta(x,z))$, so property 5) holds for $\mathbf{F}_0$.

Suppose that for all $j < n$, we have defined $\mathbf{F}_j$ satisfying properties 1) through 5). We will now define $\mathbf{F}_n$. To do this, we first define the auxiliary function f_n by:

$$f_n(0) = \mu t \in \mathbf{F}_{n-1}(\exists x \le t(\exists y < t\, \theta(x,y) \wedge \forall z < g_n(x) \neg\theta(x,z))) \text{ and}$$

$$f_n(j+1) = \mu t \in \mathbf{F}_{n-1}(t > f_n(j) \wedge \exists x \le t(\exists y < t\, \theta(x,y) \wedge \forall z < g_n(x) \neg\theta(x,z))).$$

Since f_n is Δ_1^0 definable in g_n and $\mathbf{F}_{n-1}$, $f_n \in \mathbf{S}_\Gamma$. Also, f_n is strictly increasing. Furthermore, f_n is total. To see this, suppose that j is the least integer for which $f_n(j)$ is undefined. Let $t_1 = \min(\mathbf{F}_{n-1})$ if $j = 0$ and $t_1 = f_n(j-1)$ otherwise. Then

$$\Gamma \models \forall t \in \mathbf{F}_{n-1}(t > t_1 \rightarrow \forall x < t(\exists y < t\, \theta(x,t) \rightarrow \exists z < g_n(x) \theta(x,z))).$$

Since $\mathbf{F}_{n-1}$ is infinite,

$$\Gamma \models \forall x(\exists y\, \theta(x,y) \rightarrow \exists z < g_n(x) \theta(x,z)).$$

Thus, $\mathbf{X} = \{x \in \omega : \exists z < g_n(x) \theta(x,z)\} \in \mathbf{S}_\Gamma$, contradicting the choice of $\mathbf{X}$.

We now define $\mathbf{F}_n$ in terms of f_n and $\mathbf{F}_{n-1}$ by

$$\mathbf{F}_n = \{k \in \mathbf{F}_{n-1} : k > n \wedge \exists x \le k(\exists y < f_n(k) \theta(x,y) \wedge \forall z < g_n(x) \neg\theta(x,z))\}.$$

Since $\mathbf{F}_n$ is Δ_1^0 definable in $\mathbf{F}_{n-1}$ and f_n, $\mathbf{F}_n \in \mathbf{S}_\Gamma$. Clearly, $\min(\mathbf{F}_n) > n$ and $\mathbf{F}_n \subseteq \mathbf{F}_{n-1}$. To see that $\mathbf{F}_n$ is infinite, fix $m \in \omega$ and choose $k \in \mathbf{F}_{n-1}$ such that

$k > \max(\{m, n\})$. Then $f_n(k) \in \mathbf{F}_{n-1}$, $f_n(k) > n$, and

$$\Gamma \models \exists x \leq f_n(k)(\exists y < f_n(k)\theta(x, y) \wedge \forall z < g_n(x) \neg\theta(x, z)).$$

Since $f_n(k) < f_n(f_n(k))$,

$$\Gamma \models f_n(k) > n \wedge \exists x \leq f_n(k)(\exists y < f_n(f_n(k))\theta(x, y) \wedge \forall z < g_n(x) \neg\theta(x, z)),$$

so $f_n(k) \in \mathbf{F}_n$. Since the choice of m was arbitrary, and $f_n(k) > m$, $\mathbf{F}_n$ is infinite. Finally, for any $k \in \mathbf{F}_n$,

$$\Gamma \models \exists x \leq k(\exists y < f_n(k)\theta(x, y) \wedge \forall z < g_n(x) \neg\theta(x, z)).$$

Since $f_n \in \mathbf{S}_\Gamma$, $\mathbf{F}_n$ satisfies property 5).

Let $\mathbf{U}$ be a restricted ultrafilter on ω containing $\mathbf{F}_n$ for all $n \in \omega$. Since $<\mathbf{F}_n>_{n \in \omega}$ is a nested sequence of sets, it has the finite intersection property, so such a $\mathbf{U}$ exists. Furthermore $\mathbf{U}$ contains all final segments of ω, since for every k, $\{n \in \omega : n \leq k\} \cap \mathbf{F}_k = \emptyset$. Let $\prod_\mathbf{U} \mathbf{N}_\Gamma$ denote the Γ-ultrapower of $\mathbf{N}_\Gamma = \omega$ modulo $\mathbf{U}$.

To prove part i) of the theorem, let ϕ be the sentence

$$\forall k \exists m \forall j \forall x \leq k(\exists y < j\, \theta(x, y) \rightarrow \exists z < m\, \theta(x, z)).$$

For any $k \in \omega$, an appropriate m is given by the formula

$$m = \max(\{n \in \omega : \exists x \in \mathbf{X} \cap k+1(n = \mu t\, \theta(x, t))\}) + 1.$$

Since finite initial segments of $\mathbf{X}$ are in $\mathbf{S}_\Gamma$, $\Gamma \models \phi$. On the other hand, suppose that $\prod_\mathbf{U} \mathbf{N}_\Gamma \models \phi$. Then for some $g \in \mathbf{S}_\Gamma$,

$$\prod_\mathbf{U} \mathbf{N}_\Gamma \models \forall j \forall x \leq [f^{id}](\exists y < j\, \theta(x, y) \rightarrow \exists z < [g]\theta(x, z)).$$

Now g is g_n for some n, so, plugging in f_n for j yields

$$\prod_{\mathbf{U}}\mathbf{N}_\Gamma \models \forall x \leq [f^{\,id}](\exists y < [f_n]\theta(x,y) \rightarrow \exists z < [g_n]\theta(x,z)).$$

By Theorem 5.12ii), we have that

$$\mathbf{Y}=\{i \in \omega : \Gamma \models \forall x \leq i\,(\exists y < f_n(i)\theta(x,y) \rightarrow \exists z < g_n(i)\theta(x,z))\} \in \mathbf{U}.$$

But for all $i \in \mathbf{F}_n$,

$$\Gamma \models \exists x \leq i\,(\exists y < f_n(i)\theta(x,y) \wedge \forall z < g_n(i) \neg\theta(x,z)),$$

so $\mathbf{Y} \cap \mathbf{F}_n = \emptyset$ and $\mathbf{Y} \notin \mathbf{U}$, contradicting the preceding statement. Since ϕ is a sentence and $\prod_{\mathbf{U}}\mathbf{N}_\Gamma$ does not model ϕ, we have $\prod_{\mathbf{U}}\mathbf{N}_\Gamma \models \neg\phi$ as desired.

We now turn to the construction of the second restricted ultrafilter, $\mathbf{V}$. Let θ, $\mathbf{X}$ and $<g_i>_{i \in \omega}$ be as before. Let $<\mathbf{H}_n>_{n \in \omega}$ be an enumeration of all elements of $\mathbf{S}_\Gamma$ which are disjoint from $\mathbf{X}$. We will define a filter base $<\mathbf{G}_n>_{n \in \omega}$ such that for each n the following five properties hold.

1) $\min(\mathbf{G}_n) > n$.

2) $\mathbf{G}_n$ is infinite.

3) $\mathbf{G}_{n+1} \subseteq \mathbf{G}_n$.

4) $\mathbf{G}_n \cap \mathbf{H}_n = \emptyset$.

5) $\mathbf{G}_n \cap \{i \in \mathbf{N}_\Gamma : \Gamma \models \exists t < g_n(i)\theta(i,t)\} = \emptyset$.

Let $\mathbf{G}_0 = \{k \in \mathbf{N}_\Gamma : k \neq 0 \wedge k \notin \mathbf{H}_0\}$. Clearly, properties 1) and 4) are satisfied. Since $\mathbf{X} \subseteq \mathbf{G}_0$ and $\mathbf{X}$ is infinite, so is $\mathbf{G}_0$, so property 2) holds. Since g_0 is constantly 0, property 5) is satisfied. Given $<\mathbf{G}_j>_{j<n}$ satisfying properties 1) through 5), $\mathbf{G}_n$

is defined by

$$\mathbf{G}_n = \{k \in \mathbf{G}_{n-1} : k > n \wedge k \notin \mathbf{H}_n \wedge \forall t < g_n(k) \neg\theta(k,t)\}.$$

Clearly, $\mathbf{G}_n$ satisfies properties 1), 3), 4) and 5). Let $g = \max_{i \leq n}(g_i)$. If $\mathbf{G}_n$ is finite, then $\mathbf{X}$ differs from $\{i \in \mathbf{N}_\Gamma : \Gamma \models \exists t < g(i)\theta(i,t)\}$ by a finite set, contradicting the fact that $\mathbf{X} \notin \mathbf{S}_\Gamma$.

Let $\mathbf{V}$ be a restricted ultrafilter on ω containing $\mathbf{G}_n$ for all $n \in \omega$. Property 1) insures that $\mathbf{V}$ contains all final segments of ω. Let $\prod_\mathbf{V}\mathbf{N}_\Gamma$ denote the Γ-ultrapower of $\mathbf{N}_\Gamma$ modulo $\mathbf{V}$.

To prove part ii) of the theorem, let θ and f^{id} be as above. Suppose that $\prod_\mathbf{V}\mathbf{N}_\Gamma \models \exists t\, \theta([f^{id}],t)$. Then for some $g \in \mathbf{S}_\Gamma$, $\prod_\mathbf{V}\mathbf{N}_\Gamma \models \theta([f^{id}],[g])$. By Theorem 5.12ii), there is a set $\mathbf{Y}_1$ such that

$$\mathbf{Y}_1 = \{i \in \mathbf{N}_\Gamma : \Gamma \models \theta(i, g(i))\} \in \mathbf{V}$$

Pick n such that $g_n(i) = g(i)+1$ for all $i \in \mathbf{N}_\Gamma$. Let $\mathbf{Y}_2$ be defined by

$$\mathbf{Y}_2 = \{i \in \mathbf{N}_\Gamma : \Gamma \models \exists t < g_n(i)\theta(i,t)\}.$$

Then $\mathbf{Y}_1 \subseteq \mathbf{Y}_2 \in \mathbf{S}_\Gamma$, so by Lemma 5.7ii), $\mathbf{Y}_2 \in \mathbf{V}$. But this is absurd, since $\mathbf{Y}_2 \cap \mathbf{G}_n = \emptyset$. Thus $\prod_\mathbf{V}\mathbf{N}_\Gamma \models \forall t\, \neg\theta([f^{id}],t)$.

Let $\mathbf{Z}$ be any set in $\mathbf{V}$. If for every $i \in \mathbf{Z}$, $\Gamma \models \forall t\, \neg\theta(i,t)$, then $\mathbf{Z}$ is $\mathbf{H}_n$ for some n. This is impossible since $\mathbf{G}_n \cap \mathbf{H}_n = \emptyset$. Thus, for some $i \in \mathbf{Z}$, $\Gamma \models \exists t\, \theta(f^{id}(i),t)$, as desired. ■

In spite of the indications from Theorem 5.14, there is hope for a stronger analog of Łoś's theorem. Clearly, more set comprehension than that provided by

$\mathbf{WKL}_0$ is necessary. The following theorem shows that if $\Gamma\models\mathbf{ACA}_0$, then we get the usual version of Łoś's theorem.

Theorem 5.15: Let $\Gamma\models\mathbf{ACA}_0$ and let $\mathbf{U}$ be a restricted ultrafilter on $\mathbf{N}_\Gamma$. Let $\prod_\mathbf{U}\mathbf{N}_\Gamma$ be the Γ-ultrapower of $\mathbf{N}_\Gamma$ modulo $\mathbf{U}$. Then:

i) Given any formula $\phi(x_1, \ldots, x_n)$ of $\mathbf{L}_1$ and $[f_1], \ldots, [f_n]\in\prod_\mathbf{U}\mathbf{N}_\Gamma$ we have

$$\prod_\mathbf{U}\mathbf{N}_\Gamma\models\phi([f_1], \ldots, [f_n]) \text{ iff } \{i\in\mathbf{N}_\Gamma:\Gamma\models\phi(f_1(i), \ldots, f_n(i))\}\in\mathbf{U}.$$

ii) Given any sentence ϕ of $\mathbf{L}_1$, we have

$$\Gamma\models\phi \text{ iff } \prod_\mathbf{U}\mathbf{N}_\Gamma\models\phi.$$

Proof: Part ii) follows immediately from part i). The proof of part i) is similar to that of Theorem 5.12ii). The first change necessary is to generalize the proof of the induction clause for negation to all formulas. Suppose $\phi([f])$ is a formula of $\mathbf{L}_1$ with a parameter from $\prod_\mathbf{U}\mathbf{N}_\Gamma$. Then

$$\prod_\mathbf{U}\mathbf{N}_\Gamma\models\neg\phi([f])$$

iff (definition of $\models$)

$$\prod_\mathbf{U}\mathbf{N}_\Gamma \text{ does not } \models\phi([f])$$

iff (induction hypothesis)

$$\mathbf{X}=\{i\in\mathbf{N}_\Gamma:\Gamma\models\phi(f(i))\}\notin\mathbf{U}$$

Since $\Gamma\models\mathbf{ACA_0}$, $\mathbf{X}\in\mathbf{S}_\Gamma$. Since $\mathbf{U}$ is a restricted ultrafilter on $\mathbf{N}_\Gamma$,

$$\mathbf{X}\notin\mathbf{U} \text{ and } \mathbf{X}\in\mathbf{S}_\Gamma$$

$$\text{iff (Lemma5.7iii))}$$

$$\mathbf{X}^c=\{i\in\mathbf{N}_\Gamma:\Gamma\models\neg\phi(f\ (i\ ))\}\in\mathbf{U}.$$

The only other modification of the proof is in the clause for existential quantification. Here, $\mathbf{ACA_0}$ is used to show that for any formula $\phi(x\ ,y\ )$ of $\mathbf{L_1}$ and any parameter $[f\ ]\in\prod_\mathbf{U}\mathbf{N}_\Gamma$, if $\{i\in\mathbf{N}_\Gamma:\Gamma\models\exists x\ \phi(x\ ,f\ (i\ ))\}\in\mathbf{U}$, then there is a $g\in\mathbf{S}_\Gamma$ such that $\{i\in\mathbf{N}_\Gamma:\Gamma\models\phi(g\ (i\ ),f\ (i\ ))\}\in\mathbf{U}$. Universal quantification can be eliminated by rewriting $\forall x\ \phi$ as $\neg\exists x\ \neg\phi$. ■

Corresponding to the stronger version of Theorem 5.12, we have the following stronger version of Corollary 5.13.

Corollary 5.16: Let $\Gamma\models\mathbf{ACA_0}$ and let $\mathbf{U}$ be a restricted ultrafilter on $\mathbf{U}$. Let $\prod_\mathbf{U}\mathbf{N}_\Gamma$ be the Γ-ultrapower of $\mathbf{N}_\Gamma$ modulo $\mathbf{U}$. The $\prod_\mathbf{U}\mathbf{N}_\Gamma$ is elementarily equivalent to the first order part of Γ. In particular, $\prod_\mathbf{U}\mathbf{N}_\Gamma\models\mathbf{PA}$.

Proof: Elementary equivalence follows immediately from Theorem 5.15ii). Since the first order part of any model of $\mathbf{ACA_0}$ is a model of $\mathbf{PA}$, the first order part of Γ models $\mathbf{PA}$. By elementary equivalence, $\prod_\mathbf{U}\mathbf{N}_\Gamma\models\mathbf{PA}$. ■

It is sometimes useful to consider expansions of Γ-ultrapowers to models of supersets of $\mathbf{L_1}$. The most natural such expansion is to include constant symbols for elements of $\mathbf{N}_\Gamma$ and function symbols for functions (coded) in $\mathbf{S}_\Gamma$. If $\mathbf{f}$ is a function symbol representing $f\in\mathbf{S}_\Gamma$, the natural interpretation of $\mathbf{f}([g\ ])$ in $\prod_\mathbf{U}\mathbf{N}_\Gamma$ is $[f\ (g\ )]$.

Set notation can be introduced into the language by treating $n \in \mathbf{X}$ as an abbreviation for $\chi_X(n)=1$, where χ_X is a function symbol representing the characteristic function of $\mathbf{X}$ in $\mathbf{S}_\Gamma$. Note that we are introducing set constants, not set variables. If $\mathbf{L}_1^*$ is such an expansion of $\mathbf{L}_1$, we define the complexity of a formula ϕ of $\mathbf{L}_1^*$ exactly as before. That is, new constant symbols and function symbols are treated just like 0 or +, not as some new sort of parameter. The expansion of the language in no way changes the structure of the resulting ultrapower. We summarize this in the following porism.

Porism 5.17: Let Γ be a model of $\mathbf{RCA}_0$. Let $\mathbf{L}_1^*$ be an expansion of $\mathbf{L}_1$ to include new constant symbols for elements for $\mathbf{N}_\Gamma$ and new function symbols for functions (coded) in $\mathbf{S}_\Gamma$. Then Theorem 5.12, Corollary 5.13, Theorem 5.15, and Corollary 5.16 hold with $\mathbf{L}_1$ replaced by $\mathbf{L}_1^*$.

5.3. Canonical Clones

In this section, we return to the construction of clones. Using Γ-ultrapowers, many interesting models of $\mathbf{L}_1$ can be obtained. To grow clones from these models, we need only distinguish appropriate initial segments. One excellent candidate is defined below.

Definition 5.18: Let $\Gamma \models \mathbf{RCA}_0$ and let $\prod_{\mathbf{U}}\mathbf{N}_\Gamma$ be the restricted ultrapower of $\mathbf{N}_\Gamma$ modulo a restricted ultrafilter $\mathbf{U}$. The canonical initial segment of $\prod_{\mathbf{U}}\mathbf{N}_\Gamma$ is denoted by $\mathbf{I}_{\Gamma,\mathbf{U}}$ and defined by

$$\mathbf{I}_{\Gamma,\mathbf{U}}=\{[g]\in\textstyle\prod_{\mathbf{U}}\mathbf{N}_\Gamma : \exists c \in \mathbf{N}_\Gamma \ \prod_{\mathbf{U}}\mathbf{N}_\Gamma \models [g]<[f^c]\},$$

where f^c is the function which is constantly c.

Intuitively, $\mathbf{I}_{\Gamma,\mathbf{U}}$ is the downward closure in $\prod_{\mathbf{U}}\mathbf{N}_\Gamma$ of the set of constant functions. Although $\mathbf{I}_{\Gamma,\mathbf{U}}$ is clearly an initial segment, it is not necessarily a proper initial segment. If **U** has the following property, then $\mathbf{I}_{\Gamma,\mathbf{U}}$ is proper. The terminology used here is the same as that of Kirby [25].

Definition 5.19: A restricted ultrafilter **U** on $\mathbf{N}_\Gamma$ is uniform if it contains all final segments of $\mathbf{N}_\Gamma$, i.e.

$$\forall k \in \mathbf{N}_\Gamma \{ j \in \mathbf{N}_\Gamma : \Gamma \models j > k \} \in \mathbf{U}.$$

Lemma 5.20: Let $\Gamma \models \mathbf{RCA}_0$ and let **U** be a restricted ultrafilter on $\mathbf{N}_\Gamma$. Then **U** is uniform if and only if $\mathbf{I}_{\Gamma,\mathbf{U}}$ is a proper initial segment of $\prod_{\mathbf{U}}\mathbf{N}_\Gamma$.

