Disguising induction: Proofs of the pigeonhole principle for trees

Jeffry L. Hirst [1]

ABSTRACT. We examine the relationship between a pigeonhole principle for trees and induction on Σ^0_2 formulas. This analysis is carried out in the framework of reverse mathematics utilizing a hierarchy of axiom systems formulated by Harvey Friedman.

Let $2^{<\mathbb{N}}$ denote the set of all finite sequences of zeros and ones. We often use σ to denote both a finite sequence and the associated finite function, so $\sigma(0)$ is the first element of the sequence, and $\sigma(\mathrm{lh}(\sigma)-1)$ is the last. If τ consists of σ with appended elements we write $\sigma \subseteq \tau$, and write $\sigma \subset \tau$ when τ is a proper extension of σ. Viewing $2^{<\mathbb{N}}$ as a partial order ordered by the $\subseteq$ relation, we can think of any subset of $2^{<\mathbb{N}}$ as a subtree. A bijection between a subset $S \subseteq 2^{<\mathbb{N}}$ and $2^{<\mathbb{N}}$ that preserves extension is an order isomorphism. Using this terminology, we can formulate the following pigeonhole principle on binary trees.

TT(1): Suppose $f : 2^{<\mathbb{N}} \to n$ for some $n \in \mathbb{N}$. Then there is a subtree $S \subseteq 2^{<\mathbb{N}}$ order isomorphic to $2^{<\mathbb{N}}$ and a $c < n$ such that $f(\sigma) = c$ for every $\sigma \in S$.

This pigeonhole principle follows immediately from a version of Hindman's theorem. If we let FIN denote the collection of all nonempty finite subsets of $\mathbb{N}$, then the familiar finite sum form of Hindman's theorem [9] is equivalent to the following statement. (See [1].)

HT: Suppose $f : \mathsf{FIN} \to n$ for some $n \in \mathbb{N}$. Then there is a sequence $\langle X_i \rangle_{i\in\mathbb{N}}$ of elements of FIN and a $c < n$ such that

- if $i < j$ then $\max(X_i) < \min(X_j)$, and

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- for every finite nonempty set $J \subset \mathbb{N}$, $f(\cup_{j\in J} X_j) = c$.

Here is a proof that HT implies $\mathsf{TT}(1)$. Suppose $f : 2^{<\mathbb{N}} \to n$. For each $X \in \mathsf{FIN}$, let σ_X be a sequence in $2^{<\mathbb{N}}$ of length $\max(X)+1$ such that for each i, $\sigma_X(i) = 1$ if and only if $i \in X$. Define $g : \mathsf{FIN} \to n$ by $g(X) = f(\sigma_X)$. Apply HT to g, and let $\langle X_i \rangle_{i\in\mathbb{N}}$ be the resulting sequence of finite subsets and let c be the associated color. Define $Y_{\langle\ \rangle} = \emptyset$, and for each nonempty $\tau \in 2^{<\mathbb{N}}$, let Y_τ be the union of X_0 or X_1, X_2 or X_3, X_4 or X_5 and so on, where the set X_{2i} is included if $\tau(i) = 0$ and X_{2i+1} is included if $\tau(i) = 1$. More formally, for nonempty $\tau \in 2^{<\mathbb{N}}$, let $Y_\tau = \cup_{i<\mathrm{lh}(\sigma)} X_{2i+\tau(i)}$. Then the set $S = \{\sigma_{Y_\tau} \mid \tau \in 2^{<\mathbb{N}}\}$ is a subtree of $2^{<\mathbb{N}}$ and the map taking τ to σ_{Y_τ} is an order isomorphism between $2^{<\mathbb{N}}$ and S. Furthermore, for each τ, $f(\sigma_{Y_\tau}) = g(Y_\tau) = c$, so S is the desired monochromatic subtree.

