Reverse Mathematics of Matroids

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Abstract. Matroids generalize the familiar notion of linear dependence from linear algebra. Following a brief discussion of founding work in computability and matroids, we use the techniques of reverse mathematics to determine the logical strength of some basis theorems for matroids and enumerated matroids. Next, using Weihrauch reducibility, we relate the basis results to combinatorial choice principles and statements about vector spaces. Finally, we formalize some of the Weihrauch reductions to extract related reverse mathematics results. In particular, we show that the existence of bases for vector spaces of bounded dimension is equivalent to the induction scheme for Σ^0_2 formulas.

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The study of computable and computably enumerable matroids links the work in this paper to the theme of this volume. The following incomplete survey establishes a framework for this connection and provides a few pointers into the substantial literature on computability and matroids.

In a seminal paper on computable and c.e. vector spaces, Metakides and Nerode [14] defined a vector space V_∞, the $\aleph_0$-dimensional vector space over a countable computable field F consisting of ω-sequences of elements of F with finite support, with point-wise operations. The lattice of c.e. subspaces of V_∞ is denoted $\mathcal{L}(V_\infty)$. A vector space V over a computable field F is *c.e. presented* if it has an effective enumeration of the vectors, partial recursive addition and scalar multiplication operations, and a c.e. congruence relation $\equiv$ such that the quotient $V/\!\equiv$ is a vector space. Metakides and Nerode proved that a vector space is c.e. presented if and only if it is computably isomorphic to V_∞/W for some $W \in \mathcal{L}(V_\infty)$.

Many proofs of results for $\mathcal{L}(V_\infty)$ rely on the structure of V_∞, hampering their adaptation to $\mathcal{L}(F_\infty)$, the lattice of c.e. algebraically closed subfields of a sufficiently computable algebraically closed field F_∞ with countably infinite transcendence degree. Matroids restrict interest to dependence properties common to both vector spaces and algebraic extensions, so proofs based on matroids can often be adapted to both vector space and field settings.

In computability theoretic papers, matroids are often described in terms of *Steinitz systems*. These are also called Steinitz *closure* systems [15] or Steinitz *exchange* systems [16]. Downey [8] defines a Steinitz system as a set U and a closure operator cl mapping subsets of U to subsets of U such that if $A, B \subset U$,

(1) $A \subset \mathrm{cl}(A)$,
(2) $A \subset B$ implies $\mathrm{cl}(A) \subset \mathrm{cl}(B)$,
(3) $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$,
(4) $x \in \mathrm{cl}(A)$ implies that, for some finite $A' \subset A$, $x \in \mathrm{cl}(A')$, and
(5) (exchange) $x \in \mathrm{cl}(A \cup \{y\}) - \mathrm{cl}(A)$ implies $y \in \mathrm{cl}(A \cup \{x\})$.

As an intuitive example, we can think of U as a vector space and $\mathrm{cl}(A)$ as the linear span of the vectors in the set A. The Steinitz system (U, cl) has *computable dependence* if U is computable and there is a uniformly effective procedure that, when applied to $a, b_1, \ldots b_n \in U$, computes whether $a \in \mathrm{cl}(\{b_1, \ldots b_n\})$.

A central goal in computable matroid research is to discover algebraic properties of matroids with significant computability theoretic consequences. For example, the Steinitz system (U, cl) has the *closure intersection property* if whenever

- D is closed, that is, $\mathrm{cl}(D) = D$,
- A is independent over D, that is, for every $a \in A$, $a \notin \mathrm{cl}(D \cup A \setminus \{a\})$,
- B is independent over D, and
- $\mathrm{cl}(A \cup D) \cap \mathrm{cl}(B \cup D) = \mathrm{cl}(D)$,

then $A \cup B$ is independent over D. The system is *semiregular* (called *Downey's semiregularity* by Nerode and Remmel [16]) if no finite dimensional closed set is the union of two closed proper subsets. Downey established in his thesis [6] (abstracted in [7]) that if (U, cl) is semiregular and has the closure intersection property then the theory of $\mathcal{L}(\mathcal{U})$ is undecidable.

1 Reverse Mathematics

In his development of the theory of matroids, Whitney [18, Section 6] formulates matroids in terms of a ground set of elements and a specification of every set as being either dependent or independent. We define an *enumerated* matroid (*e-matroid*) to consist of a set and an enumeration of its finite dependent sets.

Definition 1. A (nontrivial) *e-matroid* is a pair (M, e) consisting of a set M and a function $e \colon \mathbb{N} \to [M]^{<\mathbb{N}}$ satisfying:

(1) The empty set is independent.

$$(\forall n)[e(n) \neq \emptyset]$$

(2) Finite supersets of dependent sets are dependent.

$$(\forall n)(\forall Y \in [M]^{<\mathbb{N}})[e(n) \subseteq Y \to \exists m(e(m) = Y)]$$

(3) If X is an independent set that is smaller than an independent set Y, then Y contains an element that is independent of X.

$$(\forall X, Y \in [M]^{<\mathbb{N}})(\text{ if } |X| < |Y| \text{ and } (\forall n)[e(n) \neq X \wedge e(n) \neq Y] \\ \text{then } (\exists y \in Y)(\forall n)[e(n) \neq X \cup \{y\}])$$

An infinite set is independent if and only if each of its finite subsets is independent. We assume $e(0)$ is defined, so for every e-matroid, $M \neq \emptyset$ and there is at least one finite dependent set.