Proof: Let Γ and **U** be as stated. First suppose that **U** is uniform. For any $j \in \mathbf{N}_\Gamma$, $\{k \in \mathbf{N}_\Gamma : \Gamma \models j < f^{id}(k)\} \in \mathbf{U}$, so $[f^{id}] \notin \mathbf{I}_{\Gamma,\mathbf{U}}$, and $\mathbf{I}_{\Gamma,\mathbf{U}}$ is proper. On the other hand, suppose that for some $k \in \mathbf{N}_\Gamma$, $\{ j \in \mathbf{N}_\Gamma : \Gamma \models j > k \} \notin \mathbf{U}$. By Lemma 5.7iii), $\{ j \in \mathbf{N}_\Gamma : \Gamma \models j \leq k \} \in \mathbf{U}$. By $\mathbf{I}\Sigma_1^0$ in Γ, for any total function $f \in \mathbf{S}_\Gamma$, there is a $b \in \mathbf{N}_\Gamma$ such that $\Gamma \models \forall j \leq k \;\; f(j) < b$, so $\mathbf{I}_{\Gamma,\mathbf{U}}$ is not proper. ■

Although Lemma 5.20 shows that uniformity is the desired property, it may seem odd that we do not simply specify that **U** is nonprincipal. If Γ is an ω-model, then **U** is uniform if and only if it is nonprincipal. However, if $\mathbf{N}_\Gamma$ is nonstandard, then requiring **U** to be nonprincipal does not guarantee that $\mathbf{I}_{\Gamma,\mathbf{U}}$ is proper. This fact follows immediately from Lemma 5.20 and the following lemma.

Lemma 5.21: Let $\Gamma \models \mathbf{RCA}_0$ with $\mathbf{N}_\Gamma$ nonstandard. Then there is a restricted ultrafilter **U** on $\mathbf{N}_\Gamma$ such that **U** is nonprincipal but not uniform.

Proof: Let Γ be as stated. Let **V** be and nonprincipal ultrafilter on ω . Define **U** by

$$\mathbf{U}=\{\mathbf{X}\in\mathbf{S}_\Gamma:\exists\mathbf{Y}\in\mathbf{V}\ \mathbf{Y}\subseteq\mathbf{X}\}.$$

It is easy to verify that **U** is an ultrafilter. Since **V** is nonprincipal and for any $\mathbf{X}\in\mathbf{U}$, $\mathbf{X}\cap\omega\neq\emptyset$, **U** is also nonprincipal. For any $b\in\mathbf{N}_\Gamma$ such that b is nonstandard, $\{j\in\mathbf{N}_\Gamma:j\leq b\}\in\mathbf{U}$, so **U** is not uniform. ■

The fact that $\mathbf{I}_{\Gamma,\mathbf{U}}$ is proper does little to determine its structure. The following property on **U** yields a natural bijection between $\mathbf{N}_\Gamma$ and $\mathbf{I}_{\Gamma,\mathbf{U}}$. Such ultrafilters are commonly used in the literature on models of arithmetic.

Definition 5.22: A restricted ultrafilter **U** on $\mathbf{N}_\Gamma$ is additive if **U** is uniform and for any $b\in\mathbf{N}_\Gamma$ and $f\in\mathbf{S}_\Gamma$ such that $f:\mathbf{N}_\Gamma\rightarrow b$, there is an $a\in\mathbf{N}_\Gamma$ such that $\{j\in\mathbf{N}_\Gamma:f(j)=a\}\in\mathbf{U}$.

Lemma 5.23: Let **U** be a restricted ultrafilter on $\mathbf{N}_\Gamma$. The following are equivalent:

i) **U** is additive.

ii) $[f]\in\mathbf{I}_{\Gamma,\mathbf{U}}$ implies $\exists k\in\mathbf{N}_\Gamma[f]=_\mathbf{U}[f^k]$.

Proof: Let **U** be a restricted ultrafilter. First we suppose that **U** is additive and $[f]\in\mathbf{I}_{\Gamma,\mathbf{U}}$. By the definition of $\mathbf{I}_{\Gamma,\mathbf{U}}$, there is some b such that $\{j\in\mathbf{N}_\Gamma:\Gamma\models f(i)<b\}=\mathbf{X}\in\mathbf{U}$. Let $\overline{f}\in\mathbf{S}_\Gamma$ be the function defined by $\overline{f}(i)=f(i)$ for $i\in\mathbf{X}$ and $\overline{f}(i)=0$ otherwise. Since $\overline{f}:\mathbf{N}_\Gamma\rightarrow b$, by the additivity of **U**, for some $\mathbf{Y}\in\mathbf{U}$,

$$\exists a\in\mathbf{N}_\Gamma\ \forall i\in\mathbf{Y}\ \Gamma\models\overline{f}(i)=a.$$

Thus $\forall i\in\mathbf{Y}\cap\mathbf{X}\ \Gamma\models f(i)=a$, and since $\mathbf{Y}\cap\mathbf{X}\in\mathbf{U}$, $[f]=_\mathbf{U}[f^a]$.

To prove the converse, suppose that **U** is not additive. Pick $f \in \mathbf{S}_\Gamma$ and $b \in \mathbf{N}_\Gamma$ such that $f : \mathbf{N}_\Gamma \to b$ and

$$\forall a < b \; \{i \in \mathbf{N}_\Gamma : f(i) = a\} \notin \mathbf{U}.$$

Then $\mathbf{N}_\Gamma = \{i \in \mathbf{N}_\Gamma : \Gamma \models f(i) < b\} \in \mathbf{U}$ so $[f] \in \mathbf{I}_{\Gamma,\mathbf{U}}$, but $\forall a \in \mathbf{N}_\Gamma [f] \neq_{\mathbf{U}} [f^a]$. ∎

The following lemma gives a well known necessary and sufficient condition on Γ for the existence of an additive ultrafilter on $\mathbf{N}_\Gamma$.

Lemma 5.24 Let $\Gamma \models \mathbf{RCA}_0$ be a countable model. The following are equivalent.

i) $\Gamma \models \mathbf{B}\Pi_1^0$.

ii) There is an additive ultrafilter on $\mathbf{N}_\Gamma$.

Proof: Let $\Gamma \models \mathbf{RCA}_0 + \mathbf{B}\Pi_1^0$ be a countable model. Let $< f_i >_{i \in \omega}$ be an enumeration of all $f \in \mathbf{S}_\Gamma$ such that $\exists b \in \mathbf{N}_\Gamma \; f : \mathbf{N}_\Gamma \to b$. We will define a filter base for **U**. Let $\mathbf{X}_0 = \mathbf{N}_\Gamma$ and suppose that $\mathbf{X}_j$ has been constructed and is unbounded in $\mathbf{N}_\Gamma$. By $\mathbf{B}\Pi_1^0$, there is an element $a \in \mathbf{N}_\Gamma$ such that $\{i \in \mathbf{X}_j : \Gamma \models f_j(i) = a\}$ is unbounded in $\mathbf{X}_j$. Let $\mathbf{X}_{j+1} = \{i \in \mathbf{X}_j : f_j(i) = a\}$. Let **U** be any restricted ultrafilter on $\mathbf{N}_\Gamma$ containing $\mathbf{X}_j$ for every $j \in \omega$. **U** is clearly additive.

To prove the converse, suppose that **U** is an additive restricted ultrafilter on $\mathbf{N}_\Gamma$. Fix $\theta \in \Sigma_0^0$ (with possible set parameters from $\mathbf{S}_\Gamma$) and $b \in \mathbf{N}_\Gamma$ such that $\Gamma \models \forall x < b \; \exists t \; \forall i \; \theta(x,t,i)$. We will assume that

$$\Gamma \models \forall z \; \exists x < b \; \forall t < z \; \exists i \; \neg\theta(x,t,i),$$

and derive a contradiction. By Δ_1^0 comprehension in Γ, define functions g and f in $\mathbf{S}_\Gamma$ such that

$$g(z)=\mu j(\exists x<b\ \forall t<z\ \exists i<j\ \neg\theta(x,t,i)\ \text{and}$$

$$f(z)=\mu k<b(\forall t<z\ \exists i<g(z)\neg\theta(k,t,i)).$$

Since $f:\mathbf{N}_\Gamma\to b$, for some $\mathbf{X}\in\mathbf{S}_\Gamma$ and $a\in\mathbf{N}_\Gamma$, $\forall i\in\mathbf{X}\ \Gamma\models f(i)=a$. By the hypothesis, for some fixed $t\in\mathbf{N}_\Gamma$, $\Gamma\models\forall i\ \theta(a,t,i)$. Since $\mathbf{U}$ is uniform, there is an $s\in\mathbf{X}$ such that $\Gamma\models s>t$. By the definition of f, $\Gamma\models\exists i<g(s)\neg\theta(a,t,i)$, yielding the contradiction. Note that this portion of the proof does not use the assumption that Γ is countable. ■

The next definition formalizes the construction of clones from canonical initial segments.

Definition 5.25: Let $\Gamma\models\mathbf{RCA}_0$ and let $\mathbf{U}$ be a uniform restricted ultrafilter on $\mathbf{N}_\Gamma$. The canonical clone of Γ modulo $\mathbf{U}$, denoted by $\Psi_{\Gamma,\mathbf{U}}$, is defined by

$\Psi_{\Gamma,\mathbf{U}}=<\mathbf{I}_{\Gamma,\mathbf{U}}, \mathbf{R}_{\prod_\mathbf{U}\mathbf{N}_\Gamma}\mathbf{I}_{\Gamma,\mathbf{U}}>$.

The most interesting canonical clones are those in which $\Psi_{\Gamma,\mathbf{U}}\models\mathbf{I}\Sigma_1^0$. The following two definitions are used in a lemma giving a necessary and sufficient condition on $\mathbf{U}$ insuring this much induction in $\Psi_{\Gamma,\mathbf{U}}$.

Definition 5.26: Let $a\in\mathbf{N}_\Gamma$. A binary function $f\in\mathbf{S}_\Gamma$ such that $f:a\times\mathbf{N}_\Gamma\to\mathbf{N}_\Gamma$ is called locally increasing if

$$\Gamma\models\forall x\,\forall y\,\forall i\,(x<y<a\to f(x,i)\leq f(y,i)).$$

Definition 5.27: A restricted ultrafilter $\mathbf{U}$ on $\mathbf{N}_\Gamma$ is called separating if $\mathbf{U}$ is uniform and for every $a\in\mathbf{N}_\Gamma$ and every locally increasing total binary function $f\in\mathbf{S}_\Gamma$ such that $f:a\times\mathbf{N}_\Gamma\to\mathbf{N}_\Gamma$, there is a $t\in\mathbf{N}_\Gamma$ such that for all $j\in\mathbf{N}_\Gamma$,

$$\{i\in\mathbf{N}_\Gamma:\Gamma\models\forall x<a\,(f(x,i)\leq t\vee f(x,i)>j\}\in\mathbf{U}.$$

Lemma 5.28: Suppose that $\Gamma \models \mathbf{RCA}_0$ and $\mathbf{U}$ is a restricted ultrafilter on $\mathbf{N}_\Gamma$. The following are equivalent:

i) $\mathbf{U}$ is separating.

ii) $\Psi_{\Gamma,\mathbf{U}} \models \mathbf{I}\Sigma_1^0$.

Proof: To prove i) implies ii), suppose that $\mathbf{U}$ is separating. Let $\theta \in \Sigma_0^0$, $a \in \mathbf{N}_{\Psi_{\Gamma,\mathbf{U}}}$, and $\mathbf{X} \in \mathbf{S}_{\Psi_{\Gamma,\mathbf{U}}}$ such that

1) $\Psi_{\Gamma,\mathbf{U}} \models \exists t\, \theta(0,t,\mathbf{X})$ and

2) $\Psi_{\Gamma,\mathbf{U}} \models \forall x < a\, (\exists t\, \theta(x,t,\mathbf{X}) \rightarrow \exists t\, \theta(x+1,t,\mathbf{X}))$.

Let $g \in \mathbf{S}_\Gamma$ be such that $[g]$ codes $\mathbf{X}$ in $\prod_\mathbf{U} \mathbf{N}_\Gamma$. Let $\theta_1 \in \mathbf{L}_1$ be the formula which replaces the use of $\in$ and $\mathbf{X}$ in θ by the equivalent $\mathbf{L}_1$ formulas using $[g]$. Let $h \in \mathbf{S}_\Gamma$ code the greatest element $[h] \in \prod_\mathbf{U} \mathbf{N}_\Gamma$ such that

$$\prod_\mathbf{U} \mathbf{N}_\Gamma \models [h] < a \wedge \forall x \le [h]\, \exists t < [f^{\,id}]\, \theta_1(x,t,[g]).$$

Without loss of generality we may assume that $\Gamma \models \forall i\, \forall x \le h(i)\, \exists t < i\, \theta_1(x,t,g(i))$. Let $b \in \mathbf{N}_\Gamma$ be some element such that $\Psi_{\Gamma,\mathbf{U}} \models [h] < [f^{\,b}]$. Define $f : b+1 \times \mathbf{N}_\Gamma \rightarrow \mathbf{N}_\Gamma$ in $\mathbf{S}_\Gamma$ by

$$f(x,n) = \begin{cases} \mu k \le n\, (\forall i \le x\, \exists t < k\;\; \theta_1(i,t,g(n))) \\ \qquad \text{if } x \le h(n) \text{ and such a } k \text{ exists.} \\ n \;\text{ if } b \ge x > h(n). \end{cases}$$

By definition, f is total and locally increasing. Since $\mathbf{U}$ is separating, we can find $m \in \mathbf{N}_\Gamma$ such that

$$\forall j \in \mathbf{N}_\Gamma \{ i \in \mathbf{N}_\Gamma : \Gamma \models \forall x \le b\, f(x,i) \le m \vee f(x,i) > j \} \in \mathbf{U}.$$

Using m, we can define $w \in \mathbf{S}_\Gamma$ such that $w : \mathbf{N}_\Gamma \to \mathbf{N}_\Gamma$ by

$$w(i) = \mu x \leq b\,(f\,(x,i) > m).$$

If $\Psi_{\Gamma,\mathbf{U}} \models 0 = [w]$, then $\Psi_{\Gamma,\mathbf{U}} \models \forall t\, \neg\theta(0,t,\mathbf{X})$, contradicting 1). If $\Psi_{\Gamma,\mathbf{U}} \models 0 < [w] < a$, then $\Psi_{\Gamma,\mathbf{U}} \models \exists t\, \theta([w]-1,t,\mathbf{X}) \wedge \forall t\, \neg\theta([w],t,\mathbf{X})$, contradicting 2). Thus $\Psi_{\Gamma,\mathbf{U}} \models [w] \geq a$, so $\Psi_{\Gamma,\mathbf{U}} \models \forall x < a\, \exists t < [f^{\,m}]\theta(x,t,\mathbf{X})$. Since the choice of a was arbitrary, $\Psi_{\Gamma,\mathbf{U}} \models \mathrm{I}\Sigma_1^0$.

To prove that ii) implies i), suppose that $\mathbf{U}$ is not separating. Choose $f \in \mathbf{S}_\Gamma$ and $a \in \mathbf{N}_\Gamma$ such that $f : a \times \mathbf{N}_\Gamma \to \mathbf{N}_\Gamma$ is total and locally increasing and

$$\forall t \in \mathbf{N}_\Gamma \exists j \in \mathbf{N}_\Gamma \{i \in \mathbf{N}_\Gamma : \Gamma \models \exists x < a\,(f\,(x,i) > t \wedge f\,(x,i) \leq j)\} \in \mathbf{U}.$$

Without loss of generality we may assume that $\Gamma \models \forall i f\,(0,i) = 0$. Let $g \in \mathbf{S}_\Gamma$ be given by $g(i) = \prod_{x<a} p_x^{(x,f\,(x,i))_p}$ and let $\mathbf{X} \in \mathbf{S}_{\Psi_{\Gamma,\mathbf{U}}}$ be the set coded by $[g]$ in $\prod_{\mathbf{U}} \mathbf{N}_\Gamma$. Then for all $b, c \in \mathbf{N}_{\Psi_{\Gamma,\mathbf{U}}}$,

$$\Psi_{\Gamma,\mathbf{U}} \models (b,c)_p \in \mathbf{X} \text{ iff } \{i \in \mathbf{N}_\Gamma : \Gamma \models b(i) < a \wedge f\,(b(i),i) = c(i)\} \in \mathbf{U}.$$

Since $\Gamma \models \forall i f\,(0,i) = 0$, $\Psi_{\Gamma,\mathbf{U}} \models ([f^{\,0}],[f^{\,a}])_p \in \mathbf{X}$. Suppose that $\Psi_{\Gamma,\mathbf{U}} \models [h] < [f^{\,a}] - 1 \wedge \neg \exists t\,([h],t)_p \in \mathbf{X}$. Then

$$\exists j \in \mathbf{N}_\Gamma \{i \in \mathbf{N}_\Gamma : \Gamma \models \exists x < a\,(f\,(x,i) > t \wedge f\,(x,i) \leq j)\} \in \mathbf{U}.$$

Since f is locally increasing,

$$\exists j \in \mathbf{N}_\Gamma \{i \in \mathbf{N}_\Gamma : \Gamma \models (f\,(h(i)+1,i) \leq j)\} \in \mathbf{U}.$$

By Theorem 5.12ii) and the definition of $\mathbf{N}_{\Psi_{\Gamma,\mathbf{U}}}$, $\Psi_{\Gamma,\mathbf{U}} \models \exists t\,([h]+1,t)_p \in \mathbf{X}$. Thus,

$$\Psi_{\Gamma,\mathbf{U}} \models \exists x < [f^{\,a}] - 1(\exists t\,(x,t)_p \in \mathbf{X} \to \exists t\,(x+1,t)_p \in \mathbf{X}).$$

However, if $[h] \in \mathbf{N}_{\Psi_{\Gamma,\mathbf{U}}}$, then $\exists j \in \mathbf{N}_\Gamma$ $\Psi_{\Gamma,\mathbf{U}} \models [h] < [f^j]$. Since f is locally increasing,

$$\{i \in \mathbf{N}_\Gamma : \Gamma \models f(a-1,i) > j\} \in \mathbf{U},$$

so $\Psi_{\Gamma,\mathbf{U}} \models \forall t < [h]([f^{a-1}],t)_p \notin \mathbf{X}$. Since h was chosen arbitrarily, $\Psi_{\Gamma,\mathbf{U}} \models \forall t([f^{a-1}],t)_p \notin \mathbf{X}$. Thus, $\Psi_{\Gamma,\mathbf{U}}$ does not model $\mathbf{I}\Sigma_1^0$. ■

Corollary 5.29: Suppose $\Gamma \models \mathbf{RCA}_0$ and $\mathbf{U}$ is a uniform restricted ultrafilter on $\mathbf{N}_\Gamma$. The following are equivalent:

i) $\mathbf{U}$ is separating.

ii) $\Psi_{\Gamma,\mathbf{U}} \models \mathbf{WKL}_0$.

Proof: If $\mathbf{U}$ is separating then by Lemma 5.29, $\Psi_{\Gamma,\mathbf{U}} \models \mathbf{I}\Sigma_1^0$. By Corollary 5.13, $\prod_{\mathbf{U}} \mathbf{N}_\Gamma$ models $\mathbf{P}^-$ plus $\mathbf{I}\Sigma_0$. By Theorem 5.3, $\Psi_{\Gamma,\mathbf{U}} \models \mathbf{WKL}_0$. Since $\mathbf{WKL}_0$ includes $\mathbf{I}\Sigma_1^0$, ii) implies i) is immediate from Lemma 5.29. ■

The following lemma gives a necessary and sufficient condition for the existence of separating ultrafilters. It is noteworthy that this condition is strictly stronger than the condition shown to be equivalent to the existence of additive ultrafilters in Lemma 5.24.