Timothy McNicholl [11] asked if the use of Hindman's theorem is actually necessary to prove $\mathsf{TT}(1)$. Reverse mathematics, based on the axiom systems formulated by Harvey Friedman [6, 7], provides an excellent tool set for addressing this type of question. Indeed, HT is not needed to prove $\mathsf{TT}(1)$, as was shown in [3]. We will present a proof of this result below.

We will formalize our proof in the subsystem RCA_0. This theory has variable types for natural numbers and sets of natural numbers, basic arithmetic axioms including induction restricted to Σ^0_1 formulas, and the recursive comprehension axiom, which (naïvely) asserts the existence of computable sets. A very detailed discussion of RCA_0 can be found in Simpson's book [12]. Since $\mathsf{TT}(1)$ implies the infinite pigeonhole principle, it cannot be proved in RCA_0 [10]. However, if we append the induction scheme for Σ^0_2 formulas, denoted by $\Sigma^0_2 - \mathsf{IND}$, then $\mathsf{TT}(1)$ can be proved, as stated in the following result.

THEOREM 1. (RCA_0) $\Sigma^0_2 - \mathsf{IND}$ *implies* $\mathsf{TT}(1)$.

Proof. We carry out the proof in RCA_0. Suppose $f : 2^{<\mathbb{N}} \to n$. Let $\{X_j \mid j < 2^n\}$ be the collection of all subsets of $\{0, 1, \ldots n-1\}$, enumerated so that $X_i \subseteq X_j$ implies $i \le j$. Since the entire range of f is included in some X_j, there is a j such that

$$\exists\sigma\ \forall\tau \supseteq \sigma\ (f(\tau) \in X_j),$$

which is clearly a Σ^0_2 formula when the finite sequences are identified with their natural number codes. $\Sigma^0_2 - \mathsf{IND}$ implies that there is a least such j. (For equivalent formulations of induction schema, see Theorem 2.4 of [8].) Call this least element j_0, and choose σ_0 such that $\forall\tau \supseteq \sigma_0\ (f(\tau) \in X_{j_0})$. If $c = f(\sigma_0)$, then every $\tau \supseteq \sigma_0$ can be extended to an element τ' with $f(\tau') = c$. We can construct a monochromatic subtree order isomorphic to

$2^{<\mathbb{N}}$ by the following process. Let σ_0 be the root node. Fix a level-by-level enumeration of $2^{<\mathbb{N}}$. If σ has been added to the tree, for each $i \in \{0,1\}$ let τ_i be the first enumerated extension of $\sigma^\frown i$ that has color c. Note that recursive comprehension suffices to prove the existence of this subtree. ■

Imitating [3], we will call the monochromatic tree of the preceding proof the standard tree of color c based at σ_0. The fact that HT is not required for the proof of $\mathsf{TT}(1)$ is now an easy corollary.

COROLLARY 2. $\mathsf{RCA}_0 + \mathsf{TT}(1)$ *does not prove* HT.

Proof. The natural numbers together with the computable sets form a model of $\mathsf{RCA}_0 + \Sigma^0_2 - \mathsf{IND}$, but not a model of arithmetical comprehension (since not all arithmetically definable sets are computable.) This model is not a model of HT, since $\mathsf{RCA}_0 + \mathsf{HT}$ implies arithmetical comprehension (Theorem 2.6 of [1]). ■

Although this shows that HT is not necessary to prove $\mathsf{TT}(1)$, it leads us to ask if $\Sigma^0_2 - \mathsf{IND}$ is necessary to prove $\mathsf{TT}(1)$. We can address this question by looking at various combinatorial principles that imply $\mathsf{TT}(1)$ and exploring their relationships to $\Sigma^0_2 - \mathsf{IND}$.

1 Eventually constant tails

The core of the proof of Theorem 1 is locating a node σ such that for every extension α of σ the spectrum of colors appearing above α is exactly the same as the spectrum of colors appearing above σ. We can formalize the existence of such a σ as follows:

$\mathsf{ECT}(2^{<\mathbb{N}})$**:** If $f : 2^{<\mathbb{N}} \to n$ then $\exists \sigma\ \forall \tau \supseteq \sigma\ \forall \alpha \supseteq \sigma\ \exists \beta \supset \alpha\ (f(\tau) = f(\beta))$.