Although dependence in this setting is not directly related to linear combinations, it is still possible to formulate concepts of span and bases.

Definition 2. A subset B of an e-matroid (M, e) *spans* the e-matroid if adjoining any additional element to B produces a dependent set, that is,

$$(\forall x \in M)[x \notin B \to (\exists n)(e(n) \subseteq B \cup \{x\})].$$

A subset $B \subseteq M$ is a *basis* for the e-matroid if B is independent (that is, $(\forall n)[e(n) \not\subseteq B]$) and B spans M.

We can now state our first basis theorem. The analogous result showing the equivalence of ACA_0 and the existence of bases for vector spaces is included in Theorem 4.3 of Friedman, Simpson, and Smith [9].

Theorem 3. (RCA_0) *The following are equivalent:*

(1) ACA_0.
(2) *Every e-matroid has a basis.*

Proof. To show that (1) implies (2), fix an e-matroid (M, e). Let $m_0, m_1, \ldots$ be a non-repeating enumeration of M. Consider the function $g\colon \mathbb{N} \to [M]^{<\mathbb{N}}$ defined by $g(0) = \emptyset$ and for $i > 0$,

$$g(i) = \begin{cases} g(i-1) & \text{if } (\exists n)[e(n) = g(i-1) \cup \{m_{i-1}\}], \\ g(i-1) \cup \{m_{i-1}\} & \text{otherwise.} \end{cases}$$

By arithmetical comprehension, the union of the range of g exists; call this union B. Straightforward arguments verify that B is a basis for M.

To prove the converse, by Lemma III.1.3 of Simpson [17], it suffices to use (2) to prove the existence of the range of an arbitrary injection from $\mathbb{N}$ to $\mathbb{N}$. Suppose $f\colon \mathbb{N} \to \mathbb{N}$ is an injection. Let $M = \{(i, \varepsilon) : i \in \mathbb{N} \wedge \varepsilon < 2\}$ be the ground set for an e-matroid. Let $M_0, M_1, \ldots$ be an enumeration of $[M]^{<\mathbb{N}}$. Fix a bijective pairing function mapping $\mathbb{N} \times \mathbb{N}$ onto $\mathbb{N}$. Using the notation (j, k) for both the pair and its integer code, define $e((j, k)) = \{(f(j), 0), (f(j), 1)\} \cup M_k$. Because

$(f(j),0) \in e((j,k))$, item (1) of the definition of an e-matroid holds. The inclusion of M_k in $e((j,k))$ ensures that supersets of dependent sets are dependent, satisfying item (2) of the definition. To verify item (3), suppose X and Y are finite independent sets with $|X| < |Y|$. If there is a $y \in X \cap Y$, then $X \cup \{y\} = X$ so $\forall n(e(n) \neq X \cup \{y\})$. Thus we need only consider the case where $X \cap Y = \emptyset$. We hypothesized that $|Y| > |X|$, so there must be a $y = (z, \varepsilon) \in Y$ such that for all ε', $(z, \varepsilon') \notin X$. Suppose by way of contradiction that $e(n) = X \cup \{y\}$ for some n. Then, for some j, we have $\{(f(j),0),(f(j),1)\} \subset X \cup \{y\}$. By the choice of y, we know $f(j) \neq z$, so $\{(f(j),0),(f(j),1)\} \subset X$, contradicting $(\forall n)[e(n) \neq X]$. Thus item (3) of the definition holds, and we have shown that (M,e) is an e-matroid.

Finally, we claim that if B is a basis for M, then k is in the range of f if and only if $(k,0) \notin B$ or $(k,1) \notin B$. First note that if $k = f(j)$ then, assuming 0 is the code for $\emptyset$, we have $e((j,0)) = \{(k,0),(k,1)\}$. B is a basis, so $e((j,0)) \not\subset B$, and thus $(k,0) \notin B$ or $(k,1) \notin B$. Conversely, if for example $(k,0) \notin B$, then $(\exists n)[e(n) \subseteq B \cup \{(k,0)\}]$. Because $e(n)$ is dependent and B is independent, both $(k,1) \in e(n)$ and for all $f(j) \neq k$, at least one of $(f(j),0)$ and $(f(j),1)$ is not in $e(n)$. By the definition of e, $e(n)$ must contain both $(a,0)$ and $(a,1)$ for some a in the range of f, so k is in the range of f. A similar argument holds if $(k,1) \notin B$, completing the proof of our claim. Because k is in the range of f if and only if $(k,0) \notin B$ or $(k,1) \notin B$, recursive comprehension suffices to prove the existence of the range of f, completing the reversal.

Our next result shows that if we add a hypothesis bounding the dimension of the matroid, the principle asserting the existence of a basis becomes weaker. The result also illustrates the interrelatedness of matroids and graph theory. We use the concept of rank to establish the dimensional bound.

Definition 4. We say the *rank* of an e-matroid (M,e) is *no more than* n if every subset of M of size n is dependent, that is, in the range of e.