Lemma 5.30: Let Γ be a countable model. The following are equivalent:

i) $\Gamma \models \mathbf{I}\Sigma_2^0$.

ii) There is a separating restricted ultrafilter on $\mathbf{N}_\Gamma$.

Proof: First, we will assume that $\Gamma \models \mathbf{I}\Sigma_2^0$ and construct a separating ultrafilter on $\mathbf{N}_\Gamma$. Let $\langle f_i \rangle_{i \in \omega}$ be an enumeration of all the total locally increasing binary functions $f \in \mathbf{S}_\Gamma$ mapping $a \times \mathbf{N}_\Gamma$ into $\mathbf{N}_\Gamma$ for some $a \in \mathbf{N}_\Gamma$. We will inductively define a nested filter base for a separating restricted ultrafilter on $\mathbf{N}_\Gamma$. Let $\mathbf{X}_0 = \mathbf{N}_\Gamma$. Suppose that for $j \in \omega$, $\mathbf{X}_j \in \mathbf{S}_\Gamma$ has been defined and is unbounded in $\mathbf{N}_\Gamma$. $\mathbf{X}_{j+1}$ is constructed by one of the following three cases.

Case 1: Suppose that $\Gamma \models \forall t \, \exists i \in \mathbf{X}_j \, \forall x < a \, (f_j(0,i) > t)$, where $a \times \mathbf{N}_\Gamma$ is the domain of f_j. Define $\mathbf{X}_{j+1}$ by $\mathbf{X}_{j+1} = \{x_i : i \in \mathbf{N}_\Gamma\}$ where

$$x_0 = \mu t \in \mathbf{X}_j \, (f_j(0,t) > 0), \text{ and}$$

$$x_{k+1} = \mu t \in \mathbf{X}_j \, (\forall i \leq k \, (f_j(0,t) > f_j(0,x_k))).$$

Case 2: Suppose that $\Gamma \models \exists t \, \forall i \in \mathbf{X}_j \, \forall x < a \, (f_j(x,i) < t)$, where $a \times \mathbf{N}_\Gamma$ is the domain of f_j. Set $\mathbf{X}_{j+1} = \mathbf{X}_j$.

Case 3: Suppose that neither Case 1 nor Case 2 holds. That is, $\Gamma \models \exists t \, \forall i \in \mathbf{X}_j \, (f_j(0,i) \leq t)$, and $\Gamma \models \forall t \, \exists i \in \mathbf{X}_j \, \exists x < a \, (f_j(x,i) > t)$, where $a \times \mathbf{N}_\Gamma$ is the domain of f_j. Clearly $a > 1$. Suppose for a moment that

$$\Gamma \models \forall t \, \exists b \, \forall i \in \mathbf{X}_j \, \forall x < a-1 (f_j(x,i) \geq t \vee f_j(x+1,i) < b).$$

We will now derive a contradiction via an application of $\mathrm{I}\Sigma_2^0$.

Fix $y < a-1$. Suppose that for some $t \in \mathbf{N}_\Gamma$, $\Gamma \models \forall i \in \mathbf{X}_j \, (f_j(y,i) < t)$. Then

$$\Gamma \models \exists b \, \forall i \in \mathbf{X}_j \, \forall x < a-1 (f_j(x,i) \geq t \vee f_j(x+1,i) < b).$$

Thus, $\Gamma \models \exists b \, \forall i \in \mathbf{X}_j \, (f_j(y+1,i) < b)$. Since the choice of $y < a-1$ was arbitrary, we have

$$\Gamma \models \forall y < a-1 (\exists b \, \forall i \in \mathbf{X}_j \, (f_j(y,i) < b) \rightarrow \exists b \, \forall i \in \mathbf{X}_j \, (f_j(y+1,i) < b)).$$

Since $\Gamma \models \forall x \leq a-1 \exists b \, \forall i \in \mathbf{X}_j \, (f_j(0,i) < b)$, by $\mathrm{I}\Sigma_2^0$ in Γ,

$$\Gamma \models \forall x \leq a-1 \exists b \ \forall i \in \mathbf{X}_j (f_j(x,i) < b).$$

Since f is locally increasing, taking $x = a-1$ yields

$$\Gamma \models \exists b \ \forall i \in \mathbf{X}_j \ \forall x < a (f_j(x,i) < b),$$

contradicting the negation of the hypothesis for Case 2. Thus,

$$\Gamma \models \exists t \ \forall b \ \exists i \in \mathbf{X}_j \ \exists x < a-1 (f_j(x,i) < t \wedge f_j(x+1,i) \geq b).$$

We can now define $\mathbf{X}_{j+1}$. Fix $t \in \mathbf{N}_\Gamma$ as given in the last equation. Define $\mathbf{X}_{j+1}$ by $\mathbf{X}_{j+1} = \{x_i : i \in \mathbf{N}_\Gamma\}$ where

$$x_0 = \mu n \in \mathbf{X}_j \ \exists x < a-1 (f_j(x,n) > 0) \text{ and}$$

$$x_{k+1} = \mu n \in \mathbf{X}_j \ \exists x < a-1 (n > x_k \wedge f_j(x,n) \leq t \wedge f_j(x+1,n) > k).$$

It is clear from the construction that for each $j \in \omega$, $\mathbf{X}_j \in \mathbf{S}_\Gamma$, $\mathbf{X}_j$ is unbounded in $\mathbf{N}_\Gamma$, and $\mathbf{X}_j \supseteq \mathbf{X}_{j+1}$. Let $\mathbf{U}$ be a uniform restricted ultrafilter containing $\mathbf{X}_j$ for all $j \in \omega$. It is straightforward to see that $\mathbf{U}$ is a separating ultrafilter. Pick any locally increasing total binary function, f, mapping $a \times \mathbf{N}_\Gamma$ into $\mathbf{N}_\Gamma$ for some $a \in \mathbf{N}_\Gamma$. Then f is f_j for some $j \in \omega$. If $\mathbf{X}_{j+1}$ was constructed according to Case 1, then for every $k \in \mathbf{N}_\Gamma$,

$$\{i \in \mathbf{N}_\Gamma : f(0,i) > k\} \supseteq \mathbf{X}_{j+1} - \{x_i : i \leq k\} \in \mathbf{U}.$$

Since f_j is locally increasing, for any $k \in \mathbf{N}_\Gamma$,

$$\{i \in \mathbf{N}_\Gamma : \forall x < a (f(x,i) \leq 0 \vee f(x,i) > k)\} \in \mathbf{U}.$$

If $\mathbf{X}_{j+1}$ was constructed according to Case 2, then for the bound t from that case,

$$\{i \in \mathbf{N}_\Gamma : \forall x < a\,(f\,(x,i) \le t\,)\} \supseteq \mathbf{X}_{j+1} \in \mathbf{U},$$

so for any $k \in \mathbf{N}_\Gamma$,

$$\{i \in \mathbf{N}_\Gamma : \forall x < a\,(f\,(x,i) \le t \bigvee f\,(x,i) > k\,)\} \in \mathbf{U}.$$

Finally, if $\mathbf{X}_{j+1}$ was constructed according to Case 3, using the bound t fixed in that case, for any $k \in \mathbf{N}_\Gamma$, we have

$$\{i \in \mathbf{N}_\Gamma : \forall x < a\,(f\,(x,i) \le t \bigvee f\,(x,i) > k\,)\} \supseteq \mathbf{X}_{j+1} - \{x_i : i \le k\,\} \in \mathbf{U}.$$

This completes the proof that i) implies ii).

We will now turn to the proof that ii) implies i). The hypothesis that Γ is countable is not used in this portion of the proof. Let $\mathbf{U}$ be a restricted separating ultrafilter on $\mathbf{N}_\Gamma$. By definition, $\mathbf{U}$ is automatically uniform. Let $\theta \in \Sigma_0^0$ be a formula with parameters in $\mathbf{S}_\Gamma$ such that

$$\Gamma \models \exists x\, \forall k\, \exists j\, \theta(x,k,j).$$

We will show that Γ models the existence of a least such x. This implies that $\Gamma \models \mathrm{L}\Pi_2^0$, which, by the comment following Theorem 5.1, implies that $\Gamma \models \mathrm{I}\Sigma_2^0$. Fix $a \in \mathbf{N}_\Gamma$ such that $\Gamma \models \forall k\, \exists j\, \theta(a,k,j)$. For $n \le a$ and $y \in \mathbf{N}_\Gamma$, define the function $f \in \mathbf{S}_\Gamma$ by

$$f\,(n,y) = \begin{cases} \mu t < y\,(\forall x \le n\, \forall i < y\, \exists k \le t\, \forall j < i\, \neg\theta(x,k,j)) \\ \qquad \text{if such a } t \text{ exists} \\ y \ \text{otherwise.} \end{cases}$$

For a fixed $y \in \mathbf{N}_\Gamma$ and any $n < a$, $\Gamma \models f\,(n,y) < f\,(n+1,y)$, so f is locally increasing. For fixed $y,z \in \mathbf{N}_\Gamma$ and any $n < a$,

$\Gamma \models \forall x < n \, \forall i < y+1 \, \exists k \leq z \, \forall j < i \, \neg\theta(x,k,j) \rightarrow \forall x < n \, \forall i < y \, \exists k \leq z \, \forall j < i \, \neg\theta(x,k,j)$,

so $\Gamma \models f(n,y) \leq f(n,y+1)$. Since $\mathbf{U}$ is separating, we can fix a $t \in \mathbf{N}_\Gamma$ such that for any $j \in \mathbf{N}_\Gamma$,

$$\{i \in \mathbf{N}_\Gamma : \Gamma \models \forall x < a \, (f(x,i) \leq t \vee f(x,i) > j\} \in \mathbf{U}.$$

By $\mathbf{B}\Sigma_1^0$ in Γ,

$$\Gamma \models \exists n \, \forall k \leq t \, \exists j < n \, \theta(a,k,j).$$

Pick $y \in \mathbf{N}_\Gamma$ such that $y > n$. Then

$$\Gamma \models \exists x \leq a \, \exists n < y \, \forall k \leq t \, \exists j < n \, \theta(a,k,j),$$

so $\Gamma \models f(a,y) > t$. By $\mathbf{L}\Sigma_1^0$ in Γ, there is an $m \in \mathbf{N}_\Gamma$ such that

$$\Gamma \models \exists y \, (f(m,y) > t) \wedge \forall x < m \, \forall y \, (f(x,y) \leq t).$$

The following four claims complete the proof of the lemma.

Claim 1: $\Gamma \models \forall x < m \, \exists n \leq t \, \forall i \, \exists k \leq n \, \forall j < i \, \neg\theta(x,k,j)$.

Proof: Fix $x < m$. Then $\Gamma \models \forall y f(x,y) \leq t$, so in particular, $\Gamma \models \forall y f(x,y) \neq t+1$. By $\mathbf{L}\Pi_1^0$ in Γ, there is a least $n \in \mathbf{N}_\Gamma$ such that $\Gamma \models \forall f(x,y) \neq n+1$. By the choice of n, we may fix $y \in \mathbf{N}_\Gamma$ such that $\Gamma \models f(x,y) = n$, and since f is increasing in the second component, $\Gamma \models \forall r \, (r > y \rightarrow f(x,r) = n)$. By the definition of f,

$$\Gamma \models \forall r > y \, \forall u \leq x \, \forall i \leq r \, \exists k \leq n \, \forall j < i \, \neg\theta(u,k,j),$$

so $\Gamma \models \forall i \, \exists k \leq n \, \forall j < i \, \neg\theta(x,k,j)$, as desired.

Claim 2: $\Gamma \models \forall x < m \, \exists k \, \forall j \, \neg\theta(x,k,j)$.

Proof: Fix $x<m$. Suppose that $\Gamma\models\forall k\leq t\,\exists j\,\theta(x,j,k)$. Applying $\mathbf{B}\Sigma_1^0$ in Γ, yields $\Gamma\models\exists i\,\forall k\leq t\,\exists j<i\,\theta(x,k,j)$, contradicting Claim 1. Thus, we must have $\Gamma\models\exists k\,\forall j\,\neg\theta(x,k,j)$, as desired.

Claim 3: $\Gamma\models\forall b\,\exists y f\,(m,y)>b$.

Proof: Fix b. Choose n such that $\Gamma\models f\,(m,n)>t$. Define $\mathbf{X}\in\mathbf{S}_\Gamma$ by

$$\mathbf{X}=\{i\in\mathbf{N}_\Gamma{:}\Gamma\models\forall x<a\,(f\,(x,i)\leq t\bigvee f\,(x,i)>b\,)\}.$$

By the choice of t, $\mathbf{X}\in\mathbf{U}$, so by uniformity of $\mathbf{U}$, we may choose $x\in\mathbf{X}$ such that $x>n$. Then $\Gamma\models f\,(m,x)\geq f\,(m,n)>t$, so $\Gamma\models f\,(m,x)>b$.

Claim 4: $\Gamma\models\forall k\,\exists j\,\theta(m,k,j)$.

Proof: By Claim 3, $\forall b\,\exists y f\,(m,y)>b$, so by the definition of f,

$$\Gamma\models\forall b\,\exists y\,\forall c<b\,\exists x\leq m\,\exists i<y\,\forall k\leq c\,\exists j<i\,\theta(x,k,j).$$

Fix an arbitrary $b\geq t+1$. Then

$$\Gamma\models\exists y\,\forall c<b+1\exists x\leq m\,\exists i<y\,\forall k\leq c\,\exists j<i\,\theta(x,k,j).$$

Setting $c=b$, the previous statement implies that

$$\Gamma\models\exists x\leq m\,\forall k\leq b\,\exists j\,\theta(x,k,j).$$

Since $b>t$, the proof of Claim 2 shows that $\Gamma\models\forall k\leq b\,\exists j\,\theta(m,k,j)$. The choice of b was arbitrary, so $\Gamma\models\forall k\,\exists j\,\theta(m,k,j)$ as desired.

To complete the proof that ii) implies i), we combine Claim 2 and Claim 4 to obtain

$$\Gamma\models\forall k\,\exists j\,\theta(m,k,j)\bigwedge\forall x<m\,\exists k\,\forall j\,\neg\theta(x,k,j).$$

Thus $\Gamma\models\mathbf{L}\Pi_2^0$, completing the proof of the lemma. ■

Porism 5.31: If Γ is a countable model of $\mathbf{RCA}_0$ and $I\Sigma_2^0$ then there is a restricted ultrafilter on $\mathbf{N}_\Gamma$ which is both additive and separating.

Proof: The ultrafilter is constructed by carrying out the constructions of Lemma 5.30 and Lemma 5.24 on alternate steps. ■

Corollary 5.32: There is a countable model $\Gamma \models \mathbf{WKL}_0$, such that there is an additive restricted ultrafilter on $\mathbf{N}_\Gamma$ but no separating restricted ultrafilter on $\mathbf{N}_\Gamma$.

Proof: By Theorem 1 and Corollary 30 of Paris [36], there is an initial segment $\mathbf{I}$ of a model $\mathbf{M}$ of $\mathbf{PA}$ such that $<\mathbf{I},\mathbf{R}_\mathbf{M}\mathbf{I}> \models \mathbf{B}\Pi_1^0 \wedge \neg I\Sigma_2^0$. By Theorem 5.3, $<\mathbf{I},\mathbf{R}_\mathbf{M}\mathbf{I}> \models \mathbf{WKL}_0$. By Lemma 5.24 and Lemma 5.30, there is an additive restricted ultrafilter on $<\mathbf{I},\mathbf{R}_\mathbf{M}\mathbf{I}>$, but no separating restricted ultrafilter. ■

Corollary 5.33: There is a countable model Γ of $\mathbf{WKL}_0$ such that for no restricted ultrafilter $\mathbf{U}$ on $\mathbf{N}_\Gamma$ is $\Psi_{\Gamma,\mathbf{U}}$ a model of $I\Sigma_1^0$.

Proof: Let Γ be the model of Corollary 5.32. No restricted ultrafilter on $\mathbf{N}_\Gamma$ is separating, so for any restricted ultrafilter $\mathbf{U}$, $\Psi_{\Gamma,\mathbf{U}}$ does not model $I\Sigma_1^0$. ■

The preceding corollary shows that for Γ, an arbitrary model of $\mathbf{WKL}_0$, the canonical clone of Γ need not have a nice structure. The situation is much different for countable models of $\mathbf{ACA}_0$. The following theorem shows that any countable model of $\mathbf{ACA}_0$ can grow a canonical clone which is a copy of itself. This result can be viewed as a characterization theorem. In this sense, it states that every countable model of $\mathbf{ACA}_0$ can be embedded into a model of the $\mathbf{L}_1$ theory of its first order part.

Theorem 5.34: Let $\Gamma\models\mathbf{ACA}_0$ be a countable model. Then there is an additive restricted ultrafilter $\mathbf{U}$ on $\mathbf{N}_\Gamma$ such that $\Psi_{\Gamma,\mathbf{U}}$ is a Γ-clone.

Proof: First we will construct $\Psi_{\Gamma,\mathbf{U}}$, and then we will construct the isomorphism between Γ and $\Psi_{\Gamma,\mathbf{U}}$.

The ultrafilter $\mathbf{U}$ is defined by constructing a countable filter base. Let $<f_i>_{i\in\omega}$ be an enumeration of the functions in $\mathbf{S}_\Gamma$ mapping $[\mathbf{N}_\Gamma]^3$ into 2. Let $\mathbf{X}_0=\mathbf{N}_\Gamma$. Suppose that $\mathbf{X}_j\in\mathbf{S}_\Gamma$ has been defined and is unbounded in $\mathbf{N}_\Gamma$. Let $\mathbf{X}_{j+1}$ be an unbounded subset of $\mathbf{X}_j$ which is in $\mathbf{S}_\Gamma$ and is monochromatic for f_j. Since $\Gamma\models\mathbf{ACA}_0$, such a set exists. Let $\mathbf{U}$ be an extension of $<\mathbf{X}_i>_{i\in\omega}$ to a restricted ultrafilter on $\mathbf{N}_\Gamma$, and let $\Psi_{\Gamma,\mathbf{U}}$ be the canonical clone of Γ modulo $\mathbf{U}$ as in Definition 5.25.

Let g be a unary function of $\mathbf{S}_\Gamma$ with a bounded range. Consider the function $f:[\mathbf{N}_\Gamma]^3\rightarrow 2$ given by $f(x,y,z)=1$ if $g(x)=g(y)$, and $f(x,y,z)=0$ otherwise. The ultrafilter $\mathbf{U}$ contains a monochromatic set $\mathbf{X}$ for f. By a finite pigeon-hole argument, $f([\mathbf{X}]^3)=1$. Thus for some constant function $f^a\in\mathbf{S}_\Gamma$, $[g]=_{\mathbf{U}}[f^a]$. Since the choice of g was arbitrary, by Lemma 5.23, $\mathbf{U}$ is additive. Let $\phi:\mathbf{N}_{\Psi_{\Gamma,\mathbf{U}}}\rightarrow\mathbf{N}_\Gamma$ be the natural bijection. To show that ϕ extends to an isomorphism between $\Psi_{\Gamma,\mathbf{U}}$ and Γ, it suffices to show that for all $\mathbf{X}\in\mathbf{S}_{\Psi_{\Gamma,\mathbf{U}}}$,

$$\{i\in\mathbf{N}_\Gamma:\exists x\in\mathbf{X}(\phi(x)=i)\}\in\mathbf{S}_\Gamma.$$

Intuitively, we know that Γ and $\Psi_{\Gamma,\mathbf{U}}$ have the same integers. We now show that they have the same sets.