This same principle can be expressed for colorings of $\mathbb{N}$.

$\mathsf{ECT}(\mathbb{N})$**:** If $f : \mathbb{N} \to n$ then $\exists b\ \forall x \geq b\ \exists y > x\ (f(x) = f(y))$.

Informally, $\mathsf{ECT}(\mathbb{N})$ asserts that there is a b such that whenever $[x, \infty) \subseteq [b, \infty)$ then the range of f restricted to $[x, \infty)$ is identical to the range of f restricted to $[b, \infty)$. That is, we are saying that the range of f on tails is eventually constant. The label ECT is a mnemonic for **e**ventually **c**onstant **t**ails.

$\mathsf{ECT}(2^{<\mathbb{N}})$ can be substituted for the use of $\Sigma^0_2 - \mathsf{IND}$ in the proof of Theorem 1. The next three lemmas explore the relationships between the two forms of ECT and $\Sigma^0_2 - \mathsf{IND}$, and the consequences are collected in Theorem 6.

LEMMA 3. (RCA_0) $\mathsf{ECT}(2^{<\mathbb{N}})$ *implies* $\mathsf{ECT}(\mathbb{N})$.

Proof. Working in RCA_0, fix $f : \mathbb{N} \to n$. Define $g : 2^{<\mathbb{N}} \to n$ by $g(\sigma) = f(\mathrm{lh}(\sigma))$. Apply $\mathsf{ECT}(2^{<\mathbb{N}})$ to g and obtain σ. Let $b = \mathrm{lh}(\sigma)$ and choose $x \geq b$. Choose any $\tau \supseteq \sigma$ with $\mathrm{lh}(\tau) = x$. By $\mathsf{ECT}(2^{<\mathbb{N}})$, there is a $\beta \supset \tau$ such that $g(\beta) = g(\tau)$. Thus $\mathrm{lh}(\beta) > x$ and $f(\mathrm{lh}(\beta)) = g(\beta) = g(\tau) = f(x)$. ■

LEMMA 4. (RCA_0) $\mathsf{ECT}(\mathbb{N})$ *implies* $\Sigma^0_2 - \mathsf{IND}$.

Proof. By Exercise 11.3.13 of [12], over RCA_0 the scheme $\Sigma^0_2 - \mathsf{IND}$ is equivalent to the bounded Σ^0_2 comprehension scheme, which is the assertion that

$$\forall n \exists X \forall i (i \in X \leftrightarrow (i < n \land \varphi(i)))$$

where $\varphi(i)$ is a Σ^0_2 formula not containing X free. We will use $\mathsf{ECT}(\mathbb{N})$ to derive bounded Σ^0_2 comprehension.

Suppose $\varphi(i)$ is $\exists x \forall y \theta(i,x,y)$, where θ is quantifier free. Fix n. Define $f : \mathbb{N} \to n+1$ by writing each natural number as $mn + i$ with $i < n$ and setting

$$f(mn+i) = \begin{cases} i & \text{if } \mu x < m\ \forall y < m\ \theta(i,x,y) \text{ is not equal to} \\ & \qquad\qquad \mu x < m+1\ \forall y < m+1\ \theta(i,x,y) \\ n & \text{otherwise.} \end{cases}$$

In the preceding, when $\forall x < m\ \exists y < m\ \neg\theta(i,x,y)$, we define the expression $\mu x < m\ \forall y < m\ \theta(i,x,y)$ to be equal to m.