Theorem 5. (RCA_0) *The following are equivalent:*

(1) *For every n, every e-matroid of rank no more than n has a basis.*
(2) *For every n, if $G = (V,E)$ is a countable graph and every collection of n vertices contains at least one path connected pair, then G can be decomposed into its connected components.*
(3) $\mathsf{I}\Sigma^0_2$*, the induction scheme for Σ^0_2 formulas with set parameters.*

Proof. Proofs that (2) implies (3) appear as Theorem 4.5 of Hirst [13] and also as Theorem 3.2 of Gura, Hirst, and Mummert [11]. Here, we will prove that (3) implies (1) and (1) implies (2).

To see that (3) implies (1), fix n and let (M,e) be an e-matroid of rank no more than n. Let $\psi(j)$ formalize the existence of an independent set of size $n-j$. If we use X_t to denote the finite subset of $\mathbb{N}$ encoded by t, then $\psi(j)$ can be written as $(\exists t)[|X_t| = n-j \wedge \forall k(e(k) \neq X_t)]$. Note that $\psi(j)$ is a Σ^0_2 formula, and the empty set witnesses $\psi(n)$. By the Σ^0_2 least element principle (which is easily deduced from the bounded Σ^0_2 comprehension, and is therefore a consequence of (3) by Exercise II.3.13 of Simpson [17]), there is a least j_0 such that $\psi(j_0)$. Let X_{t_0} witness $\psi(j_0)$. We claim that X_{t_0} is a basis. The range of e is closed under supersets, so no subset of X_{t_0} appears in the range of e. By the minimality of j_0, if $x \notin X_{t_0}$, then $X_{t_0} \cup \{x\}$ is dependent, so for some n, $e(n) = X_{t_0} \cup \{x\}$. Thus X_{t_0} spans M.

To show that (1) implies (2), let $G(V, E)$ be a graph in which every collection of n vertices contains at least one path connected pair. The independent sets of our e-matroid will consist of subsets of V with no path connected pairs. If G contains no edges, the identity function on V decomposes G into connected components. Suppose G has an edge connecting the vertices v_0 and v_1. Let $(V_i)_{i\in\mathbb{N}}$ be an enumeration of the finite subsets of V such that every subset appears infinitely often. Define $e(j)$ by $e(j) = V_j$ if there is some $t < j$ that encodes a path between two vertices of V_j, and $e(j) = \{v_0, v_1\}$ otherwise. It is easy to verify that (V, e) satisfies the first two clauses of the definition of an e-matroid. For the third clause, suppose X and Y are finite sets of vertices such that no pair in either set is path connected, and that $|X| < |Y|$. Suppose by way of contradiction that every vertex in Y is path connected to some vertex in X. RCA_0 can prove the existence of the function mapping each $y \in Y$ to some $x \in X$ to which it is path connected, and because $|X| < |Y|$, f must map two elements of Y to the same x. These two vertices of Y are path connected, yielding the desired contradiction. Thus (V, e) is a matroid. By (1), (V, e) has a basis, which is a maximal set of disconnected vertices in G. The function which is the identity on this basis and maps very other vertex of G to the element of the basis to which it is path connected is a decomposition of G into connected components. This decomposition is computable from the basis, so RCA_0 proves (1) implies (2).

2 Why e-Matroids?

We can define a matroid as a pair (M, D) where D is the set of all finite dependent subsets of M. In this case, D satisfies the set-based analogs of the three items in the definition of e-matroid. To express this definition within RCA_0, we represent each finite subset of M via its characteristic index. Using the set-based analog of the definition of basis, we can state and prove the following result.

Theorem 6. (RCA_0) *Every matroid has a basis.*

Proof. Let (M, D) be a matroid and let $m_1, m_2, \ldots$ be a non-repeating enumeration of M. Define a nested sequence of finite independent sets $\langle I_j \rangle_{j\in\mathbb{N}}$ as follows. Let $I_0 = \emptyset$. For $j > 0$, let $I_j = I_{j-1}$ if $I_{j-1} \cup \{m_j\} \in D$, and let $I_j = I_{j-1} \cup \{m_j\}$ otherwise. Define the basis B by $m_j \in B$ if and only if $m_j \in I_j$. To see that B is independent, suppose X is a finite dependent set. Let m_j be the element of largest index in X. If $X \setminus \{m_j\} \subset I_{j-1}$, then $m_j \notin I_j$, so $m_j \notin B$ and $X \not\subset B$. If $X \setminus \{m_j\} \not\subset I_{j-1}$ then $X \not\subset I_j$, so $X \not\subset B$. Summarizing, B has no finite dependent subsets, so B is independent. To see that B spans, fix $m_j \in M$. Either $m_j \in B$, or both $B \supset I_{j-1} \notin D$ and $I_{j-1} \cup \{m_j\} \in D$. In either case, m_j is in the span of B.

The preceding result can be viewed as a reverse mathematical reframing of the statement: *Every computably presented matroid has a computable basis.* This principle was stated by Crossley and Remmel [5, §5, Lemma 1], who describe it as common knowledge and implicit in the work of Metakides and Nerode [14]. The representations of the matroid by a computable dependence relationship or by a dependence algorithm for a Steinitz system with computable dependence are equivalent. The next theorem is a reverse mathematics analog of the fact that not every c.e. presented matroid is computably isomorphic to a computably presented matroid.