Let $g \in \mathbf{S}_\Gamma$ be a function coding a set $\mathbf{X} \in \mathbf{S}_{\Psi_{\Gamma,U}}$. Define $f : [\mathbf{N}_\Gamma]^3 \to 2$ in $\mathbf{S}_\Gamma$ by the following rule. $f(x, y, z) = 1$ if $g(y)$ and $g(z)$ code the same subset of $\mathbf{N}_\Gamma$ below x, otherwise $f(x, y, z) = 0$. By a finite pigeonhole argument, there is a set $\mathbf{H} \in \mathbf{U}$ such that $f([\mathbf{H}]^3) = 1$. Let $< h_i >_{i \in \mathbf{N}_\Gamma}$ be an enumeration of $\mathbf{H}$ and define $\mathbf{Y}$ by

$$\mathbf{Y} = \{ i \in \mathbf{N}_\Gamma : i \text{ is in the subset coded by } h_i \}.$$

Since $\mathbf{H} \in \mathbf{S}_\Gamma$, we have that $\mathbf{Y} \in \mathbf{S}_\Gamma$. Finally, it is easy to see that $i \in \mathbf{Y}$ if and only if $[f^{\,i}] \in \mathbf{X}$, completing the proof. ■

The following two results are easy consequences of Theorem 5.34. Porism 5.35 strengthens a theorem of MacDowell and Specker [28], stating that every model of **PA** has an elementary end extension.

Porism 5.35: (Extended MacDowell Specker Theorem.) Let $\Gamma \models \mathbf{ACA}_0$ be countable. Then there is an $\mathbf{L}_1$ elementary end extension $\mathbf{M}$ of $\mathbf{N}_\Gamma$ such that $\mathbf{R}_\mathbf{M}\mathbf{N}_\Gamma = \mathbf{S}_\Gamma$.

Proof: Let $\mathbf{U}$ be the restricted ultrafilter on $\mathbf{N}_\Gamma$ constructed in Theorem 5.34. Let $\mathbf{M} = \prod_\mathbf{U} \mathbf{N}_\Gamma$. By Theorem 5.15, $\mathbf{M}$ is an $\mathbf{L}_1$ elementary extension of $\mathbf{N}_\Gamma$. Since $\mathbf{U}$ is uniform, $\mathbf{M}$ is an end extension. Finally, $\mathbf{R}_\mathbf{M}\mathbf{N}_\Gamma = \mathbf{S}_{\Psi_{\Gamma,U}}$, so since $\Psi_{\Gamma,\mathbf{U}}$ is a Γ-clone, $\mathbf{R}_\mathbf{M}\mathbf{N}_\Gamma = \mathbf{S}_\Gamma$. ■

Corollary 5.36: Let $\Gamma \models \mathbf{ACA}_0$. Then there is a Th($\Gamma$)-clone.

Proof: By the Lowenheim-Skolem theorem, there is a countable model $\overline{\Gamma}$ such that Th($\overline{\Gamma}$)=Th(Γ). By Theorem 5.34, there is a $\overline{\Gamma}$-clone. ■

The next theorem is a converse to Theorem 5.34. It shows that the assumption that Γ models $\mathbf{ACA}_0$ is a necessary hypothesis in the preceding theorem. It is not known if the statement is true when the requirement that $\mathbf{U}$ is additive is removed.

Theorem 5.37: Let $\Gamma\models\mathbf{RCA}_0$ be a countable model. If there is an additive restricted ultrafilter $\mathbf{U}$ on $\mathbf{N}_\Gamma$ such that $\Psi_{\Gamma,\mathbf{U}}$ is a Γ-clone, then $\Gamma\models\mathbf{ACA}_0$.

Proof: Let Γ and $\mathbf{U}$ be as stated in the theorem. Let $\phi:\Gamma\to\Psi_{\Gamma,\mathbf{U}}$ be an isomorphism. Then ϕ is a Boolean algebra isomorphism between $\mathbf{S}_\Gamma$ and $\mathbf{S}_{\Psi_{\Gamma,\mathbf{U}}}$. Thus, $\phi(\mathbf{U})$ is a restricted ultrafilter on $\mathbf{N}_{\Psi_{\Gamma,\mathbf{U}}}$. It is easy to see that $\phi(\mathbf{U})$ is uniform.

We will now show that $\phi(\mathbf{U})$ is additive. Let $\phi(f)\in\mathbf{S}_{\Psi_{\Gamma,\mathbf{U}}}$ be a function such that for some $\phi(a)\in\mathbf{N}_{\Psi_{\Gamma,\mathbf{U}}}$, $\phi(f):\mathbf{N}_{\Psi_{\Gamma,\mathbf{U}}}\to\phi(a)$. Then $\Gamma\models f:\mathbf{N}_\Gamma\to a$, so since $\mathbf{U}$ is additive, for some $\mathbf{X}\in\mathbf{U}$ and $j<a$, $\Gamma\models\forall i\in\mathbf{X}f(i)=j$. Thus $\Psi_{\Gamma,\mathbf{U}}\models\forall i\in\phi(\mathbf{X})\phi(f)(i)=\phi(j)$.

Let $\mathbf{M}$ denote $\prod_{\mathbf{U}}\mathbf{N}_\Gamma$. To simplify notation, we will identify $\mathbf{U}$ with $\phi(\mathbf{U})$ and $\mathbf{N}_\Gamma$ with $\mathbf{N}_{\Psi_{\Gamma,\mathbf{U}}}$. Construct a new model $\mathbf{K}$ of $\mathbf{L}_1$ by taking the ultrapower of $\mathbf{M}$ modulo $\mathbf{U}$, i.e. $\mathbf{K}=\Pi_{\mathbf{U}}\mathbf{M}$. Since $\mathbf{U}$ is additive, $\mathbf{N}_\Gamma$ is (isomorphic to) an initial segment of $\mathbf{K}$. Since $\mathbf{U}$ is uniform, the identity function in $\mathbf{M}$ defines an element $k\in\mathbf{K}$ such that $k\notin\mathbf{N}_\Gamma$ and $\mathbf{K}\models k<b$ for all $b\in\mathbf{M}-\mathbf{I}$. Let $\mathbf{X}\in\mathbf{R}_{\mathbf{K}}\mathbf{N}_\Gamma$. Then an overspill argument shows that $\mathbf{X}$ can be coded by some element $[f]\in\mathbf{K}$ such that $\mathbf{K}\models[f]<k$. Thus $f\in\mathbf{R}_{\mathbf{M}}\mathbf{N}_\Gamma$. Let $\overline{\mathbf{X}}\in\mathbf{R}_{\mathbf{M}}\mathbf{N}_\Gamma$ such that $\overline{\mathbf{X}}$ is coded by $[f]$ in $\mathbf{M}$. Then for all $j\in\mathbf{N}_\Gamma$,

$$\mathbf{K}\models j\in\mathbf{X} \quad \text{iff}$$

$$\exists\mathbf{Y}\in\mathbf{U}\forall i\in\mathbf{Y}\ \ \mathbf{K}\models p_j\mid f(i) \quad \text{iff}$$

$$\exists\mathbf{Y}\in\mathbf{U}\forall i\in\mathbf{Y}\ \ \mathbf{M}\models p_j\mid f(i) \quad \text{iff}$$

$$M \models j \in \overline{\mathbf{X}}.$$

Thus $\mathbf{R_M N_\Gamma} = \mathbf{R_K N_\Gamma}$. By Paris and Kirby's Theorem 7 [26] this suffices to show that $\mathbf{N_\Gamma}$ is strong in M. By Theorem 5.4, $<\mathbf{N_\Gamma}, \mathbf{R_M N_\Gamma}> \models \mathbf{ACA_0}$. Since $<\mathbf{N_\Gamma}, \mathbf{R_M N_\Gamma}>$ is $\mathbf{\Psi_{\Gamma,U}}$ and $\Psi_{\Gamma,U}$ is a Γ-clone, $\Gamma \models \mathbf{ACA_0}$. ■

CHAPTER 6

RAMSEY'S THEOREM

In this chapter, we examine the provability of certain restrictions of Ramsey's theorem in subsystems of $\mathbf{Z}_2$. The first two sections contain results on Ramsey's theorem for singletons and pairs. The third section contains a theorem on min-homogeneous sets for regressive partitions which provides an interesting contrast to the usual Ramsey's theorem. The final section contains conjectures on the proof theoretic strength of Ramsey's theorem for pairs. Since it has long been known that Ramsey's theorem for k-tuples (where $k \geq 3$) is equivalent to $\mathbf{ACA}_0$ over $\mathbf{RCA}_0$, the situations for singletons and pairs are the only ones of interest.

We will make use of several standard notational conventions. For instance, $[\mathrm{N}]^k$ denotes the set of k element subsets of $\mathbf{N}$. An element $\{x_0, x_1, \ldots, x_k\}$ of $[\mathrm{N}]^k$ is always treated as a sequence in increasing order. This convention allows us to write partitions of $[\mathrm{N}]^k$ as k-ary functions whenever convenient. The formula $f([\mathrm{X}]^k)=c$ means for every $\mathrm{Y}\in[\mathrm{X}]^k$, $f(\mathrm{Y})=c$. Finally, we use the following notation to avoid restating the entire Ramsey's theorem simply to change the exponent or the number of colors.

Notation 6.1: $\mathrm{RT}(n, k)$ dnotes the formula of L_2 representing the statement: For every $f:[\mathrm{N}]^n \rightarrow k$ there is an infinite set $\mathbf{X}$ and a $c<k$ such that $f([\mathrm{X}]^n)=c$.

Notation 6.2: $\mathrm{RT}(n)$ denotes the formula $\forall k\, \mathrm{RT}(n, k)$.

6.1. Singletons

If $\mathbf{N}$ is colored with a finite number of colors, it seems obvious that an infinite monochromatic set must exist. Certainly, if the number of colors is an element of ω, this is the case.

Theorem 6.3: For all $n \in \omega$, $\mathbf{RCA}_0 \vdash \mathbf{RT}(1,n)$.

Proof: Fix $n \in \omega$. We will work in $\mathbf{RCA}_0$. Suppose $f : \mathbf{N} \to n$. If we have that $\forall t < n \exists y \forall x > y (f(x) \neq t)$, then $\bigwedge_{t<n} \exists y \forall x > y (f(x) \neq t)$, which implies that $\exists y \forall t < n (f(y) \neq t)$, a contradiction. Thus, $\exists t < n \forall y \exists x > y (f(x) = t)$. The sets $\mathbf{X}_t = \{x \in \mathbf{N} : f(x) = t\}$ all exist by Δ_1^0 comprehension, and one of these must be the desired homogeneous set. ■

In the proof of Theorem 6.3, we pass from a quantifier bounded by the number of colors, n, to a finite conjunction with n conjuncts. If n is nonstandard, this technique no longer works. The general form for such quantifier swapping amounts to a bounding scheme. With this in mind, the following theorem is no great surprise.

Theorem 6.4: ($\mathbf{RCA}_0$) The following are equivalent:

i) $\mathbf{RT}(1)$.

ii) $\mathbf{B}\Pi_1^0$.

Proof: To prove that i) implies ii), assume $\mathbf{RT}(1)$. Let θ be a Σ_0^0 formula (possibly with set parameters). Fix y and suppose that $\forall x < y \exists z \forall w \, \theta(x,z,w)$. Define f by

$$f(t) = \begin{cases} \mu n < t \, (\forall x < y \exists z < n \forall w < t \, \theta(x,z,w)) & \text{if such an } n \text{ exists,} \\ t & \text{otherwise.} \end{cases}$$

Suppose $\mathbf{X}$ is an infinite set such that $f(\mathbf{X})=t_0$ for some t_0. Then $\forall x<y\,\exists z<t_0\forall w\,\theta(x,z,w)$ as desired.

Suppose, on the other hand, that no such monochromatic set exists. By $\mathbf{RT}(1)$, the range of f is unbounded, i.e. $\forall n\,\exists t\,(f(t)>n)$. By Δ_1^0 comprehension, we may construct a sequence $<t_i>_{i\in\mathrm{N}}$ such that for each $i\in\mathbf{N}$, $t_i<t_{i+1}$ and $f(t_i)<f(t_{i+1})$. Define g by

$$g(i)=\mu x<y\,(\forall z<f(t_i)-1\,\exists w<t_i\,\neg\theta(x,z,w)).$$

Let $\mathbf{T}$ be an unbounded monochromatic set for g, and let $x_0=g(\mathbf{T})$. Choose z_0. Since $\mathbf{T}$ is unbounded, there is some $i\in\mathbf{T}$ such that $f(t_i)-1>z_0$. By the definition of g, $\exists w<t_i\,\neg\theta(x_0,z_0,w)$. Hence, $\forall z\,\exists w\,\neg\theta(x_0,z,w)$, contradicting our very first assumption.

To prove ii) implies i), we will assume $\mathbf{B}\Pi_1^0$. Let $f:\mathbf{N}\rightarrow y$. Suppose that $\forall x<y\,\exists z\,\forall w\,(w>z\rightarrow f(w)\neq x)$. By $\mathbf{B}\Pi_1^0$,

$$\exists t\,\forall x<y\,\exists z<t\,\forall w\,(w>z\rightarrow f(w)\neq x).$$

In particular, $\exists t\,\forall x<y\,\forall w\,(w>t\rightarrow f(w)\neq x)$. Let t_0 be such a t. Then we have $\forall x<y f(t+1)\neq x$, contradicting the definition of f. Thus, it must be the case that $\exists x<y\,\forall z\,\exists w\,(w>z\rightarrow f(w)=x)$. Let x_0 be such an x. Then $\{t:f(t)=x_0\}$ is the desired monochromatic set. ■

By applying techniques from Chapter 5, it is now easy to show that $\mathbf{RT}(1)$ is not provable in $\mathbf{WKL}_0$. In some ways, it is startling that a statement as natural as $\mathbf{RT}(1)$ is not provable in a system as powerful as $\mathbf{WKL}_0$.

Corollary 6.5: $\mathbf{WKL_0} \nvdash \mathbf{RT}(1)$.

Proof: By Propositional 2 of Paris and Kirby [26], there is an initial segment **I** of a model **M** of **PA** such that $<\mathbf{I},\mathbf{R_M I}> \models \mathbf{I}\Sigma_1^0 \wedge \neg \mathbf{B}\Pi_1^0$. By Theorem 5.3, $<\mathbf{I},\mathbf{R_M I}> \models \mathbf{WKL_0}$. By Theorem 6.4, $<\mathbf{I},\mathbf{R_M I}> \models \neg \mathbf{RT}(1)$. ■

Since Theorem 6.3 implies that every ω-model of $\mathbf{WKL_0}$ is a model of $\mathbf{RT}(1)$, we may immediately draw two additional conclusions. First, $\mathbf{RT}(1)$ is independent of $\mathbf{WKL_0}$. Secondly, $\mathbf{WKL_0}$ is not ω-consistent.

Many theorems of ordinary mathematics are equivalent to $\mathbf{WKL_0}$ or $\mathbf{ACA_0}$. $\mathbf{RT}(1)$ is an exception. As we have said, it is independent of $\mathbf{WKL_0}$. Furthermore, since every ω-model of $\mathbf{WKL_0}$ is a model of $\mathbf{RT}(1)$, $\mathbf{RT}(1)$ does not imply $\mathbf{ACA_0}$ over $\mathbf{RCA_0}$. Since it does not fit nicely into the program of reverse mathematics, one is tempted to say that $\mathbf{RT}(1)$ is an unimportant fluke. However, it does crop up occasionally in the literature, in a variety of guises. As an example, we consider a lemma of Rado.

Theorem 6.6: $(\mathbf{RCA_0})$ The following are equivalent:

i) $\mathbf{RT}(1)$.

ii) (Rado's Lemma) Let $\mathbf{A} = < a_i >_{i \in \mathbf{N}}$ be a sequence of codes for finite nonempty subsets of $\mathbf{N}$. There is a choice function for $\mathbf{A}$ satisfying

$$1) \quad \forall y \exists b \forall x (x > b \rightarrow f(x) \neq y)$$

if and only if for each infinite set $\mathbf{I} \subseteq \mathbf{N}$, the set $\mathbf{A}(\mathbf{I}) = \{ j \in \mathbf{N} : \exists i \in \mathbf{I}\, j \in \mathbf{X}_{a_i} \}$ is infinite, where $\mathbf{X}_{a_i}$ is the set coded by a_i.

Proof: To prove that i) implies ii), assume **RT**(1) and let **A** be as in the hypothesis of ii). We must prove both parts of the biconditional statement. Suppose first that for some infinite $\mathbf{I} \subseteq \mathbf{N}$, the set $\mathbf{A}(\mathbf{I})$ is finite. Then $\mathbf{A}(\mathbf{I})$ is bounded by some integer d. Let f be any choice function for **A**. By **RT**(1), there is an infinite set $\mathbf{J} \subseteq \mathbf{I}$ and an integer $c < d$ such that for all $x \in \mathbf{J}$, $f(x) = c$. Thus f does not satisfy property 1).

Now we must prove the other half of the biconditional statement. Suppose that for every infinite set $\mathbf{I} \subseteq \mathbf{N}$, the set $\mathbf{A}(\mathbf{I})$ is also infinite. Define a choice function f by the rule: $f(i)$ is the greatest element of $\mathbf{X}_{a_i}$. If f does not satisfy property 1), then there is an integer d and an infinite set **I** such that for all $i \in \mathbf{I}$, the maximum element in the set coded by a_i is d. But in this case, $\mathbf{A}(\mathbf{I})$ is finite, contradicting our hypothesis. Thus, f satisfies property 1).

To prove that ii) implies i), suppose that **RT**(1) is false. Choose $f : \mathbf{N} \to a$ such that for each $b < a$ the set $\{n \in \mathbf{N} : f(n) = b\}$ is not infinite. Define $\mathbf{A} = < a_i >_{i \in \mathbf{N}}$ by letting a_i be the code for $\{f(i)\}$. Then f is a choice function for **A** satisfying property 1). However, $\mathbf{A}(\mathbf{N})$ is the range of f which is finite. Thus ii) does not hold. ■

Corollary 6.7: $\mathbf{WKL}_0 \nvdash$ Rado's Lemma.

Proof: Immediate from Theorem 6.6 and Corollary 6.5. ■

6.2. Pairs

Given the results of the previous section, one might hope that Ramsey's theorem for pairs is better behaved. Unfortunately, pairs are even more unruly than singletons. It is open whether or not **RT**(2,2) or **RT**(2) imply $\mathbf{ACA}_0$. We do know that $\mathbf{WKL}_0$ does not prove **RT**(2,2) or **RT**(2). This result can be approached in three ways. The first, and most obvious, is via the results of the previous section.

Theorem 6.8: $\mathbf{RCA}_0 \vdash \mathbf{RT}(2,2) \rightarrow \mathbf{RT}(1)$.

Proof: Assume **RT**(2,2) and suppose $f : \mathbf{N} \rightarrow a$. Define $g : [\mathbf{N}]^2 \rightarrow 2$ by

$$g(n,m) = \begin{cases} 0 \text{ if } f(n) = f(m). \\ 1 \text{ if } f(n) \neq f(m). \end{cases}$$

Let **H** be an infinite monochromatic set for g. A finite pigeonhole argument shows that $g([\mathbf{H}]^2) = 0$. Thus, for some $b < a$, $f(\mathbf{H}) = b$. ■

Corollary 6.9: $\mathbf{WKL}_0 \nvdash \mathbf{RT}(2,2)$.