Note that if there is an x such that $\forall y \theta(i,x,y)$ then $\Sigma^0_1 - \mathsf{IND}$ implies there is a least such element; call it x_0. Applying $\mathsf{B}\Sigma^0_0$, which is also a consequence of $\Sigma^0_1 - \mathsf{IND}$ (see [8], Chapter I, section 2), we can find an m_0 so large that $\forall t < x_0\ \neg\forall y < m_0\ \theta(i,t,y)$. Then for any $m > m_0$,

$$\mu x < m\ \forall y < m\ \theta(i,x,y)\ =\ x_0\ =\ \mu x < m+1\ \forall y < m+1\ \theta(i,x,y),$$

so $f(mn+i) = n$. Consequently, on any final segment of $[m_0 n + i, \infty)$, i is not in the range of f. Summarizing, if $\exists x \forall y \theta(i,x,y)$, then eventually i is omitted from all tails of f.

On the other hand, suppose $\neg\exists x \forall y \theta(i,x,y)$ and fix an element $b > 0$. If for every $m > b$ we have $\forall x < m\ \exists y < m\ \neg\theta(i,x,y)$, then for all $m > b$ we have $f(mn+i) = i$ and so eventually i is in every tail. Suppose that for some $m > b$ we have $\exists x < m\ \forall y < m\ \theta(i,x,y)$. Let x_0 be the least element less than m satisfying $\forall y < m\ \theta(i,x_0,y)$. Since $\neg\exists x \forall y\ \theta(i,x,y)$, let y_0 be the least element such that $\neg\theta(i,x_0,y_0)$. Then $\mu x < y_0 \forall y < y_0\ \theta(i,x,y) = x_0$, but $\mu x < y_0 + 1\ \forall y < y_0 + 1\ \theta(i,x,y) > x_0$. Hence $f(y_0 n + i) = i$ for arbitrarily large values of n. Summarizing, if $\neg\exists x \forall y \theta(i,x,y)$ then eventually i is in every tail.

Apply $\mathsf{ECT}(\mathbb{N})$ to f to find a b such that the range of f is constant on final segments of $[b, \infty)$. By bounded Σ^0_1 comprehension (which is a consequence of $\Sigma^0_1 - \mathsf{IND}$ [12]) the set

$$Y = \{i < n \mid \exists t > b \ f(t) = i\}$$

exists. By recursive comprehension, the set $X = \{i < n \mid i \notin Y\}$ also exists, and by the preceding paragraphs, $i \in X$ if and only if $i < n$ and $\exists x \forall y \theta(i, x, y)$, as desired. ■

LEMMA 5. (RCA_0) $\Sigma^0_2 - \mathsf{IND}$ *implies* $\mathsf{ECT}(2^{<\mathbb{N}})$.

Proof. Suppose $f : 2^{<\mathbb{N}} \to n$. Let $\langle X_i \mid i < 2^n \rangle$ enumerate the subsets of n in an order that preserves containment. By the Σ^0_2 least element principle (which is equivalent over RCA_0 to $\Sigma^0_2 - \mathsf{IND}$ [8]), there is a least j such that we can find a node σ so that $\forall \tau \supseteq \sigma (f(\tau) \in X_j)$. Since j is the least such integer, for every $\tau \supseteq \sigma$, the spectrum of colors appearing above τ must match that above σ. ■

THEOREM 6. (RCA_0) *The following are equivalent:*

1. $\mathsf{ECT}(2^{<\mathbb{N}})$.
2. $\mathsf{ECT}(\mathbb{N})$.
3. $\Sigma^0_2 - \mathsf{IND}$.

Proof. Immediate from Lemmas 3, 4, and 5. ■

Thus, the ECT principles are essentially disguised forms of Σ^0_2 induction. This is a common attribute among many principles that imply $\mathsf{TT}(1)$, as shown in the next section.

2 A result of Corduan, Groszek, and Mileti

Doctors Corduan, Groszek, and Mileti reveal a strong connection between $\mathsf{TT}(1)$ and $\Sigma^0_2 - \mathsf{IND}$ in the following conservation result, which appears in [4].

THEOREM 7. *If $\mathfrak{T}$ is any extension of* RCA_0 *by* Π^1_1 *axioms, then $\mathfrak{T}$ proves* $\mathsf{TT}(1)$ *if and only if $\mathfrak{T}$ proves* $\Sigma^0_2 - \mathsf{IND}$.