Theorem 7. (RCA_0) *The following are equivalent:*

(1) ACA_0.
(2) *Every e-matroid is isomorphic to a matroid. That is, if (M, e) is an e-matroid, then there is a matroid (N, D) and a bijection $h\colon M \to N$ such that for all finite sets $X \subset M$, there is an n such that $e(n) = X$ if and only if $\{h(x) : x \in X\} \in D$.*

Proof. To see that (1) implies (2), suppose (M, e) is an e-matroid. The range of e is arithmetically definable using e as a parameter, so ACA_0 proves the existence of the range as a set D. Then (M, D) is a matroid and the identity is the desired isomorphism.

To prove the converse, we capitalize on the construction from the proof of the reversal of Theorem 3. As in that proof, fix an injection f and construct the associated e-matroid (M, e). Apply (2) above to find a matroid (N, D) and an isomorphism $h\colon M \to N$. By the construction of (M, e), for each $k \in \mathbb{N}$, k is in the range of f if and only if $\{(k, 0), (k, 1)\}$ is in the range of e, which holds if and only if $\{h((k, 0)), h((k, 1))\} \in D$. Thus, the range of f is computable from D and h, completing the proof of the reversal.

In terms of Turing degrees, the previous theorem only shows that each c.e. presented matroid is computable from $\mathbf{0}'$. The next corollary shows that, if a c.e presented matroid is isomorphic to a computable matroid, the isomorphism may necessarily be noncomputable.

Corollary 8. *There is a c.e. presented matroid M, which is isomorphic to a computable matroid, such that if φ is any isomorphism between M and a computable matroid then $\mathbf{0}'$ is Turing computable from φ.*

Proof. Let f be any computable injection with a range that computes $\mathbf{0}'$. Use the construction of (M, e) from the proof of the reversal of Theorem 3. This is the desired c.e. presented matroid. The proof of Theorem 7 shows that any isomorphism between (M, e) and a computable matroid computes the range of f and consequently computes $\mathbf{0}'$. Since the range of f is both infinite and co-infinite, (M, e) is isomorphic to the computable matroid with ground set $\mathbb{N}$ and D consisting of all finite supersets of sets of the form $\{3k, 3k + 1\}$ where $k \in \mathbb{N}$.

A recent paper of Harrison-Trainor, Melnikov, and Montalbán [12] presents more results and applications for c.e. presented matroids. The pregeometries of their section 2 are Steinitz systems.

3 Weihrauch Reducibility

In Theorem 6, we used reverse mathematics to study the problem of finding a basis for an e-matroid. In this section, we study the same problem using Weihrauch reducibility. For additional information on Weihrauch reducibility, see Brattka and Gherardi [2] and Dorais, Dzhafarov, Hirst, Mileti, and Shafer [4]. The following simplified definition of Weihrauch problems will be sufficient for our purposes.

Definition 9. A *Weihrauch problem* is a subset of $\mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$, $\mathbb{N}^\mathbb{N} \times \mathbb{N}$, $\mathbb{N} \times \mathbb{N}^\mathbb{N}$, or $\mathbb{N} \times \mathbb{N}$. For a Weihrauch problem P, the "problem" is: given an "instance" $I \in \mathrm{dom}(P)$, produce a "solution" S with $(I, S) \in P$.

A Weihrauch problem P is *Weihrauch reducible* to a Weihrauch problem Q, written $P \leq_\mathrm{W} Q$, if there are computable functions or functionals Φ, Ψ such that, for all $S \in \mathrm{dom}(P)$, $\Phi(S) \in \mathrm{dom}(Q)$, and for all R such that $(\Phi(S), R) \in Q$, we have $(S, \Psi(R, S)) \in P$. If this can be done with a functional Ψ that does not depend on S, we say that P is *strongly Weihrauch reducible* to Q, written $P \leq_\mathrm{sW} Q$. The relations $\leq_\mathrm{W}$ and $\leq_\mathrm{sW}$ are reflexive and transitive, and thus they induce equivalence relations, which are denoted $\equiv_\mathrm{W}$ and $\equiv_\mathrm{sW}$, respectively.

The *parallelization* of a Weihrauch problem P is the problem

$$\widehat{P} = \{(f, g) : (f(n), g(n)) \in P \text{ for all } n \in \mathbb{N}\}$$

whose instances are sequences of instances of P and whose solutions are sequences of solutions corresponding to those instances.

Definition 10. We define the following Weihrauch principles. The first two are well known in the literature [1].