Proof: Suppose $\mathbf{WKL}_0 \vdash \mathbf{RT}(2,2)$. Then by Theorem 6.8, $\mathbf{WKL}_0 \vdash \mathbf{RT}(1)$, contradicting Corollary 6.5.

Since every ω-model of $\mathbf{WKL}_0$ models **RT**(1), it seems feasible that the same would hold for **RT**(2,2). However, monochromatic sets for **RT**(2,2) can code considerably more information than that used in the proof of Theorem 6.8. Using the clever construction of Jockusch, we can prove the following theorem. This result can be viewed as the second proof that $\mathbf{WKL}_0$ does not prove **RT**(2).

Theorem 6.10: There is an ω-model of $\mathbf{WKL}_0$ which is not a model of **RT**(2,2).

Proof: By Theorem 1.6, there is an ω-model Γ of $\mathbf{WKL}_0$ such that if $\mathbf{A} \in \mathbf{S}_\Gamma$, and **a** is the degree of **A**, then $\mathbf{a}' \leq 0'$. By Theorem 3.1 of Jockusch [19], there is a recursive $f : [\omega]^2 \rightarrow 2$ such that no infinite monochromatic set for f is recursive in $0'$. Since f is recursive, $f \in \mathbf{S}_\Gamma$, so $\Gamma \models \mathbf{WKL}_0$, but $\Gamma \models \neg \mathbf{RT}(2,2)$. ■

Since $\mathbf{RT}(2,2)$ is more powerful than $\mathbf{RT}(1)$, it would be nice to know what analog of Theorem 6.4 is provable. Unfortunately, the only known arguments make use of an arbitrary numbers of colors. The following is the best result known.

Theorem 6.11: $\mathbf{RCA}_0 \vdash \mathbf{RT}(2) \rightarrow \mathrm{B}\Pi_2^0$.

Proof: We will work in $\mathbf{RCA}_0$. Assume $\mathbf{RT}(2)$ and $\neg \mathrm{B}\Pi_2^0$. Then for some Σ_0^0 formula θ (possibly with set parameters) and some $y \in \mathbf{N}$, we have

$$1)\quad \forall x < y\, \exists z\, \forall u\, \exists v\, \theta(x, z, u, v)\text{ , and}$$

$$2)\quad \forall t\, \exists x < y\, \forall z < t\, \exists u\, \forall v\, \neg\theta(x, z, u, v).$$

Define $f : [\mathbf{N}]^2 \rightarrow \mathbf{N}$ by

$$f(t, s) = \begin{cases} \mu z < s\, \forall x < y\, \forall u < t\, \exists v < s\, \theta(x, z, u, v) & \text{if such a } z \text{ exists,} \\ s & \text{otherwise.} \end{cases}$$

Suppose that the range of f is bounded. Then by $\mathbf{RT}(2)$, we can find an infinite set $\mathbf{X}$ and a $z_0 \in \mathbf{N}$ such that $f([\mathbf{X}]^2) = z_0$. By property 2),

$$\exists x < y\, \forall z < z_0 + 1\, \exists u\, \forall v\, \neg\theta(x, z, u, v).$$

In particular, for some $x_0 < y$, $\exists u\, \forall v\, \neg\theta(x_0, z_0, u, v)$. Thus for some $u_0 \in \mathbf{N}$, $\forall v\, \neg\theta(x_0, z_0, u_0, v)$. Pick $s, t \in \mathbf{X}$ such that $u_0 < t < s$. Then since $f(t, s) = z_0$, we have $\exists v < s\, \theta(x_0, z_0, u_0, v)$, a contradiction.

Now suppose that the range of f is unbounded. We will derive another contradiction. Define $h : y \times [\mathbf{N}]^2 \rightarrow \mathbf{N}$ for $x < y$ and $s, t \in \mathbf{N}$ by

$$h(x, t, s) = \begin{cases} \mu z < s\, \forall u < t\, \exists v < s\, \theta(x, z, u, v) & \text{if such a } z \text{ exists.} \\ s & \text{otherwise.} \end{cases}$$

Define $g:[\mathbf{N}]^2 \to y$ by

$$g(t,s)=\mu x<y(h(x,t,s)=\max(\{h(j,t,s):j<y\})).$$

By $\mathbf{RT}(2)$, we can find an infinite set $\mathbf{Y}$ and an $x_0<y$ such that $g([\mathbf{Y}]^2)=x_0$. By property 1), for some $z_0 \in \mathbf{N}$,

$$\forall u \exists v\, \theta(x_0,z_0,u,v).$$

By property 2), for some $x_1<y$,

$$\forall z<z_0+1 \exists u \forall v \neg\theta(x_1,z,u,v).$$

By Theorem 6.8 and Theorem 6.4, we may apply $\mathbf{B}\Pi_1^0$ to find a $t \in \mathbf{Y}$ such that

$$\forall z<z_0+1 \exists u<t \forall v \neg\theta(x_1,z,u,v).$$

Since $\mathbf{RCA}_0$ implies $\mathbf{B}\Sigma_1^0$, we may find an $s \in \mathbf{Y}$ such that $s>t$, $s>z_0+1$, and

$$\forall u<t \exists v<s\, \theta(x_0,z_0,u,v).$$

Thus $h(x_0,t,s) \leq z_0$, but $h(x_1,t,s) \geq z_0+1$, contradicting $g(t,s)=x_0$. This shows that the range of f is neither bounded nor unbounded. Thus the original assumption of $\mathbf{RT}(2)$ and $\neg\mathbf{B}\Pi_1^0$ is false. ■

The Ackermann function can be used to give a further indication of the proof theoretic strength of $\mathbf{RT}(2)$. Although the function in the following theorem is not the usual Ackermann diagonal function, it grows at approximately the same rate.

Theorem 6.12: $(\mathbf{RCA}_0)$ $\mathbf{RT}(2) \vdash \forall x \exists z(g(x,x)=z)$, where the function g is defined by the three relations $g(0,x)=x+1$, $g(x+1,0)=g(x,1)$, and $g(x+1,y+1)=g(x,g(x+1,y))$.

Proof: We will work in $\mathbf{RCA}_0$ using $\mathbf{RT}(2)$. Let $(m)_n^2$ denote the n^{th} digit in the binary expansion of m. Define $f_1:[\mathbf{N}]^2 \to 2^{t+1}$ by

$$f_1(x,y)=m \text{ iff } \forall n \le t\,((m)_n^2=1 \leftrightarrow \exists w \le x\, \forall z \le y\,(g(n,w) \ne z)).$$

f_1 exists by Δ_1^0 comprehension. By $\mathbf{RT}(2)$, there is an infinite monochromatic set $\mathbf{X}$ for f_1. Let $<x_i>_{i \in \mathbf{N}}$ be an increasing enumeration of $\mathbf{X}$. Consider the following three cases.

Case 1: Suppose $f_1([\mathbf{X}]^2)=2^{t+1}-1$. Then, in particular, $\exists w \le x_0 \forall z \le x_1 g(0,w) \ne z$. Since $g(0,w)=w+1$, we have $x_1 < w+1 \le x_0+1$, a contradiction.

Case 2: Suppose $f_1([\mathbf{X}]^2)=m$, where $0<m<2^{t+1}-1$. Let $r=\mu n \le t\,((m)_n^2=1)$. Then for $x,y \in \mathbf{X}$ such that $x<y$, we have

1) $\exists w \le x\, \forall z \le y\,(g(r,w) \ne z)$, and

2) $\forall n < r\, \forall w \le x\, \exists z \le y\,(g(n,w)=z)$.

Define $f_2:\mathbf{N}^2 \to 2^{x_0+1}$ by

$$f_2(y)=m \text{ iff } \forall n \le x_0((m)_n^2=1 \leftrightarrow \forall z \le y\,(g(r,n) \ne z)).$$

By Theorem 6.8, there is an infinite monochromatic set for f_2. Let $\mathbf{Y}=<y_i>_{i \in \mathbf{N}}$ be an increasing enumeration of such a set. If $f_2(\mathbf{Y})=0$, then $\forall w \le x_0 \exists z \le y_0(g(r,w)=z)$, contradicting 1). Let $s=\mu n \le x_0(f_2(y_0))_n^2 \ne 0$. Then

3) $\forall z\,(g(r,s) \ne z)$, and

4) $\forall n < s\, \exists z \le y_0(g(r,n)=z)$.

By definition, $g(r,s)=g(r\dot{-}1,g(r,s\dot{-}1))$. By 4), $\exists z\leq y_0(g(r,s\dot{-}1)=z)$, say $g(r,s\dot{-}1)=z_0$. Since $\mathbf{X}$ is infinite, there is some $x_j\in\mathbf{X}$ such that $x_j>z_0$. Thus, by 2), $\exists z\leq x_{j+1}(g(r\dot{-}1,z_0)=z)$. Since $g(r\dot{-}1,z_0)=g(r,s)$, it follows immediately that $\exists z\leq x_{j+1}(g(r,s)=z)$, contradicting 3).

Case 3: Suppose $f_1([\mathbf{X}]^2)=0$. Choose x_j such that $x_j>t$. Then $f_1(x_j,x_{j+1})=0$, so we have $\forall n\leq t\,\forall w\leq x_j\,\exists z\leq x_{j+1}(g(n,w)=z)$. In particular, $\forall n\leq t\,\exists z(g(n,n)=z)$, as desired. ■

Theorem 6.12 leads to a proof of a slightly weaker result than Corollary 6.8. We state the result and its proof, since the method used is proof theoretical, as opposed to the model theoretic methods of Corollary 6.8 or the recursion theoretic methods of Theorem 6.10.

Corollary 6.13: $\mathbf{WKL}_0\nvdash\mathbf{RT}(2)$.

Proof: Harrington proved that $\mathbf{WKL}_0$ is a conservative extension of primitive recursive arithmetic for Π^0_2 sentences. Robinson [42] showed that the diagonal function $g(x,x)$ of Theorem 6.12 is not primitive recursive. It follows immediately that $\mathbf{WKL}_0\nvdash\forall x\,\exists z(g(x,x)=z)$. Given this, the corollary follows easily from Theorem 6.12. ■

6.3. Regressive Partitions

In this section we will consider a variation on the Erdös-Rado theorem similar to one used by Kanamori and McAloon [22]. Let f be a partition of $[\mathbf{N}]^n$ for some $n\in\omega$, and let $h:\mathbf{N}\to\mathbf{N}$. The partition f is h-regressive if and only if for every

$S \in [N]^n$, $f(S) < h(\min(S))$. A set $X \subseteq N$ is min-homogeneous for f if for each S and T in $[X]^n$, $\min(S) = \min(T)$ implies $f(S) = f(T)$. The following theorem provides an interesting contrast to the situation for the usual Ramsey's theorem. Here, the situation for pairs is clear.

Theorem 6.14: ($\mathbf{RCA}_0$) The following are equivalent:

i) $\mathbf{ACA}_0$.

ii) For every $n \in \omega$ and every f which is an h-regressive partition of $[N]^n$, there is an infinite min-homogeneous set for f.

iii) Let f be an h-regressive partition of $[N]^2$. Then there is an infinite min-homogeneous set for f.

Proof: First, we will prove that i) implies ii). If $n = 1$, then N is min-homogeneous for f. By Theorem 1.5, it suffices to prove i) for arbitrary $n \in \omega$ using Ramsey's theorem for $(2n-1)$-tuples. Fix $n \in \omega$ such that $n > 1$. Let f be an h-regressive partition of $[N]^n$. Define a partition $g : [N]^{2n-1} \to 2$ by

$$g(x_1, x_2, \ldots, x_n, y_2, y_3, \ldots, y_n) = \begin{cases} 0 \text{ if } f(x_1, \ldots, x_n) = f(x_1, y_2, \ldots, y_n), \\ 1 \text{ otherwise.} \end{cases}$$

Let H be a monochromatic set for g.

First, we claim that $g([H]^{2n-1}) = 0$. Suppose not. Let $x_0 \in H$ and choose $h(x_0)+1$ disjoint sets $X_0, \ldots, X_{h(x_0)+1}$ in $[H]^{n-1}$ such that for all $i \leq h(x_0)+1$, $x_0 < \min(X_i)$. Since $g([\{x_0\} \cup \bigcup_{j \leq h(x_0)+1} X_i]^{2n-1}) = 1$,

$$f(\{x_0\} \cup X_i) \neq f(\{x_0\} \cup X_j)$$

for all $i < j < h(x_0)+1$. By a finite pigeonhole argument, $f(\{x_0\} \cup X_i) \geq h(x_0)$ for some i, contradicting the fact that f is h-regressive. Thus, $g([\mathbf{H}]^{2n-1})=0$.

Now we claim that $\mathbf{H}$ is min-homogeneous for f. Fix $x_0 \in \mathbf{H}$ and choose $X_1, X_2 \in [\mathbf{H}]^{n-1}$ such that $x_0 < \min(X_1 \cup X_2)$. Choose $X_3 \in [\mathbf{H}]^{n-1}$ such that $\min(X_3) > \max(X_1 \cup X_2)$. Since $g(\{x_0\} \cup X_1 \cup X_3) = g(\{x_0\} \cup X_2 \cup X_3) = 0$, we have $f(\{x_0\} \cup X_1) = f(\{x_0\} \cup X_3) = f(\{x_0\} \cup X_2)$, as desired.

Since iii) is a special case of ii) we need only show that iii) implies i). Let $g : \mathbf{N} \rightarrow \mathbf{N}$ be an injection. By Theorem 1.4, it suffices to prove the existence of the range of g using iii). Define the auxiliary function r by $r(i,n)=1$ if $\exists j < n\, (g(j)=i)$, and $r(i,n)=0$ otherwise. Define the partition $f : [\mathbf{N}]^2 \rightarrow \mathbf{N}$ by

$$f(m,n) = \sum_{i=0}^{m} 2^i r(i,n).$$

The partition f is clearly 2^{m+1}-regressive. By iii), there is an infinite set $\mathbf{X}$ which is min-homogeneous for f. Let $\langle x_i \rangle$ be an enumeration of $\mathbf{X}$. It is easy to see that $n \in \mathbf{Ran}(f)$ if and only if $f(x_n, x_{n+1})$ is congruent to 1 mod 2^{n+1}. Since $\mathbf{Ran}(f)$ is Δ_1^0 in $\mathbf{X}$, $\mathbf{Ran}(f)$ exists. ■

The following recursion theoretic porism is the Clote-style $0'$ basis result [4] corresponding to the Kanamori and McAloon independence result [22].

Porism 6.15: There is a recursive function h, and a recursive h-regressive partition $f : [\mathbf{N}]^2 \rightarrow \mathbf{N}$, such that $0'$ is recursive in every infinite min-homogeneous set for f.

Proof: Let g be a recursive function with $0'$ recursive in its range. Construct f as in the proof of the preceding theorem. ■

6.4. Conjectures

In this section, we present two conjectures concerning the strength of **RT**(2,2). Previous work of Jockusch, Kirby, and Paris supports these conjectures. A program is proposed which could lead to the proof of one or both of the conjectures. First, we state both conjectures.

Conjecture 6.16: $\mathbf{WKL}_0 + \mathbf{RT}(2,2) \vdash \mathbf{ACA}_0$.

Conjecture 6.17: $\mathbf{RCA}_0 + \mathbf{RT}(2,2) \nvdash \mathbf{ACA}_0$.

The proof of Conjecture 6.16 would provide a solution to the well-known 2–3 problem. In 1972, Kirby and Paris [26] wished to determine if every initial segment **I** of a model **M** of **PA** satisfying $<\mathbf{I},\mathbf{R_M I}> \models \mathbf{RT}(2,2)$ is strong. At this time, no counterexample has been found. By Theorem 5.3, Conjecture 6.16 implies that no counterexample exists.

It seems that if a proof of Conjecture 6.16 exists, it would already have been found. However, most efforts in this direction have actually been attempts to prove that $\mathbf{RCA}_0 + \mathbf{RT}(2,2) \vdash \mathbf{ACA}_0$. Jockusch [19] conjectured that there is no recursive partition of pairs such that $0'$ is recursive in every infinite monochromatic set. This conjecture would be an easy corollary to a proof of Conjecture 6.17.

Clearly, if both Conjecture 6.16 and Conjecture 6.17 are true, then $\mathbf{WKL}_0$ plays an instrumental role in the proof of Conjecture 6.16. One way to prove Conjecture 6.16 is to fix a function f and find a partition $g:[\mathbf{N}]^2 \to k$ for some $k \in \omega$ such that the range of f is Δ_1^0 definable in every infinite monochromatic set for g. We propose to use $\mathbf{WKL}_0$ to build such a partition. To carry out the proof, some sort of finite combinatorial lemma is needed.

The combinatorial lemma should be a type of finite anti-Ramsey theorem. This lemma will involve a notion which we will call *big* finite sets. *Bigness* must satisfy the following four properties.

1) "$\mathbf{X}$ is *big* " is Δ_1^0 definable.

2) Every infinite set contains a *big* set.

3) If $x_0, x_1, \ldots, x_n$ is *big*, then every infinite set contains an element y such that $x_0, \ldots, x_{n-1}, y$ is *big*.

4) There is a $k \in \omega$ such that for any $n \in \mathbf{N}$ and $f : n \rightarrow n$, there is a partition $g : [n]^2 \rightarrow k$ such that if $x_0, x_1, \ldots, x_j \subseteq n$ is *big* and monochromatic for g, then $\forall t\, (x_{j-1} < t < x_j \rightarrow f(t) \geq x_0)$.

The proof of Conjecture 6.16 follows immediately from the proof that any predicate satisfies properties 1) through 4). For a fixed $f : \mathbf{N} \rightarrow \mathbf{N}$, one constructs the tree containing k-colorings of $[n]^2$ given by property 4) for each n. $\mathbf{WKL}_0$ is used to find a path through this tree, which yields a k-coloring of $[\mathbf{N}]^2$. $\mathbf{RT}(2,2)$ is applied (possibly k times) to find an infinite monochromatic set, $\mathbf{H}$, for this partition. Properties 1) and 2) are used to find an infinite sequence of *big* subsets of $\mathbf{H}$ with increasing minimums. By properties 3) and 4), each big set determines the range of f below its minimum. In this way, the range of f is shown to exist.

It seems reasonable that a predicate satisfying properties 1) through 4) exists. Properties 1), 2), and 3) are easily satisfied. For instance, the predicate "$\mathbf{X}$ has cardinality 2" satisfies these properties. However, this predicate does not satisfy property 4). Another indication of the feasibility comes from weakening property 4) to require partitions $g : [n]^3 \rightarrow 2$. With this weakening, the predicate "$\mathbf{X}$ has cardinality at least min (**X**)+3" satisfies all four properties. However, **RT**(3,2) must be used in place of **RT**(2,2) in the argument outlined above. This yields a proof that $\mathbf{WKL}_0 + \mathbf{RT}(3,2) \vdash \mathbf{ACA}_0$, a weaker result than Theorem 1.5.

If the appropriate predicate can be found, it should be straightforward to prove or disprove Conjecture 6.17. Given an actual partition to manipulate, it should be possible to carry out recursion theoretic analysis to obtain information about Conjecture 6.17. Unfortunately, determining whether or not a given predicate satisfies property 4) is very difficult. Computer verifications are impractical except for cases which are easily checked by more direct proofs.