Since the usual infinite pigeonhole principle, $\mathsf{RT}(1)$, can be expressed as a Π^1_1 formula and is known to be equivalent to the scheme $\mathsf{B}\Pi^0_1$ and therefore strictly weaker than $\Sigma^0_2 - \mathsf{IND}$, an immediate corollary of Theorem 7 is that $\mathsf{RT}(1)$ does not imply $\mathsf{TT}(1)$ over RCA_0. This result appears as Corollary

3.8 in [4] and provides the current best strict lower bound on the strength of $\mathsf{TT}(1)$.

Using Theorem 7, we can show that many informal proofs of $\mathsf{TT}(1)$ make use of disguised forms of $\Sigma^0_2 - \mathsf{IND}$. For example, in the next paragraph we present an alternative proof of a portion of Theorem 6. This new argument sidesteps the technical details of Lemma 4, but also does not yield information about $\mathsf{ECT}(\mathbb{N})$.

Proof.[Alternative proof that $\mathsf{ECT}(2^{<\mathbb{N}})$ implies $\Sigma^0_2 - \mathsf{IND}$ over RCA_0.] By Theorem 7, since $\mathsf{ECT}(2^{<\mathbb{N}})$ is a Π^1_1 sentence it suffices to prove that $\mathsf{ECT}(2^{<\mathbb{N}})$ implies $\mathsf{TT}(1)$ over RCA_0. Working in RCA_0, suppose $f : 2^{<\mathbb{N}} \to n$ and apply $\mathsf{ECT}(2^{<\mathbb{N}})$ to find a σ such that every extension of σ can be further extended to a node β with $f(\beta) = f(\sigma)$. As in the proof of Theorem 1, RCA_0 proves the existence of the standard tree of color $f(\sigma)$ based at σ. ■

The following limit principle arises as a natural intermediate step in a proof of $\mathsf{TT}(1)$ from stable Ramsey's Theorem for pairs (denoted SRT^2).

L**:** Given $f : \mathbb{N}^2 \to a$ such that $\lim_n f(x,n)$ exists for every $x \in \mathbb{N}$, there is a least b such that for some x, $\lim_n f(x,n) = b$.

Let L^+ denote the stronger version of L resulting from replacing "every" in the hypothesis by the word "some." Rather than deducing L from SRT^2, we will prove L^+ from $\Sigma^0_2 - \mathsf{IND}$. This is sharper, since SRT^2 is strictly stronger than $\Sigma^0_2 - \mathsf{IND}$ [2].

LEMMA 8. (RCA_0) $\Sigma^0_2 - \mathsf{IND}$ *implies* L^+. *Consequently,* $\Sigma^0_2 - \mathsf{IND}$ *also proves* L.

Proof. Suppose $f : \mathbb{N}^2 \to a$ and $\lim_n f(x,n)$ exists for some $x \in \mathbb{N}$. Thus there is a $b < a$ such that $\exists x \exists t \forall n (n > t \to f(x,n) = b)$. By the Σ^0_2 least element principle, which is equivalent to $\Sigma^0_2 - \mathsf{IND}$, there is a least such b. Thus L^+ holds. Since predicate calculus proves that L^+ implies L, the last sentence of the lemma follows immediately. ■

By proving that L implies $\mathsf{TT}(1)$, we create an opportunity for applying Theorem 7.

LEMMA 9. (RCA_0) L *implies* $\mathsf{TT}(1)$.