- $\mathsf{C}_\mathbb{N}$: closed choice for subsets of $\mathbb{N}$.

$$\mathsf{C}_\mathbb{N} = \{(f, n) : f \in \mathbb{N}^\mathbb{N}, n \notin \mathrm{range}(f)\}$$

- $\widehat{\mathsf{C}}_\mathbb{N}$: the parallelization of $\mathsf{C}_\mathbb{N}$.

$$\widehat{\mathsf{C}}_\mathbb{N} = \{(f, g) : ((f)_n, g(n)) \in \mathsf{C}_\mathbb{N} \text{ for all } n \in \mathbb{N}\ \}$$

- GAC: the graph antichain problem. For a countable graph G, an antichain is a set of vertices no two of which are connected by a path in G. Letting $\mathrm{Max}(G)$ be the set of maximal antichains of G, we have

$$\mathsf{GAC} = \{(G, A) : G \text{ is a countable graph}, A \in \mathrm{Max}(G)\}$$

- EMB: the e-matroid basis problem.

$$\mathsf{EMB} = \{(M, B) : M \text{ is a countable e-matroid}, B \text{ is a basis for } M\}$$

- VSB: the vector space basis problem, for countable vector spaces over countable fields, coded as in Definition III.4.1 of Simpson [17].

$$\mathsf{VSB} = \{(V, B) : V \text{ is a countable vector space and } B \text{ is a basis for } V\}$$

For each $n > 1$ in $\mathbb{N}$, we define the following restricted principles:

- GAC_n: the restriction to GAC to graphs with n connected components.
- EMB_n: the restriction of EMB to e-matroids with a basis of size n.
- VSB_n: the restriction of VSB to vector spaces with dimension n.

In previous work [11], we considered another well known Weihrauch problem, LPO.

$$\mathsf{LPO} = \{(f, n) : f \in [\mathbb{N}]^{<\mathbb{N}} \text{ and } f(n) = 0 \leftrightarrow (\exists m)[f(m) = 0]\}$$

The following lemma shows that the parallelization of LPO is strict Weihrauch equivalent to the parallelization of $\mathsf{C}_\mathbb{N}$. This equivalence is implicit in work of Brattka and Gherardi [2, 3], but the reductions obtained by combining their results are very indirect. The next lemma provides a pair of direct reductions.

Lemma 11. *$\widehat{\mathsf{C}_\mathbb{N}}$ is strongly Weihrauch equivalent to $\widehat{\mathsf{LPO}}$.*

Proof. First, suppose we are given an instance f of $\mathsf{C}_\mathbb{N}$. The function f enumerates the complement of some nonempty set. We form a sequence (p_n) of instances of LPO such that p_n has 0 in its range if and only if n is in the range of f. Then, given solutions to the instance $(p_n)_{n\in\mathbb{N}}$ of $\widehat{\mathsf{LPO}}$, we can search effectively for the least n such that p_n does not have 0 in its range, which will be the least n not in the range of f. Thus, by effective dovetailing, $\widehat{\mathsf{C}_\mathbb{N}}$ is strict Weihrauch reducible to $\widehat{\mathsf{LPO}}$.

For the converse, we first reduce LPO to $\mathsf{C}_\mathbb{N}$, as follows. Given an instance p of LPO, we enumerate in stages the complement of a nonempty set $A = A(p)$. If $p(0) > 0$, we enumerate 1 into the complement of A. Then if $p(1) > 0$ we enumerate 2 into the complement of A. We continue in this way. If we ever find that $p(n) = 0$ for some n, we enumerate 0 into the complement of A, after which we do not enumerate anything else into the complement, so we will have $A = \{n + 1, n + 2, \ldots\}$. On the other hand, if 0 is not in the range of p, then we continue enumerating elements into the complement of A, so that we will obtain $A = \{0\}$. Hence, if we view A as an instance of $\mathsf{C}_\mathbb{N}$, we can determine whether $(\exists m)[p(m) = 0]$ by looking at the value of any solution. Thus LPO is strict Weihrauch reducible to $\mathsf{C}_\mathbb{N}$, and so the parallelization of LPO is strict Weihrauch reducible to the parallelization of $\mathsf{C}_\mathbb{N}$.

Theorem 12. *The following strong Weihrauch equivalences hold:*

$$\mathsf{GAC} \equiv_{\mathrm{sW}} \mathsf{EMB} \equiv_{\mathrm{sW}} \mathsf{VSB} \equiv_{\mathrm{sW}} \widehat{\mathsf{C}_\mathbb{N}}.$$

Proof. Gura, Hirst, and Mummert [11] proved that $\mathsf{GAC} \equiv_{\mathrm{sW}} \widehat{\mathsf{C}_\mathbb{N}}$. Therefore, it is sufficient to establish the following four reductions:

$$\mathsf{GAC} \leq_{\mathrm{sW}} \mathsf{EMB} \leq_{\mathrm{sW}} \widehat{\mathsf{C}_\mathbb{N}}, \qquad \widehat{\mathsf{C}_\mathbb{N}} \leq_{\mathrm{sW}} \mathsf{VSB} \leq_{\mathrm{sW}} \mathsf{EMB}.$$

Three of these reductions are straightforward. First, to show that $\mathsf{VSB} \leq_{\mathrm{sW}} \mathsf{EMB}$, modify the construction used to prove (1) implies (2) in Theorem 5. Given a vector space with vector set V and zero vector 0_V, let $(V_i)_{i\in\mathbb{N}}$ be an enumeration of all the finite subsets of V in which each subset appears infinitely often. Define $e\colon \mathbb{N} \to [V]^{<\mathbb{N}}$ by setting $e(j) = V_j = \{v_0, \dots, v_k\}$ if there is a sequence of field elements $\{a_0, \dots, a_k\}$ with canonical code less than j such that $\sum_{i\leq k} a_i v_i = 0$, and set $e_j = \{0_V\}$ otherwise. Because e enumerates the finite dependent subsets of V, it is easy to verify that (V, e) is a matroid and any basis for the matroid is a basis for the vector space.