Finally, it is noteworthy that the proof that a predicate satisfies properties 1) through 4) is actually a sort of finite anti-Ramsey theorem. Property 4) states that for some finite coloring there is no monochromatic set of a certain sort. The usual approach is to use infinite combinatorial theorems to prove finite results, as in the original proof of Ramsey's theorem [40]. If the program to prove Conjecture 6.16 is successful, it will be an interesting reversal of this technique.

CHAPTER 7

HINDMAN'S THEOREM

In 1974, Hindman proved an infinite version of a theorem of Folkman [18]. Hindman's theorem asserts that for any finite partition of ω, there is an infinite subsequence of ω such that the set of finite sums of elements from the subsequence forms a monochromatic set. Hindman's theorem was formalized and analyzed in subsystems of $\mathbf{Z}_2$ by Blass, Hirst, and Simpson [3]. At the conclusion of this work, the exact proof theoretic strength of Hindman's theorem was still undetermined.

After reviewing previously known results, this chapter examines two attempts to determine the proof theoretic strength of Hindman's theorem. In the second section, a version of Hindman's theorem emphasizing algebraic content is explored. The last two sections are the result of applying the model theoretic methods of Chapter 5 to the problem. Although the exact strength of Hindman's theorem remains a mystery, both avenues lead to interesting results.

7.1. Previous results

This section reviews the work of Blass, Hirst, and Simpson [3]. All the proofs have been omitted, with the exception of Lemma 7.3. We begin with two formal versions of Hindman's theorem. The first version, which we will call **HT**, is Hindman's original statement involving finite sums of integers. The second version, called **HTU**, was used by Baumgartner [1], and involves unions of sets. In this version, the theorem is stripped of any reference to the algebraic structure of $\mathbf{N}$, and becomes purely combinatorial. The following definitions give the two versions of the theorem.

Notation 7.1: The following statement will be denoted by **HT**: If $f:\mathbf{N}\rightarrow l$ is a finite partition of **N**, then there is an infinite set $\mathbf{X}\subseteq\mathbf{N}$ such that for some $i<l$, $f(\mathbf{FS}(\mathbf{X}))=i$, where **FS(X)** denotes the set of all sums of nonempty finite subsets of **X**.

Notation 7.2 The following statement will be denoted by **HTU**: If $f:\mathbf{P}_{<\mathbf{N}}(\mathbf{N})\rightarrow l$ is a finite partition of $\mathbf{P}_{<\mathbf{N}}(\mathbf{N})$, then there exists an infinite set **X** of pairwise disjoint elements of $\mathbf{P}_{<\mathbf{N}}(\mathbf{N})$ such that for some $i<l$, $f(\mathbf{FU}(\mathbf{X}))=i$, where **FU(X)** denotes the set of all unions of nonempty finite subsets of **X**.

We will now show that **HT** and **HTU** are actually the same statement.

Lemma 7.3: ($\mathbf{RCA}_0$) The following are equivalent:

i) **HT**.

ii) **HTU**.

Proof: The proof that i) implies ii) uses the proof of Lemma 2.3 in [17]. Assume **HT** and let $f:\mathbf{P}_{<\mathbf{N}}(\mathbf{N})\rightarrow l$ be a finite partition of $\mathbf{P}_{<\mathbf{N}}(\mathbf{N})$. Let $\tau:\mathbf{N}\rightarrow\mathbf{P}_{<\mathbf{N}}(\mathbf{N})$ be the bijection defined by

$$\tau(n)=\{x<n:\mathrm{int}(\frac{n}{2^x})=1(\bmod 2)\}.$$

The partition f induces a partition $g:\mathbf{N}\rightarrow l$ defined by $g(n)=f(\tau(n))$. By **HT**, there is a $c<l$ and an infinite set $\mathbf{X}\subseteq\mathbf{N}$ such that $g(\mathbf{FS}(\mathbf{X}))=c$. Let $<x_i>_{i\in\mathbf{N}}$ be an increasing enumeration of **X**. We will construct a sequence $<y_i>_{i\in\mathbf{N}}$, such that

$$\mathbf{FS}(<y_i>_{i\in\mathrm{N}})\subseteq\mathbf{FS}(<x_i>_{i\in\mathrm{N}}),$$

and $2^s \mid y_{n+1}$ whenever $2^{s-1}\leq y_n$. Set $y_1=x_1$ and $k_1=2$. Suppose y_n and k_n have been chosen. Let b be the maximum integer such that $2^{b-1}\leq y_n$. Consider the set $\mathbf{S}=\{x_i : k_n\leq i\leq k_n+2^{2b}\}$. By the finite pigeonhole principle, there is a $j<2^b$ and a $\mathbf{T}\subseteq\mathbf{S}$ such that $|\mathbf{T}|=2^b$ and $x\in\mathbf{T}$ if and only if $x=j \pmod{2^b}$. Pick the least such j and $\mathbf{T}$. Let $y_{n+1}=\sum_{x\in\mathbf{T}} x$ and let $k_{n+1}=k_n+2^{2b}+1$. Clearly, $<y_n>_{n\in\mathrm{N}}$ is a sequence with the desired properties. Furthermore, for $i\neq j$, $\tau^{-1}(y_i)\cap\tau^{-1}(y_j)=\emptyset$, and $\tau^{-1}(y_i+y_j)=\tau^{-1}(y_i)\cup\tau^{-1}(y_j)$. Since for any finite set $\mathbf{S}\subseteq\mathbf{N}$,

$$f\,(\bigcup_{j\in\mathbf{S}}\tau^{-1}(y_j))=g\,(\sum_{j\in\mathbf{S}} y_j)=c\ ,$$

the set $\mathbf{Y}=\{\tau^{-1}(y_i) : i\in\mathbf{N}\}$ is the desired monochromatic set for **HTU**.

To prove that ii) implies i), use τ^{-1} to induce a partition on $\mathbf{P}_{<\mathrm{N}}(\mathrm{N})$ and map the monochromatic set for **HTU** back onto **N** by τ. Since the monochromatic set for **HTU** consists of disjoint sets, the result follows even more easily than before. ■

Throughout the remainder of this chapter, **HT** and **HTU** are used interchangeably. We retain both notations only to clarify the applications of Hindman's theorem in proofs.

The following three theorems capsulate the results in [3] concerning the proof theoretic strength of Hindman's theorem.

Theorem 7.4: $\mathbf{RCA_0}\vdash\mathbf{HT}\rightarrow\mathbf{ACA_0}$.

Theorem 7.5: ($\mathbf{ACA_0^+}$) If $f_i:\mathbf{N}\rightarrow l_i$ is a countable sequence of finite partitions of **N**, then there is an infinite sequence $<x_i>_{i\in\mathrm{N}}$ such that for each i, there is some $c_i<l_i$ such that $f_i(\mathbf{FS}(<x_j>_{j\geq i}))=c_i$.

Corollary 7.6: $\mathbf{ACA_0^+} \vdash \mathbf{HT}$.

7.2. An Algebraic Version

In this section, we present an algebraic version of Hindman's theorem. The original intent was to emphasize the algebraic content of Hindman's theorem, and then analyze its proof theoretic strength using work done by Friedman, Simpson, and Smith on countable algebra in subsystems of $\mathbf{Z}_2$ [10]. Unfortunately, this reformulation does not lend itself to such analysis. Furthermore, results later in the section indicate that stronger algebraic analogs are unlikely.

The following reformulation of Hindman's theorem is in terms of Boolean rings. Chapter 4 contains all the relevant definitions and fundamental results. As in Chapter 4, we will use the phrase "infinite Boolean ring" to refer to a countably infinite Boolean ring.

Definition 7.7: The following statement will be denoted by **HTA**: If $f : \mathbf{R} - \{0\} \to l$ is a finite partition of an infinite Boolean ring $\mathbf{R}$, then there is an $i < l$ and an infinite subring $\mathbf{S} \subseteq \mathbf{R}$ such that $f(\mathbf{S}) = i$.

We will now show that this algebraic reformulation of Hindman's theorem is equivalent to the versions in the preceding section.

Theorem 7.8: ($\mathbf{RCA_0}$) The following are equivalent:

i) **HT**.

ii) **HTA**.

Proof: To see that i) implies ii), assume **HTU**. Let $f : \mathbf{R} \rightarrow l$ be a finite partition of an infinite Boolean ring **R**. By Corollary 7.5, **HTU** implies $\mathbf{ACA}_0$. Thus, we may apply Corollary 4.13 to find $<s_i>_{i \in \mathrm{N}}$, an infinite sequence of pairwise zero divisors in **R**. Define a partition $g : \mathbf{P}_{<\mathrm{N}}(\mathrm{N}) \rightarrow l$ by $g(\mathrm{A}) = f\,(\sum_{i \in \mathrm{A}} s_i)$, for all $\mathrm{A} \in \mathbf{P}_{<\mathrm{N}}(\mathrm{N})$. By **HTU**, there is an infinite sequence of disjoint elements of $\mathbf{P}_{<\mathrm{N}}(\mathrm{N})$, $<\mathbf{A}_i>_{i \in \mathrm{N}}$, such that g is monochromatic on the finite unions of elements from $<\mathbf{A}_i>_{i \in \mathrm{N}}$. Let $\mathbf{A}_J$ denote $\bigcup_{j \in \mathrm{J}} \mathbf{A}_j$, and define **S** by

$$\mathbf{S} = \{ \sum_{i \in \mathbf{A}_J} s_i : \ \mathrm{J} \in \mathbf{P}_{<\mathrm{N}}(\mathrm{N}) \}.$$

S is closed under sums since **R** has characteristic 2. By the definition of s_i, **S** is also closed under products. Thus **S** is a subring. By definition of f and $\mathbf{A}_i$, for some $k < l$, $g(\mathbf{S}) = k$.

To show that ii) implies i), assume **HTA**. Let **R** denote the Boolean ring on $\mathbf{P}_{<\mathrm{N}}(\mathrm{N})$ where $+$ denotes symmetric difference and $*$ denotes intersection. Let $f : \mathbf{P}_{<\mathrm{N}}(\mathrm{N}) \rightarrow l$ be a finite partition of $\mathbf{P}_{<\mathrm{N}}(\mathrm{N})$. Obviously, f is also a finite partition of **R**. Let **S** be an infinite monochromatic subring. Since **S** is a Boolean ring of finite sets, by Lemma 4.11, there is a sequence $<s_i>_{i \in \mathrm{N}}$ of elements of **S** which are pairwise zero divisors. Note that $<s_i>_{i \in \mathrm{N}}$ is also a sequence of disjoint elements of $\mathbf{P}_{<\mathrm{N}}(\mathrm{N})$. Since $s_i + s_j = s_i \cup s_j$ for all $i \neq j$, every finite union of elements from $<s_i>_{i \in \mathrm{N}}$ is in **S**. Thus, for some $k < l$, $f\,(\mathbf{FU}(<s_i>_{i \in \mathrm{N}})) = k$. ■

While Boolean rings are not a commonly studied structure, they seem to be ideally suited to the statement of Hindman's theorem. The following negative results show that the class of rings considered cannot be arbitrarily broadened.

Theorem 7.9: **HTA** can not be extended to include all rings.

Proof: Consider the integers. Pick any of the partitions commonly used as counterexamples for an infinite version of van der Waerden's theorem [13]. No infinite monochromatic subring can exist. ■

Theorem 7.10: **HTA** can not be extended to include all rings of finite characteristic.

Proof: Let **R** be an infinite ring of characteristic 3. Let $<r_i>_{i \in \mathrm{N}}$ be an enumeration of its elements. Color r_1 blue and r_1+r_1 red. For $j>1$, if r_j hasn't been colored, color r_j blue and r_j+r_j red. There is no monochromatic subring for this coloring. ■

Another way to extend the strength of **HTA** is to restrict the class of acceptable monochromatic substructures. The most obvious candidate, an ideal, is eliminated by the following theorem.

Theorem 7.11: **HTA** can not be strengthened by requiring monochromatic ideals rather than subrings.

Proof: Let **R** be the Boolean ring on $\mathbf{P}_{<\mathrm{N}}(\mathbf{N})$ with + denoting symmetric difference and $*$ denoting intersection. Partition **R** by coloring each element by the parity of its cardinality. **R** has no monochromatic ideals. ■

One might hope to "improve" **HTA** by restricting attention to Boolean algebras, and seeking monochromatic subalgebras. This also fails. Simply color each element the opposite color from its complement. However, the following extremely unnatural statement is true.

Theorem 7.12: ($\mathbf{RCA}_0$) The following are equivalent:

i) **HTA**.

ii) Let **B** be a countably infinite Boolean algebra and f :**B**$\rightarrow l$ a finite partition of **B** such that $\forall b \in$**B**$(f(b)=f(b^c))$. ($f(0)$ and $f(1)$ are undefined.) Then there is an infinite Boolean subalgebra of **B** which is monochromatic for f.

Proof: To prove that i) implies ii), let **B** and f be as in the statement of ii). Apply **HTA** to find **S**, a monochromatic subring of **B**. The set

$$\mathbf{C}=\{x \in \mathbf{B}: x \in \mathbf{S} \bigvee 1+x \in \mathbf{S}\}$$

is a monochromatic subalgebra of **B**.

To prove that ii) implies i), by Theorem 7.8 it suffices to prove **HTU** using ii). This is done by applying ii) to the Boolean algebra of finite and cofinite subsets of **N** and imitating the proof of Theorem 7.8. ∎

It is also possible to somewhat strengthen the conclusion of **HTA** if we restrict the hypothesis. The following theorem uses the structure of Boolean rings of finite sets to achieve this end.

Porism 7.13: ($\mathbf{RCA}_0$) The following are equivalent:

i) **HTA**.

ii) Let **R** be an infinite Boolean ring of finite sets and f :**R**$\rightarrow l$ a finite partition of **R**. Then there is an infinite monochromatic subring of **R** which is isomorphic to **R**.

Proof: To prove that i) implies ii), apply i) to a Boolean ring of finite sets to find **S**, a monochromatic subring of **R**. **S** is a Boolean ring of finite sets. Using the construction in the proof of Theorem 4.10, it is easy to show that **R** is isomorphic to **S**.

To prove that ii) implies i), use ii) to prove **HTU** as in the proof of Theorem 7.8. ■

Generalizing the Boolean ring of finite sets in the statement of Porism 7.13 to an arbitrary Boolean ring results in a false statement. To see this, let **R** be a Boolean algebra and f be a partition as in the example preceding Theorem 7.12. Since no monochromatic subalgebra exists, no monochromatic subring isomorphic to **R** exists.

7.3. Galvin Ultrafilters

In this section we begin to apply the methods of Chapter 5 to Hindman's theorem. Two concepts, pegged Folkman sequences and Galvin ultrafilters, are introduced. In an ultrapower modulo a Galvin ultrafilter, pegged Folkman sequences of nonstandard length exist. This section uses this fact to prove the slightly strengthened version of Folkman's theorem given in Theorem 7.19. The application of this material to Hindman's theorem is explained in the next section.

Definition 7.14: Let $f : \mathbf{N} \rightarrow l$ be a finite partition. Given an element $y \in \mathbf{N}$, the pegged Folkman sequence of y (for f) is a strictly increasing sequence $< x_i >_{i<k}$, defined by

$$x_i = \mu z < y \, (f \ (\mathbf{FS}(< x_j >_{j<i} \cup \{z, y\})) = f \ (y)).$$

The pegged Folkman sequence of y may be empty. In this case, $k = 0$. The length, k, of the pegged Folkman sequence is denoted by $\mathbf{L}(f, y)$.

Definition 7.15: A (restricted) ultrafilter $\mathbf{U}$ on a set $(\mathbf{N}_\Gamma)$ is called Galvin if

$$\forall \mathbf{X} \in \mathbf{U} \exists y \in \mathbf{X} \ \{x - y : x > y \wedge x \in \mathbf{X}\} \in \mathbf{U}.$$

Galvin and Hindman referred to Galvin ultrafilters as almost downward translation invariant ultrafilters. Although their terminology is more descriptive, ours is much shorter. Assuming the continuum hypothesis, **CH**, Hindman's theorem can be viewed as a proof of the existence of Galvin ultrafilters on ω. This yields the following lemma.

Lemma 7.16: Assuming **CH**, there is a Galvin ultrafilter on ω.

Proof: For the proof that, assuming **CH**, Hindman's theorem is equivalent to the existence of a Galvin ultrafilter on ω, see [17]. For a proof of Hindman's theorem, see [18]. ■

The next step is to use a Galvin ultrafilter on ω to show the existence of arbitrarily long pegged Folkman sequences. This is done by taking an ultrapower of an ω-model by a Galvin ultrafilter. In such an ultrapower, the pegged Folkman sequence of $[id]$ is unbounded in ω. An application of Łoś's theorem proves the theorem. We state the result for models of $\mathbf{RCA}_0$, since in this way we can eventually eliminate the use of **CH**.

Lemma 7.17: (**CH**) Let $\Gamma \models \mathbf{RCA}_0$ be an ω-model. Let $f : \mathbf{N} \rightarrow l$ be a finite partition such that $f \in \mathbf{S}_\Gamma$. Then there are arbitrarily long pegged Folkman sequences for f, that is,

$$\Gamma \models \forall x \exists y \, (\mathbf{L}(f, y) > x).$$

Proof: Let Γ and f be as stated. Let $\mathbf{V}$ be a Galvin ultrafilter on ω as provided by Lemma 7.16. Let $\mathbf{U} = \mathbf{S}_\Gamma \cap \mathbf{V}$. Then $\mathbf{U}$ is a restricted Galvin ultrafilter on $\mathbf{N}_\Gamma$.

Let $[id]$ denote the element of $\prod_U \mathbf{N}_\Gamma$ containing the identity function. Let $<y_i>$ be the pegged Folkman sequence for $[id]$ in $\prod_U \mathbf{N}_\Gamma$. We must determine the appropriate range for the indices. It $<y_i>$ is an infinite sequence, then we are done. To see this, let $\prod_U \mathbf{N}_\Gamma{}^*$ be the expansion of $\prod_U \mathbf{N}_\Gamma$ to a model with interpretations for f and $\mathbf{L}$. If $<y_i>$ is an infinite sequence, then for any $n \in \omega$, $\prod_U \mathbf{N}_\Gamma{}^* \models \exists y\, (\mathbf{L}(f\ ,y\ ) > n\ )$. Thus, by Porism 5.17 and Theorem 5.12 part iii), for any $n \in \omega$, $\Gamma \models \exists y\, (\mathbf{L}(f\ ,y\ ) > n\ )$. Since Γ is an ω-model, $\Gamma \models \forall x\ \exists y\, (\mathbf{L}(f\ ,y\ ) > x\ )$, as desired.