Proof. Let $f : 2^{<\mathbb{N}} \to a$ and let $\langle \sigma_i \rangle_{i \in \mathbb{N}}$ be an enumeration of $2^{<\mathbb{N}}$. Let $\langle Y_i \rangle_{i < 2^a}$ be an enumeration of the power set of $\{0, 1, \dots a-1\}$ such that $Y_i \subseteq Y_j$ implies $i \leq j$. Define $g : \mathbb{N}^2 \to 2^a$ by

$$g(m,n) = \mu t(\{f(\tau) \mid \tau \supseteq \sigma_m \wedge \mathrm{lh}(\tau) \leq n\} = Y_t).$$

Note that recursive comprehension suffices to prove the existence of g and that for fixed m, the function $g(m,n)$ is increasing. By the Π^0_1 least element principle (which is provable in RCA_0), for each m there is a least upper bound on the range of $g(m,n)$. Consequently, for each m, $\lim_n g(m,n)$ exists. By L, there is a least b and a $\sigma_m \in 2^{<\mathbb{N}}$ such that $\lim_n g(m,n) = b$. Since b is least, for every σ_j extending σ_m, we have $\lim_n g(j,n) = b$ also. In particular, every node extending σ_m can be extended to a node of color $f(\sigma_m)$. Thus, the standard tree of color $f(\sigma_m)$ extending σ_m (as constructed in the proof of Theorem 1) is isomorphic to $2^{<\mathbb{N}}$. ∎

Combining Theorem 7 with the preceding lemmas, we see that L and L^+ are disguised forms of $\Sigma^0_2 - \mathsf{IND}$.

COROLLARY 10. (RCA_0) *The following are equivalent:*

1. $\Sigma^0_2 - \mathsf{IND}$.
2. L^+.
3. L.

Proof. Lemma 8 shows that over RCA_0, item 1 implies item 2, and item 2 implies item 3. Since L is Π^1_1, Theorem 7 applied to Lemma 9 shows that RCA_0 proves that item 3 implies item 1. ∎

We close the section by noting that any proof of $\mathsf{TT}(1)$ relying on the standard tree construction from the proof of Theorem 1 inherently uses $\Sigma^0_2 - \mathsf{IND}$.

THEOREM 11. (RCA_0) *The following are equivalent:*

1. $\Sigma^0_2 - \mathsf{IND}$.
2. *If $f : 2^{<\mathbb{N}} \to a$ then there is a node σ such that the standard tree of color $f(\sigma)$ based at σ is isomorphic to $2^{<\mathbb{N}}$.*

Proof. The proof of Theorem 1 shows that item 1 implies item 2. To prove the reversal, note that over RCA_0, item 2 is equivalent to the statement "if $f : 2^{<\mathbb{N}} \to a$ then there is a node σ such that for every n there is a stage t such that the algorithm for constructing the standard tree of color $f(\sigma)$ based at σ halts and produces a tree containing an initial segment order isomorphic to the full binary tree of height n." Since this statement is Π^1_1 and implies $\mathsf{TT}(1)$ over RCA_0, by Theorem 7 it implies $\Sigma^0_2 - \mathsf{IND}$. Thus, item 2 implies item 1 over RCA_0. ∎

3 Null stable colorings

In this section, we present one more disguised form of induction, a related though weaker combinatorial principle that is actually equivalent to $\mathsf{TT}(1)$, and a proof of $\mathsf{TT}(1)$ from a stable version of Ramsey's theorem for pairs in trees. All these results depend on the following notion. Suppose we have $f : 2^{<\mathbb{N}} \to 2$. We say that f is a *null stable* coloring if for every $\sigma \in 2^{<\mathbb{N}}$ there is a $\sigma' \supseteq \sigma$ such that for every $\tau \supseteq \sigma'$ we have that $f(\tau) = 0$. Note that by definition, every null stable coloring is a 2-coloring. Intuitively, f is null stable if above each node we can find a node above which f is constantly 0. Given a finite sequence of null stable colorings, we could sequentially apply the definition of null stable for each coloring and eventually arrive at a node which is colored 0 for all of the colorings. The next theorem shows that the existence of such a node is a disguised form of $\Sigma^0_2 - \mathsf{IND}$.