Second, to show that $\mathsf{GAC} \leq_{\mathrm{sW}} \mathsf{EMB}$, let $G = (V, E)$ be a graph. We wish to ensure that G has at least one edge. To this end, choose a vertex $v_1 \in V$ and add a new vertex v_0 to V and a new edge (v_0, v_1) to E, yielding a graph $G' = (V', E')$. Construct a matroid (V', e) as in the proof that (1) implies (2) in Theorem 5. (Note that in that argument, the bound on the number of components is used only to bound the rank of the matroid.) As in that proof, any basis for (V', e) is a maximal set of disconnected vertices of G'. If v_0 is in the basis, it can be replaced by v_1 to form a new basis which is a maximal set of disconnected vertices of G.

Third, to show that $\mathsf{EMB} \leq_{\mathrm{sW}} \widehat{\mathsf{C}}_{\mathbb{N}}$, let (M, e) be a countable e-matroid. Construct an enumeration e' of the finite sets in $\mathrm{Range}(e) \cup \{F \mid F \not\subseteq M\}$. Then $M' = (\mathbb{N}, e')$ is an e-matroid with domain $\mathbb{N}$ which has exactly the same independent sets and exactly the same bases as M. Fix an enumeration $(F_n)_{n\in\mathbb{N}}$ of $[\mathbb{N}]^{<\mathbb{N}}$. Define an instance $(f_n)_{n\in\mathbb{N}}$ of $\widehat{\mathsf{C}}_{\mathbb{N}}$ by

$$f_n(j) = \begin{cases} j+1 & \text{if } (\forall t < j)[e_{M'}(t) \neq F_n], \\ 0 & \text{otherwise.} \end{cases}$$

Note that F_n is independent if and only if $\mathrm{Range}(f_n) = \mathbb{N} \setminus \{0\}$. Also, if F_n is dependent, then $0 \in \mathrm{Range}(f_n)$. Thus, if g is a solution to this instance of $\widehat{\mathsf{C}}_{\mathbb{N}}$, then for every $n \in \mathbb{N}$, F_n is independent if and only if $g(n) = 0$. To simplify notation, if F is finite, we can let n be the smallest value such that $F_n = F$, and write $g(F) = g(n)$. We define the basis in stages. Let $B_0 = \{0\}$ if $g(\{0\}) = 0$ and $B_0 = \emptyset$ otherwise. If B_j is defined, let $B_{j+1} = B_j \cup \{j+1\}$ if $g(B_j \cup \{j+1\}) = 0$ and $B_{j+1} = B_j$ otherwise. Then $B = \{j \mid j \in B_j\}$ is a basis for M' and thus also for M.

It remains to show that $\widehat{\mathsf{C}}_{\mathbb{N}} \leq_{\mathrm{sW}} \mathsf{VSB}$. We adapt the construction presented by Simpson [17, Theorem III.4.3] showing that the principle "every countable vector space over $\mathbb{Q}$ has a basis" is equivalent to ACA_0 in the sense of reverse mathematics. The proof presented by Simpson shows, more specifically, that given an injective function $f\colon \mathbb{N} \to \mathbb{N}$ we may uniformly compute a $\mathbb{Q}$-vector space V_f such that the range of f is uniformly computable from any basis of V_f. This shows, in particular, that $\mathsf{C}_{\mathbb{N}} \leq_{\mathrm{sW}} \mathsf{VSB}$.

To complete the proof, it is sufficient for us to verify that $\widehat{\mathsf{VSB}} \leq_{\mathrm{sW}} \mathsf{VSB}$, because then we have $\widehat{\mathsf{C}}_{\mathbb{N}} \leq_{\mathrm{sW}} \widehat{\mathsf{VSB}} \leq_{\mathrm{sW}} \mathsf{VSB}$. The proof uses an effective direct sum construction. Given a sequence $(V_n)_{n\in\mathbb{N}}$ of countable vector spaces, we may assume without loss of generality that their underlying sets of vectors are pairwise disjoint. We may then form a countable vector space V whose elements are finite formal $\mathbb{Q}$-linear combinations of the form

$$a_1u_1 + \cdots + a_mu_m$$

where $a_i \in \mathbb{Q}$ and $u_i \in V_i$ for $i \leq m$. The scalar multiplication on V is the obvious one, and the vector addition is so that

$$\left(\sum_{i\leq m} a_iu_i\right) + \left(\sum_{i\leq n} b_iv_i\right) = \sum_{i\leq \max m,n} (a_iu_i + b_iv_i)$$

where each addition $a_iu_i + b_iv_i$ is carried out in V_i, and terms that did not appear in the left are treated vacuously as zero vectors. Then V is a countable vector space that is uniformly computable from the sequence $(V_n)_{n\in\mathbb{N}}$. Moreover, if B is a basis of V then $B \cap V_i$ is a basis of V_i for each $i \in \mathbb{N}$. To see this, note that on one hand $B \cap V_i$ must span V_i for each i, and on the other hand any dependency of the set $B \cap V_i$ within V_i would induce a dependency of B within V.