It remains to show that $<y_i>$ is an infinite sequence. We will actually show that $<y_i> \cap \omega$ is unbounded in ω. Since $\mathbf{N}_\Gamma = \omega$, $\mathbf{U}$ is additive, so for some $c < l$, $\{x \in \mathbf{N}_\Gamma : f\ (x\ ) = c\ \} \in \mathbf{U}$, and equivalently, $\prod_U \mathbf{N}_\Gamma{}^* \models f\ ([id\ ]) = c$. Let $<y_i>_{i<k}$ be a (possibly empty) initial segment of $<y_i> \cap \omega$. By the definition of $<y_i>$, we have

$$\prod_U \mathbf{N}_\Gamma{}^* \models f\ (\mathbf{FS}(\{y_0, \ldots, y_{k-1}, [id\ ]\})) = c.$$

By Porism 5.17 and Theorem 5.12 part iii), we may define $\mathbf{X} \in \mathbf{S}_\Gamma$ such that

$$\mathbf{X} = \{x > y_{k-1} : \Gamma \models f\ (\mathbf{FS}(\{y_0, \ldots, y_{k-1}, x\ \})) = c\ \} \in \mathbf{U}.$$

Since $\mathbf{U}$ is a Galvin ultrafilter, we can find an element $y_k \in \mathbf{X}$ and a set $\mathbf{Y} \in \mathbf{U}$ such that

$$\mathbf{Y} = \{x - y_k : x > y_k \wedge x \in \mathbf{X}\} \in \mathbf{U}.$$

Since $y_k \in \mathbf{X}$, $\Gamma \models f\ (\mathbf{FS}(\{y_0, \ldots, y_{k-1}, y_k\ \})) = c$. Furthermore, $x \in \mathbf{Y}$ implies $x \in \mathbf{X}$ and $x + y_k \in \mathbf{X}$, so for any $x \in \mathbf{Y}$,

$$\Gamma \models f\ (\mathbf{FS}(\{y_0, \ldots, y_{k-1}, x\ \})) = f\ (\mathbf{FS}(\{y_0, \ldots, y_{k-1}, x + y_k\ \})) = c.$$

Since $\mathbf{Y} \in \mathbf{U}$, by Porism 5.17 and Theorem 5.12 part ii), y_k witnesses that

$$\prod_{\mathbf{U}} \mathbf{N}_{\Gamma}^* \models \exists y \leq y_k \; f\,(\mathbf{FS}(\{y_0, \ldots, y_{k-1}, y, [id]\})) = c.$$

Thus $<y_i>_{i<k}$ is a proper initial segment of $<y_i> \cap \omega$. As stated above, this suffices to complete the proof. ■

Porism 7.18: If Γ is a countable ω-model of $\mathbf{RCA}_0$, **CH** is not needed to prove Lemma 7.17.

Proof: **CH** is used only to find the Galvin ultrafilter **V** on the full power set of ω. Any countable ω-model of $\mathbf{RCA}_0$ can be expanded to a countable ω-model of Milliken's theorem, and a restricted Galvin ultrafilter **V** can be found on this model, using the construction in Theorem 7.25. ■

Lemma 7.17 can be considerably strengthened. The following theorem shows that there is a recursive function which gives an upper bound for the location of a pegged Folkman sequence of a given length.

Theorem 7.19: There is a total recursive function g such that for any finite partition $f:\mathbf{N}\rightarrow k$, and for any $n \in \omega$, there is a $y < g(\max(n, k))$ such that $\mathbf{L}(f, y) > n$.

Proof: We will work momentarily in the real world. Suppose that there is no bounding function whatsoever. Let k be the least integer such that for all b there is a partition $f: b \rightarrow k$ such that $\forall y < b\, \mathbf{L}(f, y) < k$. Thus for every b there is a partition $f: b \rightarrow k$ such that $\forall y < b\, \mathbf{L}(f, y) < k$.

Now, move to Γ, a countable ω-model of $\mathbf{WKL}_0$. The tree **T** consisting of all finite sequences σ of integers less than k such that $\forall y < \mathrm{lh}(\sigma) \mathbf{L}(\sigma, y) < k$ is defined in

Γ and infinite. By $\mathbf{WKL}_0$, $\mathbf{T}$ has an infinite path, which is a partition violating Lemma 7.17. Thus, a total bounding function exists, and can be defined by

$$g(n) = \mu t (\forall f : t \to n \; \exists y < t \; \mathbf{L}(f, y) > n).$$

Since the formula defining g consists of a μ-operator applied to a Σ_0^0 formula, g is recursive. Note also that since Γ is countable, the use of Lemma 7.17 may be replaced by Porism 7.18, so **CH** is not needed. ■

7.4. Milliken's Theorem

Milliken's theorem is a combination of Ramsey's theorem and Hindman's theorem. In Theorem 7.22, we relate Milliken's theorem to the extended version of Hindman's theorem stated in Theorem 7.5. This allows us to prove analogs of Theorem 7.4 and Corollary 7.6. We conclude the section by relating Milliken's theorem and Galvin ultrafilters, completing the development started in the previous section.

As with Hindman's theorem, Milliken's theorem may be expressed either in terms of sums or in terms of unions of finite sets. For $\mathbf{X} \subseteq \mathbf{N}$ and $n \in \omega$, we will use the notation $<\mathbf{X}>_\Sigma^n$ to denote the collection of all sets $\mathbf{Y} = \{ \sum_{x \in \mathbf{A}_i} x : i < n \}$, where for each i, $\mathbf{A}_i \in \mathbf{P}_{<\mathrm{N}}(\mathbf{X})$, and $i < j < n$ implies $\max(\mathbf{A}_i) < \min(\mathbf{A}_j)$. For $\mathbf{X} \subseteq \mathbf{P}_{<\mathrm{N}}(\mathbf{N})$ and $n \in \omega$, we will use $<\mathbf{X}>_\cup^n$ to denote the collection of all sets $\mathbf{Y} = \{\cup \mathbf{A}_i : i < n\}$ where $\mathbf{A}_i \in \mathbf{P}_{<\mathrm{N}}(\mathbf{X})$ for each i, and $i < j < n$ implies $\max(\cup \mathbf{A}_i) < \min(\cup \mathbf{A}_j)$. With this notation, it is easy to state the two versions of Milliken's theorem.

Definition7.20: i) The following statement will be denoted by **MT**: For $n \in \omega$, if $f : [\mathbf{N}]^n \to l$ is a finite partition of $[\mathbf{N}]^n$, then there is an infinite set $\mathbf{X} \subseteq \mathbf{N}$ such that for some $i < l$, $f(<\mathbf{X}>^n_\Sigma) = i$.

ii) The following statement will be denoted by **MTU**: For $n \in \omega$, if $f : [\mathbf{P}_{<\mathbf{N}}(\mathbf{N})]^n \to l$ is a finite partition of $[\mathbf{P}_{<\mathbf{N}}(\mathbf{N})]^n$, then there is an infinite set $\mathbf{X}$ of pairwise disjoint elements of $\mathbf{P}_{<\mathbf{N}}(\mathbf{N})$ such that for some $i < l$, $f(<\mathbf{X}>^n_\cup) = i$.

As for Hindman's theorem, the sum and union versions of Milliken's theorem are essentially the same statement. The following Lemma parallels Lemma 7.3.

Lemma 7.21: ($\mathbf{RCA_0}$) The following are equivalent:

i) **MT**.

ii) **MTU**.

Proof: Use the bijection between $\mathbf{P}_{<\mathbf{N}}(\mathbf{N})$ and $\mathbf{N}$ introduced in the proof of Lemma 7.3. ■

We now prove that Milliken's theorem is equivalent to the strong version of Hindman's theorem given in Theorem 7.5, and derive analogs of Theorem 7.4 and Corollary 7.6.

Theorem 7.22: ($\mathbf{RCA_0}$) The following are equivalent:

i) **MT**.

ii) (Theorem 7.5) If $f_i : \mathbf{N} \to l_i$ is a countable sequence of finite partitions of $\mathbf{N}$, then there is an infinite sequence $< x_i >_{i \in \mathbf{N}}$ such that for each i, there is some $c_i < l_i$ such that $f_i(\mathbf{FS}(< x_j >_{j \geq i})) = l_i$.

Proof: To prove that i) implies ii), assume **MT** and let $f_i:\mathbf{N}\rightarrow l_i$ be a countable sequence of finite partitions of **N**. Define $g:[\mathbf{N}]^3\rightarrow 2$ by

$$g(x,y,z)=\begin{cases}0 \text{ if } \forall i<x\,(f_i(y)=f_i(z))\\ 1 \text{ otherwise.}\end{cases}$$

By **MT**, we can find an infinite sequence $\mathbf{X}=<x_i>_{i\in\mathbf{N}}$ such that $x_i>i$ for all i, and g is monochromatic on $<\mathbf{X}>^3_\Sigma$. An easy pigeonhole argument shows that $g(<\mathbf{X}>^3_\Sigma)=0$. To see that **X** is the desired set, fix i and choose

$$\mathbf{A}_1,\mathbf{A}_2\in\mathbf{P}_{<\mathbf{N}}(\mathbf{X}-\{x_j:j<i\}).$$

Let $x\in\mathbf{X}$ such that $x>\max(\mathbf{A}_1\cup\mathbf{A}_2)$. By the definitions of g and **X**, $g(x_{i-1},\Sigma\mathbf{A}_1,x)=g(x_{i-1},\Sigma\mathbf{A}_2,x)$, so

$$f_i(\Sigma\mathbf{A}_1)=f_i(x)=f_i(\Sigma\mathbf{A}_2),$$

as desired.

To prove that ii) implies i), we assume ii) and use an external induction argument to prove that for each $n\in\omega$, Milliken's theorem for n-tuples holds. To simplify notation, we will use the finite union variants of both i) and ii). We leave it to the reader to verify that the proof of Lemma 7.3 can be applied uniformly to ii).

Since Milliken's theorem for singletons is Hindman's theorem, the case for $n=1$ is trivial. Suppose that Milliken's theorem for k-tuples holds for all $k<n$. Let $f:[\mathbf{P}_{<\mathbf{N}}(\mathbf{N})]^n\rightarrow l$ be a finite partition of $[\mathbf{P}_{<\mathbf{N}}(\mathbf{N})]^n$. For each $j\in\mathbf{N}$, let $l_j=|[\mathbf{P}_{<j}(j)]^{n-1}|$. Choose an enumeration e such that for all $j\in\mathbf{N}$,

$$\exists i<l_j\; e(j,i)=\mathbf{X}\leftrightarrow\mathbf{X}\in[\mathbf{P}_{<j}(j)]^{n-1}.$$

Define a sequence of partitions $g_j:\mathbf{P}_{<\mathbf{N}}(\mathbf{N})\rightarrow\prod_{i<l_j}p_i^l$ by

$$g_j(\mathbf{X})=\begin{cases}0 \text{ if } \min(\mathbf{X})<j \bigvee [\mathbf{P}_{<j}(j)]^{n-1}=\emptyset \\ \prod_{i<l_j} p_i^{f\,(e(i,j)\cup\{\mathbf{X}\})} \text{ otherwise.}\end{cases}$$

Here p_i denotes the i^{th} prime. Intuitively, g_j codes the action of f on $(n-1)$-tuples of subsets of j. Applying ii) yields an infinite increasing disjoint sequence, $\mathbf{X}=<\mathbf{X}_i>_{i\in\mathrm{N}}\subseteq\mathbf{P}_{<\mathrm{N}}(\mathrm{N})$, such that

$$\forall j \exists b_j < \prod_{i<l_j} p_i^l \,(g_j(\mathbf{FU}(<\mathbf{X}_i>_{i\geq j}))=b_j).$$

Define $h:<\mathbf{X}>_{\cup}^{n-1}\rightarrow l$ as follows. For $(\mathbf{A}_1, \ldots, \mathbf{A}_{n-1})\in<\mathbf{X}>_{\cup}^{n-1}$, let $\mathbf{Y}$ be the least element of $\mathbf{X}$ such that for all $j\leq n-1$, $\max(\mathbf{A}_j)<\min\mathbf{Y}$. Set

$$h(\mathbf{A}_1, \ldots, \mathbf{A}_{n-1})=f\,(\mathbf{A}_1, \ldots, \mathbf{A}_{n-1},\mathbf{Y}).$$

We now apply Milliken's theorem for $(n-1)$-tuples to h, to find an infinite increasing disjoint sequence $\mathbf{Z}=<\mathbf{Z}_i>_{i\in\mathrm{N}}\subseteq\mathbf{FU}(\mathbf{X})$ such that for some $c<l$, we have $h(<\mathbf{Z}>_{\cup}^{n-1})=c$.

We claim that $f\,(<\mathbf{Z}>_{\cup}^{n})=c$. To see this, let $\{\mathbf{V}_1, \ldots, \mathbf{V}_n\}\in<\mathbf{Z}>_{\cup}^{n}$, and let $\mathbf{W}$ be the least element of $\mathbf{X}$ such that $\min(\mathbf{W})>\max(\mathbf{V}_{n-1})$. Let $t=\min(\mathbf{V}_n\cup\mathbf{W})$. Since $\mathbf{W},\mathbf{V}_n\in\mathbf{FU}(\mathbf{X})$,

$$g_t(\mathbf{V}_n)=g_t(\mathbf{W}),$$

so in particular,

$$f\,(\mathbf{V}_1, \ldots, \mathbf{V}_{n-1},\mathbf{V}_n)=f\,(\mathbf{V}_1, \ldots, \mathbf{V}_{n-1},\mathbf{W}).$$

By the definition of $\mathbf{W}$, $h(\mathbf{V}_1, \ldots, \mathbf{V}_{n-1})=f\,(\mathbf{V}_1, \ldots, \mathbf{V}_{n-1},\mathbf{W})$, and since $\{\mathbf{V}_1, \ldots, \mathbf{V}_{n-1}\}\in<\mathbf{Z}>_{\cup}^{n-1}$, $h(\mathbf{V}_1, \ldots, \mathbf{V}_{n-1})=c$. Summarizing, we have

$$f\ (\mathbf{V}_1, \ldots, \mathbf{V}_n) = c$$

for all $\{\mathbf{V}_1, \ldots, \mathbf{V}_n\} \in <\mathbf{Z}>^n_{\mathbf{U}}$. ■

Corollary 7.23: $\mathbf{ACA}_0^+ \vdash \mathbf{MT}$.

Proof: Immediate from Theorem 7.22 and Theorem 7.5. ■

Corollary 7.24: $\mathbf{RCA}_0 \vdash \mathbf{MT} \rightarrow \mathbf{ACA}_0$.

Proof: Immediate from Theorem 7.22 and Theorem 7.4. An alternate proof is to note that Milliken's theorem for triples implies **RT**(3), simply by considering only those elements of $<\mathbf{X}>^3_\Sigma$ in which $|\mathbf{A}_i|=1$ for all $i<3$. The result then follows immediately from Theorem 1.5. ■

We will now examine the connection between Milliken's theorem and Galvin ultrafilters. Immediately following the proof of Theorem 7.25 is a discussion of its impact on the program of applying model theoretic methods to determine the strength of **HT**.

Theorem 7.25: Let Γ be a countable model of $\mathbf{RCA}_0$. Then the following are equivalent:

i) $\Gamma \models \mathbf{MT}$.

ii) There is an additive restricted Galvin ultrafilter **U** on $\mathbf{N}_\Gamma$ such that $\mathbf{S}_\Gamma = \mathbf{S}_{\Psi_{\Gamma,\mathbf{U}}}$, i.e. such that $\Psi_{\Gamma,\mathbf{U}}$ is a Γ-clone.

Proof: To prove that i) implies ii), we imitate Hindman's proof that Hindman's theorem implies the existence of Galvin ultrafilters. **MT** is used to insure that $\mathbf{S}_\Gamma = \mathbf{S}_{\Psi_{\Gamma,\mathbf{U}}}$.

The following notation, some new, some old and abused, is helpful. $\mathbf{N}_\Gamma$ will denote the nonzero elements of $\mathbf{N}_\Gamma$. $\mathbf{P}$ will denote $\mathbf{P}_{<\mathbf{N}_\Gamma}(\mathbf{N}_\Gamma)$. For $\mathbf{X} \in \mathbf{S}_\Gamma$ and $x \in \mathbf{N}_\Gamma$, $\mathbf{X}-x = \{y-x : y \in \mathbf{X} \wedge y > x\}$. For $\mathbf{X} \in \mathbf{S}_\Gamma$, $\mathbf{X}^c = \{x \in \mathbf{N}_\Gamma : x \notin \mathbf{X}\}$. Given $k \in \mathbf{N}_\Gamma$, (k) will denote the ideal generated by k, i.e. $(k) = \{n * k : n \in \mathbf{N}_\Gamma\}$. For a set $\mathbf{X} \in \mathbf{S}_\Gamma$ and any $j \in \mathbf{N}_\Gamma$, the j^{th} projection of $\mathbf{X}$ is defined by $\mathbf{X}(j) = \{k \in \mathbf{N}_\Gamma : (j, k)_p \in \mathbf{X}\}$, where $(\ ,\ )_p$ is the pairing function used in Chapter 3. Finally, for $x, y \in \mathbf{N}_\Gamma$, $x \in y$ will mean that the x^{th} prime divides y, i.e. that x is in the set coded by y in the prime power coding of Chapter 5.

Now we will slowly define $\mathbf{U}$. Let $<\mathbf{A}_n>_{n \in \omega}$ be an enumeration of the sets in $\mathbf{S}_\Gamma$ with $\mathbf{A}_0 = \mathbf{N}_\Gamma$. Let $<f_n>_{n \in \omega}$ be an enumeration of the total functions in $\mathbf{S}_\Gamma$ such that f_0 is the zero constant function. For each $n \in \omega$, we will inductively define sets Π_n, $\mathbf{Z}_n$, and $\mathbf{X}_n$ such that the following five properties hold.

1) $\mathbf{Z}_n = \mathbf{A}_n$ or $\mathbf{Z}_n = \mathbf{A}_n^c$.

2) If $i < j$ then $\{\Pi_i(k) : k \in \mathbf{N}_\Gamma\} \subseteq \{\Pi_j(k) : k \in \mathbf{N}_\Gamma\}$.

3) For some $t \in \mathbf{N}_\Gamma$, $\Pi_n(t) = \mathbf{Z}_n$.

4) If $\mathbf{F} \in \mathbf{P}$ then there is a Γ-infinite subset $\mathbf{B}$ of $\bigcap_{j \in \mathbf{F}} \Pi_n(j)$ such that for each $x \in \mathbf{B}$ there is a t_x satisfying

$$\Pi_n(t_x) \subseteq \bigcap_{j \in \mathbf{F}} \Pi_n(j) \cap \left(\bigcap_{j \in \mathbf{F}} \Pi_n(j) - x\right).$$

5) For every $x \in \mathbf{N}_\Gamma$, $x \in \mathbf{X}_n$ if and only if $\exists j \in \mathbf{N}_\Gamma \forall y \in \Pi_n(j)(x \in f_n(y))$.

Let $\mathbf{Z}_0 = \mathbf{A}_0$ and $\Pi_0 = \bigcap_{j \in \mathbf{N}_\Gamma} \{j\} \times (j)$. Properties 1) and 2) hold trivially. Since $\mathbf{Z}_0 = \mathbf{N}_\Gamma = \Pi_0(1)$, property 3) holds. If $\mathbf{F} \in \mathbf{P}$, then $\bigcap_{j \in \mathbf{F}} \Pi_0(j) = (k)$ for some $k \in \mathbf{N}_\Gamma$.

Since $(k)\cap((k)-j)=(k)$ for every $j\in(k)$, property 4) holds. Finally, since $f_0(j)=0$ for all $j\in\mathbf{N}_\Gamma$, property 5) holds with $\mathbf{X}_0=\emptyset$.