THEOREM 12. (RCA_0) *The following are equivalent:*

1. $\Sigma^0_2 - \mathsf{IND}$.
2. *Suppose* $\langle f_i \rangle_{i<n}$ *is a sequence of null stable colorings of* $2^{<\mathbb{N}}$. *Then there is a* $\sigma \in 2^{<\mathbb{N}}$ *such that* $f_i(\sigma) = 0$ *for every* $i < n$.

Proof. We work in RCA_0. First, assume $\Sigma^0_2 - \mathsf{IND}$ and suppose $\langle f_i \rangle_{i<n}$ is a sequence of null stable 2-colorings. Define $f : 2^{<\mathbb{N}} \to 2^n$ by $f(\sigma) = \sum_{i<n} f_i(\sigma) \cdot 2^i$. By Lemma 5 and $\Sigma^0_2 - \mathsf{IND}$, we may apply $\mathsf{ECT}(2^{<\mathbb{N}})$ to f and find a σ_0 such that $\forall \alpha \supseteq \sigma_0\ \exists \beta \supset \alpha\ f(\sigma_0) = f(\beta)$. Suppose, by way of contradiction, that $f(\sigma_0) \neq 0$. Fix $i < n$ such that $f_i(\sigma_0) = 1$. Since f_i is null stable, we can find an $\alpha_0 \supseteq \sigma_0$ such that for all $\beta \supseteq \alpha_0$, $f_i(\beta) = 0$. Since σ_0 was chosen using $\mathsf{ECT}(2^{<\mathbb{N}})$, for some $\beta_0 \supset \alpha_0$, $f(\sigma_0) = f(\beta_0)$. However, $f_i(\sigma_0) = 1$ and $f_i(\beta_0) = 0$, so $f(\sigma_0) \neq f(\beta_0)$, yielding the desired contradiction. Thus we must have $f(\sigma_0) = 0$. Since $f(\sigma_0) = 0$, $f_i(\sigma_0) = 0$ for every $i < n$.

To prove that item 2 implies item 1, we will use item 2 to deduce $\mathsf{TT}(1)$ and apply Theorem 7. Assume item 2 and let $f : 2^{<\mathbb{N}} \to n$. Define the sequence $\langle f_i \rangle_{i<n}$ by setting $f_i(\sigma) = 1$ if $f(\sigma) = i$ and $f_i(\sigma) = 0$ if $f(\sigma) \neq i$. If each f_i was null stable, then by item 2 we could locate a σ such that $f_i(\sigma) = 0$ for all $i < n$, contradicting the fact that $f(\sigma) = i$ for some $i < n$. Thus there is an i_0 such that f_{i_0} is not null stable. For this i_0, we can locate a σ_0 such that for every $\sigma' \supseteq \sigma_0$ there is a $\tau \supseteq \sigma'$ such that $f_{i_0}(\tau) = 1$. Choose any $\tau \supseteq \sigma_0$ with $f_{i_0}(\tau) = 1$. Then $f(\tau) = i_0$ and every node extending τ has an extension of color i_0. Consequently the standard tree for f of color i_0 based at τ witnesses $\mathsf{TT}(1)$ for f. Since item 2 is Π^1_1, $\Sigma^0_2 - \mathsf{IND}$ follows by Theorem 7. (We could substitute a use of Theorem 11 for Theorem 7 here, if we liked.) ∎

The construction in the preceding proof of a single coloring from many 2-colorings suggests a way to exchange one application of $\mathsf{TT}(1)$ for many colors for many simultaneous applications of $\mathsf{TT}(1)$ restricted to 2-colorings. In fact, we will see that the corresponding formulations are provably equivalent. This is of some interest, since $\mathsf{TT}(1)$ restricted to any standard number of colors is a theorem of RCA_0. The next theorem capitalizes on this notion, and even restricts the simultaneous applications to $\mathsf{TT}(1)$ for null stable 2-colorings, which for single applications is very clearly a theorem of RCA_0.