We next consider the restricted versions of two principles from Theorem 12.

Theorem 13. *For $n \geq 2$, the following equivalences hold:*

$$\mathsf{GAC}_n \equiv_{\mathrm{sW}} \mathsf{EMB}_n \equiv_{\mathrm{sW}} \mathsf{C}_{\mathbb{N}}.$$

Proof. Let $n \geq 2$ be fixed for the remainder of this proof. Gura, Hirst, and Mummert [11, Theorem 6.6] proved that $\mathsf{GAC}_n \equiv_{\mathrm{sW}} \mathsf{C}_{\mathbb{N}}$. Therefore, it is sufficient to establish the reductions $\mathsf{GAC}_n \leq_{\mathrm{sW}} \mathsf{EMB}_n$ and $\mathsf{EMB}_n \leq_{\mathrm{sW}} \mathsf{C}_{\mathbb{N}}$.

The reduction $\mathsf{GAC}_n \leq_{\mathrm{sW}} \mathsf{EMB}_n$ follows from the proof of Theorem 12, because the construction there produces an e-matroid whose dimension is the same as the number of components of the graph.

To show that $\mathsf{EMB}_n \leq_{\mathrm{sW}} \mathsf{C}_{\mathbb{N}}$, let (M, e) be an e-matroid with a basis of size n. As in the proof of Theorem 12, construct an enumeration e' of the finite sets in $\mathrm{Range}(e) \cup \{F \mid F \not\subseteq M\}$, so that $M' = (\mathbb{N}, e')$ is an e-matroid with domain $\mathbb{N}$ and with exactly the same bases as (M, e). Let $(F_i)_{i\in\mathbb{N}}$ be an enumeration of $[\mathbb{N}]^n$ in which each set appears infinitely often. Let $(G_i)_{i\in\mathbb{N}}$ be an enumeration of $[\mathbb{N}]^n$ in which each set appears exactly once, and such that $G_0 = F_0$.

We define an instance f of $\mathsf{C}_{\mathbb{N}}$ inductively along with an auxiliary sequence $(m_j)_{j\in\mathbb{N}}$. At stage 0, let $m_0 = 0$ and $f(0) = 1$. At stage $j + 1$, suppose m_j and $f(j)$ have been defined. If $e'(j) = F_{m_j}$, set $f(j + 1) = m_j$, let k be the smallest integer such that $(\forall t \leq j)[e'(t) \neq G_k]$ and set

$$m_{j+1} = (\mu s)[G_k = F_s \wedge (\forall t \leq j)(s > f(t))].$$

At stage $j + 1$, if $e'(j) \neq F_{m_j}$, set $f(j + 1) = \min(\mathbb{N} \setminus (\{f(t) \mid t \leq j\} \cup \{m_j\}))$ and let $m_{j+1} = m_j$.

The range of f will include all integers except one, namely some m such that $F_m = G_k$ for the least k for which G_k is a basis for M. Thus F_m will be a basis for M, as desired.

The next lemma, which is well known, extends the list of principles in Theorem 13 slightly, simplifying the proof of the next theorem.

Lemma 14. *Let $\mathsf{C}^u_{\mathbb{N}}$ denote the restriction of $\mathsf{C}_{\mathbb{N}}$ to functions for which the complement of the range consists of a unique natural number. Then $\mathsf{C}^u_{\mathbb{N}} \equiv_{\mathrm{sW}} \mathsf{C}_{\mathbb{N}}$.*

Proof. Because $\mathsf{C}^u_{\mathbb{N}}$ restricts $\mathsf{C}_{\mathbb{N}}$ to a smaller class of inputs, $\mathsf{C}^u_{\mathbb{N}} \leq_{\mathrm{sW}} \mathsf{C}_{\mathbb{N}}$. To prove $\mathsf{C}_{\mathbb{N}} \leq_{\mathrm{sW}} \mathsf{C}^u_{\mathbb{N}}$, suppose $f\colon \mathbb{N} \to \mathbb{N}$ is not surjective. In the following construction, we will conflate the pair (i, j) with its integer code via a fixed bijection between $\mathbb{N}$ and $\mathbb{N}\times\mathbb{N}$. Define $g\colon \mathbb{N} \to \mathbb{N}$ by the following moving marker construction. Let $m_0 = (0, 0)$ be the initial marker. Suppose $m_k = (m^0_k, m^1_k)$ has been defined. If $f(k) \neq m^0_k$, set $m_{k+1} = m_k$ and set $g(k)$ to the least code for a pair not included in $\{g(j) : j < k\}$. If $f(k) = m^0_k$, define a pair (y_0, t_0) so that

$$y_0 = (\mu\, y \leq k + 1)(\forall j \leq k)[f(j) \neq y],$$
$$t_0 = (\mu\, t)(\forall j < k)[g(j) \neq (y_0, t)],$$

and then set $m_{k+1} = (y_0, t_0)$ and $g(k) = m_k$.