Suppose Π_j, $\mathbf{Z}_j$, and $\mathbf{X}_j$ are defined and satisfy properties 1) through 5) for all $j<n$. Let $\Pi_n^*\in\mathbf{S}_\Gamma$ such that

$$\forall i<n\,\forall t\,\exists k\,(\Pi_i(t)=\Pi_n^*(t)),\text{ and}$$

$$\forall i\,\forall j\,(i\neq j\rightarrow\Pi_n^*(i)\neq\Pi_n^*(j)).$$

Since $n\in\omega$, $\Pi_i\in\mathbf{S}_\Gamma$ for $i<n$ and $\Gamma\models\mathbf{ACA}_0$, such a Π_n^* exists. Let

$$\mathbf{V}=\bigcup_{j\in\mathbf{N}_\Gamma}(\{j\}\times\bigcap_{i\leq j}\Pi_n^*(i)).$$

$\mathbf{V}$ is also in $\mathbf{S}_\Gamma$. Now $\mathbf{V}(1)=\Pi_i(j)$ for some $i<n$ and $j\in\mathbf{N}_\Gamma$. By property 4), there is a least $x_1\in\mathbf{N}_\Gamma$ and a least $t\in\mathbf{N}_\Gamma$ such that $x_1\in\mathbf{V}(1)$ and $\Pi_n^*(t)\subseteq\mathbf{V}(1)\cap(\mathbf{V}(1)-x_1)$. Let $m(1)=1$ and $m(2)=t+1$. Then we have $\mathbf{V}(m(2))\subseteq\mathbf{V}(m(1))\cap(\mathbf{V}(m(1))-x_1)$. Assume that we have chosen x_s and $m(s+1)$ for each $s<r\in\mathbf{N}_\Gamma$, such that $x_x\in\mathbf{V}(m(s))$, $x_s>\sum_{k=1}^{s-1}x_k$, $m(s+1)>m(s)$, and

$\mathbf{V}(m(s+1))\subseteq\mathbf{V}(m(s))\cap(\mathbf{V}(m(s))-x_s)$. Since

$$\{\Pi_n^*(k):1\leq k\leq m(r)\}\subseteq\{\Pi_j(i):i\in\mathbf{N}_\Gamma\}$$

for some $j<n$, and since $\mathbf{V}(m(r))=\bigcap_{k\leq m(r)}\Pi_n^*(k)$, by property 4) we can find an infinite subset $\mathbf{B}$ of $\mathbf{V}(m(r))$ and, for each x in $\mathbf{B}$, a t_x such that we have $\Pi_j(t_x)\subseteq\mathbf{V}(m(r))\cap(\mathbf{V}(m(r))-x)$. Note that we can find arbitrarily large x of this sort, and that $\Pi_j(t_x)$ is $\Pi_n^*(b)$ for some b. Thus we can choose the least integer x_r and the least b such that $x_r>\sum_{k<r}x_k$ and $\Pi_n^*(b)\subseteq\mathbf{V}(m(r))\cap(\mathbf{V}(m(r))-x_r)$. Let

$m(r+1) = \max\{b, m(r)+1\}$. Then $\mathbf{V}(m(r+1)) \subseteq \mathbf{V}(m(r)) \cap (\mathbf{V}(m(r)) - x_r)$ as desired. By arithmetic comprehension in Γ, the sequences $<x_s>_{s \in \mathbf{N}_\Gamma}$ and $<m(s)>_{s \in \mathbf{N}_\Gamma}$ are in $\mathbf{S}_\Gamma$.

We now show that $\mathbf{FS}(<x_s>_{s \geq r}) \subseteq \mathbf{V}(m(r))$ for all $r \in \mathbf{N}_\Gamma$. Since for all $i < j$, $x_i \in \mathbf{V}(m(i))$ and $\mathbf{V}(m(i)) \supseteq \mathbf{V}(m(j))$, we have that $<x_s>_{s \geq r} \subseteq \mathbf{V}(m(r))$ for all $r \in \mathbf{N}_\Gamma$. Suppose that for $k \geq 2$ and for all r, we have that for every $\mathbf{J} \in \mathbf{P}_{<k}(\{x \in \mathbf{N}_\Gamma : x \geq r\})$, the inclusion $\mathbf{FS}(<x_i>_{i \in \mathbf{J}}) \subseteq \mathbf{V}(m(r))$ holds. Let $\mathbf{K} \in \mathbf{P}_{<k+1}(\mathbf{N}_\Gamma)$, $s = \min(\mathbf{K})$, and $\mathbf{L} = \{i \in \mathbf{K} : i \neq s\}$. Then it follows that $\mathbf{L} \in \mathbf{P}_{<k}(\{x \in \mathbf{N}_\Gamma : x \geq s+1\})$, so $\sum_{i \in \mathbf{L}} x_i \in \mathbf{V}(m(s+1))$ and $\mathbf{V}(m(s+1)) \subseteq \mathbf{V}(m(s)) - x_s$.

Thus,

$$\sum_{i \in \mathbf{K}} x_i = x_s + \sum_{i \in \mathbf{L}} x_i \in \mathbf{V}(m(s)),$$

so $\sum_{i \in \mathbf{K}} x_i \in \mathbf{V}(m(r))$ for all $r < s$. By induction, it follows that the inclusion $\mathbf{FS}(<x_s>_{s \geq r}) \subseteq \mathbf{V}(m(r))$ holds for all r.

The next step is to apply Milliken's theorem to a partition h of $[\mathbf{FS}(<x_s>_{s \in \mathbf{N}_\Gamma})]^3$. Since $\sum_{s < r} x_s < x_r$, there is a natural bijection between $\mathbf{FS}(<x_s>_{s \in \mathbf{N}_\Gamma})$ and $\mathbf{P}$ given by $\pi(\mathbf{A}) = \sum_{s \in \mathbf{A}} x_s$ for all $\mathbf{A} \in \mathbf{P}$. **MTU** insures that there is a sequence $<y_s>_{s \in \mathbf{N}_\Gamma}$ such that for some constant c, $h(<<y_s>_{s \in \mathbf{N}_\Gamma}>_\Sigma^3) = c$. The sequence can simultaneously be chosen so that if $i < j$, $y_i = \sum_{s \in \mathbf{I}} x_s$, and $y_j = \sum_{s \in \mathbf{J}} x_s$, then $\max_{s \in \mathbf{I}} x_s < \min_{s \in \mathbf{J}} x_s$. The partition h is defined in terms of two auxiliary functions. For $(b, c, d) \in [\mathbf{FS}<x_s>_{s \in \mathbf{N}_\Gamma}]^3$ such that $b < c < d$, define h_1 and h_2 by:

$$h_1((b,c,d))=\begin{cases}0 & \text{if } b\in \mathbf{A}_n.\\ 1 & \text{if } b\notin \mathbf{A}_n.\end{cases}$$

$$h_2((b,c,d))=\begin{cases}0 & \text{if } \forall t<b\,(t\in f_n(c) \text{ iff } t\in f_n(d)).\\ 1 & \text{otherwise.}\end{cases}$$

Let $h((b,c,d))=2h_2((b,c,d))+h_1((b,c,d))$. Let $<y_s>_{s\in\mathbf{N}_\Gamma}$ denote the monochromatic sequence of color c given by Milliken's theorem and described above. Note that $<y_s>_{s\in\mathbf{N}_\Gamma}\in\mathbf{S}_\Gamma$. An easy pigeonhole argument shows that $c\in\{0,1\}$.

We can now define $\mathbf{Z}_n$, Π_n and $\mathbf{X}_n$. Let $\mathbf{Z}_n=\mathbf{A}_n$ if $c=0$, and $\mathbf{Z}_n=\mathbf{A}_n^c$ if $c=1$. Let

$$\mathbf{S}_n=\{\sum_{s\in\mathbf{J}} y_s : \mathbf{J}\in\mathbf{P}\wedge \min(\mathbf{J})\geq n\}.$$

Define Π_n by

$$\Pi_n(t)=\begin{cases}\mathbf{Z}_n & \text{if } t=1.\\ \Pi_n^*(t/2) & \text{if } t \text{ is even.}\\ \mathbf{S}_{(t-1)/2} & \text{if } t>1 \text{ and } t \text{ is odd.}\end{cases}$$

Finally, let $\mathbf{X}_n=\{k\in\mathbf{N}_\Gamma : k\in f_n(y_{k+1})\}$. Note that Π_n, $\mathbf{Z}_n$, and $\mathbf{X}_n$ are all in $\mathbf{S}_\Gamma$.

We must now verify properties 1) through 5) for Π_n, $\mathbf{Z}_n$ and $\mathbf{X}_n$. Properties 1), 2) and 3) follow trivially from the definitions of $\mathbf{Z}_n$ and Π_n. To verify property 4), let $\mathbf{F}\in\mathbf{P}$ and write $\mathbf{F}=\mathbf{F}_1\cup\mathbf{F}_2$, where $\mathbf{F}_1$ consists of even integers and $\mathbf{F}_2$ consists of odd integers. If $\mathbf{F}_2=\emptyset$, then for some $m<n$, $\exists k\,(\Pi_n(j)\subseteq\Pi_m(k))$ for every $j\in\mathbf{F}$. By the induction hypothesis, property 4) holds. If $\mathbf{F}_2\neq\emptyset$, then for some k, $\mathbf{S}_k\subseteq\bigcap_{j\in\mathbf{F}_2}\Pi_n(j)$. If $\mathbf{F}_1=\emptyset$, then $\mathbf{S}_k\subseteq\bigcap_{j\in\mathbf{F}}\Pi_n(j)$. If $\mathbf{F}_1\neq\emptyset$, then $\mathbf{V}(l)\subseteq\bigcap_{j\in\mathbf{F}_2}\Pi_n(j)$ for some l. Since $\mathbf{S}_l\subseteq\mathbf{V}_l$, we have in either case some r such that $\mathbf{S}_r\subseteq\bigcap_{j\in\mathbf{F}}\Pi_n(j)$.

Let $\mathbf{B}=\{y_j : j \geq r\}$. Fix $y_m \in \mathbf{B}$. Then $\mathbf{S}_{m+1} \in \Pi_n$ and

$$\mathbf{S}_{m+1} \subseteq \mathbf{S}_r \cap (\mathbf{S}_r - y_m) \subseteq (\bigcap_{j \in \mathbf{F}} \Pi_n(j) - y_m).$$

Thus property 4) is satisfied. Finally, we verify property 5). Fix $x \in \mathbf{N}_\Gamma$. Then $x \in \mathbf{X}_n$ if and only if $x \in f_n(y_{x+1})$. Since $h_2(<<y_s>_{s \in \mathbf{N}_\Gamma}>^3_\Sigma)=0$, for every $z \in \mathbf{S}_{x+1}$, $x \in f_n(z)$ if and only if $x \in f_n(y_{x+1})$. Since $\mathbf{S}_{x+1}=\Pi_n(2(x+1)+1)$, property 5) is satisfied. This completes the inductive construction of Π_n, $\mathbf{Z}_n$, and $\mathbf{X}_n$.

Let $\mathbf{U}=\{\Pi_m(n) : m, n \in \mathbf{N}_\Gamma\}$. By property 4), $\mathbf{U}$ has the finite intersection property. Property 1) guarantees that $\mathbf{U}$ is a restricted ultrafilter on $\mathbf{N}_\Gamma$. Property 5) insures that $\mathbf{U}$ is additive and $\mathbf{S}_{\Psi_{\Gamma,\mathrm{U}}}=\mathbf{S}_\Gamma$. It remains only to show that $\mathbf{U}$ is Galvin.

Fix $\mathbf{A} \in \mathbf{U}$. $\mathbf{A}=\Pi_n(t)$ for some $n, t \in \mathbf{N}_\Gamma$. By property 4), there is some $x \in \mathbf{A}$ and some t_x such that $\Pi_n(t_x) \subseteq \mathbf{A} \cap (\mathbf{A}-x)$. Since $\Pi_n(t_x) \in \mathbf{U}$ and $\Pi_n(t_x) \subseteq \mathbf{A}-x$, $\mathbf{A}-x \in \mathbf{U}$, as desired.

We will now prove that ii) implies i). Assume that there is an additive restricted Galvin ultrafilter $\mathbf{U}$ on $\mathbf{N}_\Gamma$ such that $\mathbf{S}_\Gamma=\mathbf{S}_{\Psi_{\Gamma,\mathrm{U}}}$. By Theorem 7.22, it suffices to show that Γ models the extended version of Hindman's theorem stated in Theorem 7.5. This is done by a modification of the proof of Lemma 7.17.

Let $f_i : \mathbf{N} \rightarrow l_i$ be a countable sequence in $\mathbf{S}_\Gamma$ of partitions of $\mathbf{N}_\Gamma$. Since $\mathbf{R}_{\prod_\mathrm{U} \mathbf{N}_\Gamma} \mathbf{N}_\Gamma=\mathbf{S}_{\Psi_{\Gamma,\mathrm{U}}}=\mathbf{S}_\Gamma$, the sequence is coded in $\prod_\mathrm{U} \mathbf{N}_\Gamma$. By Porism 5.17, we may expand $\prod_\mathrm{U} \mathbf{N}_\Gamma$ to a model of $\mathbf{L}_1$ with a new function symbol for the sequence $<f_i>$. (From now on, we will introduce such new symbols routinely.) Let $[id]$ denote the element of $\prod_\mathrm{U} \mathbf{N}_\Gamma$ containing the identity function. Define the sequence $<y_i>$ (which is coded in $\prod_\mathrm{U} \mathbf{N}_\Gamma$) by

$$y_i = \mu z < [id](\forall j \leq i (f_j(\mathbf{FS}(<y_k>_{j \leq k < i} \cup \{z, [id]\}))) = f_j([id])).$$

Clearly, $<y_i> \cap \mathbf{N}_\Gamma \in \mathbf{S}_{\Psi_{\Gamma,\mathbf{U}}}$. We need to show that $<y_i>$ is unbounded in $\mathbf{N}_\Gamma$.

$\mathbf{U}$ is additive, so for each $j \in \mathbf{N}_\Gamma$, there is a unique $c_j < l_j$ such that $\{x \in \mathbf{N}_\Gamma : f_j(x) = c_j\} \in \mathbf{U}$, and, equivalently, $\prod_{\mathbf{U}} \mathbf{N}_\Gamma \models f_j([id]) = c_j$. Let $<y_i>_{i<k}$ be a (possibly empty) initial segment of $<y_i> \cap \mathbf{N}_\Gamma$. By the definition of y_i, we have

$$\prod_{\mathbf{U}} \mathbf{N}_\Gamma \models \forall j < k (f_j(\mathbf{FS}(<y_n>_{j \leq n < k} \cup \{[id]\})) = c_j).$$

By Theorem 5.12 part iii), we may define $\mathbf{X} \in \mathbf{S}_\Gamma$ such that

$$\mathbf{X} = \{x > y_{k-1} : \Gamma \models \forall j < k (f_j(\mathbf{FS}(<y_n>_{j \leq n < k} \cup \{x\})) = c_j)\} \in \mathbf{U}.$$

Since $\prod_{\mathbf{U}} \mathbf{N}_\Gamma \models f_k([id]) = c_k$, we can define $\mathbf{Y} \in \mathbf{S}_\Gamma$ such that $\mathbf{Y} = \{x \in \mathbf{X} : f_k(x) = c_k\}$. $\mathbf{U}$ is a Galvin ultrafilter, so we can find an element $y_k \in \mathbf{Y}$ and a set $\mathbf{Z} \in \mathbf{U}$ such that

$$\mathbf{Z} = \{x - y_k : x > y_k \wedge x \in \mathbf{Y}\} \in \mathbf{U}.$$

Since $y_k \in \mathbf{X}$, $\Gamma \models \forall j < k (f_j(\mathbf{FS}(<y_n>_{j \leq n \leq k})) = c_j)$. Furthermore, $x \in \mathbf{Z}$ implies $x \in \mathbf{X}$ and $x + y_k \in \mathbf{X}$, so for all $x \in \mathbf{Z}$,

$$\Gamma \models \forall j < k (f_j(\mathbf{FS}(<y_n>_{j \leq n < k} \cup \{x\})) = f_j(\mathbf{FS}(<y_n>_{j \leq n < k} \cup \{x + y_k\})) = c_j).$$

Thus $\prod_{\mathbf{U}} \mathbf{N}_\Gamma \models \forall j < k \ (f_j(\mathbf{FS}(<y_n>_{j \leq n \leq k} \cup \{[id]\})) = c_j)$. Finally, $y_k \in \mathbf{Y}$, so $\Gamma \models f_k(y_k) = c_k$. Also, for all $x \in \mathbf{Z}$,

$$\Gamma \models f_k(x + y_k) = f_k(x) = c_k$$

$\mathbf{Z} \in \mathbf{U}$, so by Theorem 5.12 part ii),

$$\prod_U \mathbf{N}_\Gamma \models f_k([id]) = f_k([id]+y_k) = f_k(y_k) = c_k\,.$$

Summarizing, we have

$$\prod_U \mathbf{N}_\Gamma \models \forall j \le k\, (f_j(\mathbf{FS}(<y_n>_{j \le n \le k} \cup \{[id]\})) = c_j).$$

This suffices to show that $<y_i>$ is unbounded in $\mathbf{N}_\Gamma$. We will write $<y_i>_{i \in \mathbf{N}_\Gamma}$ for $<y_i> \cap \mathbf{N}_\Gamma$.

The proof is almost finished. Note that $<y_i>_{i \in \mathbf{N}_\Gamma} \in \mathbf{S}_{\Psi_{\Gamma,U}}$ and $\mathbf{S}_{\Psi_{\Gamma,U}} = \mathbf{S}_\Gamma$, so $<y_i>_{i \in \mathbf{N}_\Gamma} \in \mathbf{S}_\Gamma$. Furthermore, for every $j \ge k \in \mathbf{N}_\Gamma$,

$$\prod_U \mathbf{N}_\Gamma \models f_k(\mathbf{FS}(<y_i>_{k \le i \le j}) = c_k),$$

so by Theorem 5.12 part ii),

$$\Gamma \models \forall k\, \exists c_k\, \forall j f_k(\mathbf{FS}(<y_i>_{k \le i \le j})) = c_k).$$

Thus Γ models the extended version of Hindman's theorem, and the proof of the theorem is complete. ■

In the previous section, we set out to use model theoretic methods to determine the proof theoretic strength of **HT**. The most natural model theoretic proof that $\mathbf{ACA}_0$ proves **HT** is as follows. Given an arbitrary countable model Γ of $\mathbf{ACA}_0$, construct an additive restricted Galvin ultrafilter **U** on $\mathbf{N}_\Gamma$ such that $\mathbf{S}_{\Psi_{\Gamma,U}} = \mathbf{S}_\Gamma$. For any finite partition f of $\mathbf{N}_\Gamma$, the pegged Folkman sequence of $[id]$ yields an unbounded sequence in $\mathbf{S}_\Gamma$ which is monochromatic for f in the sense of **HT**. Thus, Γ models **HT**, provided we can find **U**.

This argument is most appealing when Γ is an ω-model. Since the ultrafilter **U** is constructed externally, we are not confined to the formal system $\mathbf{ACA}_0$ during the proof. Any information concerning the existence and nature of ultrafilters on ω may be used in constructing $\mathbf{U}$.

Unfortunately, Theorem 7.25 casts serious doubt on the existence of ultrafilters like $\mathbf{U}$ (for arbitrary countable models of $\mathbf{ACA}_0$). Theorem 7.25 shows that if the model theoretic proof could be carried out, both **HT** and **MT** would be provable in $\mathbf{ACA}_0$. This is strong empirical evidence against the feasibility of such a model theoretic proof.

On the positive side, Theorem 7.25 is intrinsically interesting. First, it equates a statement about models of combinatorics with a statement about combinatorics of models. Secondly, it contrasts nicely with the equivalence proved by Hindman. Overall, Theorem 7.25 illuminates some of the subtle twists so common in the study of combinatorics within models of subsystems of $\mathbf{Z}_2$.

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