THEOREM 13. (RCA_0) *The following are equivalent:*

1. $\mathsf{TT}(1)$.
2. *Suppose that for each* $i < n$, $f_i : 2^{<\mathbb{N}} \to 2$. *Then there is a subtree* $S \subseteq 2^{<\mathbb{N}}$ *order isomorphic to* $2^{<\mathbb{N}}$ *such that for each* $i < n$, f_i *is constant on* S.
3. *Item 2 holds in the case where each* f_i *is null stable.*

Proof. Assuming RCA_0, item 2 can be proved by applying $\mathsf{TT}(1)$ to the function f constructed as in the first paragraph of the proof of Theorem 12. Since item 3 is a restricted form of item 2, it remains only to prove $\mathsf{TT}(1)$ from item 3.

Suppose $f : 2^{<\mathbb{N}} \to n$. Construct f_i for $i < n$ as in the reversal for Theorem 12. If one of f_i functions is not null stable, then the standard tree construction as in the proof of the reversal of Theorem 12 witnesses $\mathsf{TT}(1)$ for f, completing the proof. If all of the f_i functions are null stable, then we may apply item 3 to find a subtree S isomorphic to $2^{<\mathbb{N}}$ and monochromatic for all the f_i functions. Let σ be the root node of S. Now $f(\sigma) = i_0$ for exactly one $i_0 < n$, and for every τ in S, $f_i(\tau) = 1$ if and only if $i = i_0$. Consequently, $f(\tau) = i_0$ for every $\tau \in S$, so S witnesses $\mathsf{TT}(1)$ for f. ■

Using the ideas from this section, we close by proving $\mathsf{TT}(1)$ from a stable version of Ramsey's theorem in trees for pairs and two colors. This is a tree analog of the proof of $\mathsf{RT}(1)$ from SRT^2 in [2].

We adopt the notation from [5] for the following. A function on pairs of comparable tree nodes $f : [2^{<\mathbb{N}}]^2 \to k$ is said to be 3-*stable* if for each $\sigma \in 2^{<\mathbb{N}}$ there is a $c < k$ such that for every $\sigma' \supseteq \sigma$ there exists $\tau \supset \sigma'$ with $f(\sigma, \rho) = c$ for all $\rho \supseteq \tau$. The principle $\mathsf{S}^3\mathsf{TT}^2_2$ asserts that every 3-stable two coloring of comparable pairs from $2^{<\mathbb{N}}$ has a monochromatic subtree isomorphic to $2^{<\mathbb{N}}$.

THEOREM 14. (RCA_0) $\mathsf{S}^3\mathsf{TT}^2_2$ *implies* $\mathsf{TT}(1)$.

Proof. Assume RCA_0 and suppose $f : 2^{<\mathbb{N}} \to n$. Define $g : [2^{<\mathbb{N}}]^2 \to 2$ for $\sigma \subset \tau$ by setting $g(\sigma, \tau) = 1$ if and only if $f(\sigma) = f(\tau)$. If σ witnesses that g is not 3-stable, then the standard tree for f based at σ with color $f(\sigma)$ witnesses $\mathsf{TT}(1)$ for f. If g is 3-stable, we may apply $\mathsf{S}^3\mathsf{TT}^2_2$ to find a monochromatic subtree S for g. Select a sequence of $n + 1$ comparable nodes in T. Some pair in this sequence must be colored identically by f. Thus $g([S]^2) \equiv 1$, and S is a monochromatic subtree for f. ■

It is not known whether or not SRT^2_2 implies $\Sigma^0_2 - \mathsf{IND}$ [2]. Similarly, it is not known whether or not $\mathsf{S}^3\mathsf{TT}^2_2$ implies $\Sigma^0_2 - \mathsf{IND}$. Since $\mathsf{S}^3\mathsf{TT}^2_2$ is a Π^1_2 sentence, Theorem 7 is not applicable. Thus, Theorem 14 is a candidate for a proof of $\mathsf{TT}(1)$ that does not rely on a disguised use of $\Sigma^0_2 - \mathsf{IND}$.

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