Intuitively, if y is the smallest natural number not in the range of f, then at some stage in the construction the marker is set to (y, n) for some n, and does not move after that point. The code (y, n) is not in the range of g, but every other code and consequently every other natural number is in the range of g. Thus g satisfies the input requirements for $\mathsf{C}^u_{\mathbb{N}}$, and the process yields (y, n) as an output. The number y (retrievable by a projection function) is a solution to $\mathsf{C}_{\mathbb{N}}$ for input f.

The following theorem adds the fixed dimension vector space basis problem to the list of equivalent problems of Theorem 13

Theorem 15. *For $n \geq 2$, $\mathsf{VSB}_n \equiv_{\mathrm{sW}} \mathsf{C}_{\mathbb{N}}$.*

Proof. By Theorem 13, $\mathsf{EMB_n} \leq_{\mathrm{sW}} \mathsf{C}_{\mathbb{N}}$. In the proof of Theorem 12, the argument showing $\mathsf{VSB} \leq_{\mathrm{sW}} \mathsf{EMB}$ preserves the dimension of input vector space, and so shows $\mathsf{VSB}_n \leq_{\mathrm{sW}} \mathsf{EMB}_n$. By transitivity, $\mathsf{VSB}_n \leq_{\mathrm{sW}} \mathsf{C}_{\mathbb{N}}$.

Next we will show that $\mathsf{C}^u_{\mathbb{N}} \leq_{\mathrm{sW}} \mathsf{VSB}_2$. Our proof uses ideas and notation from the proof of Theorem III.4.2 of Simpson [17]. Fix $f\colon \mathbb{N} \to \mathbb{N}$ with the range of f including all of $\mathbb{N}$ except for one value. Let V_0 be the set of all formal sums $\sum_{i\in I} q_i x_i$ with I finite and $0 \neq q_i \in \mathbb{Q}$. We can identify formal sums with their sequence codes, yielding a well-ordering on V_0. Without loss of generality, we may assume that x_i is minimal in this ordering among all vectors with a nonzero coefficient on x_i. As in Simpson's proof, let $x'_m = x_{2f(m)} + (m+1)x_{2f(m)+1}$ and $X' = \{x'_m : m \in \mathbb{N}\}$. Let U_0 denote the subspace consisting of the linear span of X'. Note that $\sum_{i\in I} q_i x_i \in U_0$ if and only if

$$(\forall n)\left[(q_{2n} \neq 0 \to f(q_{2n+1}/q_{2n} - 1) = n) \wedge (q_{2n} = 0 \to q_{2n+1} = 0)\right],$$

so U_0 is computable from f. Let V_1 be V_0/U_0, where a vector v is in V_1 if and only if it is the element of $\{v - u : u \in U_0\}$ which is least in the well ordering on V_0. Only finitely many sequence codes are less than the code for v, so V_1 is computable.

By our choice of ordering and the construction of U_0, for every $i \in \mathbb{N}$, $x_{2i} \in V_1$. Let U_1 be the linear span of $\{x_{2i} : i \in \mathbb{N}\}$ in V_1. Then U_1 is a vector subspace of V_1 computable from f, and we may construct the quotient space $V = V_1/U_1$, using minimal representatives as before. For any $j \in \mathbb{N}$,

$$x_0 = x_{2f(j)+1} - (-\frac{1}{j+1}x_{2f(j)} - x_0) - \frac{1}{j+1}(x_{2f(j)} + (j+1)x_{2f(j)+1}).$$

The vector $-\frac{1}{j+1}x_{2f(j)} - x_0$ is in U_1 and $\frac{1}{j+1}(x_{2f(j)} + (j+1)x_{2f(j)+1})$ is in U_0, so x_0 and $x_{2f(j)+1}$ correspond to the same vector in V. The range of f excludes only one element, so the dimension of V is 2. Let $\{v_1, v_2\}$ be a basis for V. Let P be the finite collection of odd indices in the formal sums for v_1 and v_2, and let $R = \{m : 2m+1 \in P\}$. Exactly one m in R does not appear in the range of f. Thus, for exactly one m in R, $\{x_0, x_{2m+1}\}$ is linearly independent. Sequentially enumerate linear combinations of the form $q_0x_0 + q_1x_{2m+1}$, ejecting values from R corresponding to linear combinations that equal 0 in V. The last value left in R is the sole natural number that is not in the range of f. Thus $\mathsf{C}^u_{\mathbb{N}} \leq_{\text{sW}} \mathsf{VSB}_2$. By Lemma 14, $\mathsf{C}_{\mathbb{N}} \leq_{\text{sW}} \mathsf{VSB}_2$.

To prove $\mathsf{C}_{\mathbb{N}} \leq_{\text{sW}} \mathsf{VSB}_n$ for $n > 2$, add $n-1$ dummy vectors to the the basis of V_0 in the preceding argument.

The reduction of EMB_n to $\mathsf{C}_{\mathbb{N}}$ in the proof of Theorem 13 relies heavily on knowing the precise dimensions (in the appropriate sense) of the objects being studied. This suggests a variation in which we only place an upper bound on their dimensions. We begin with definitions of bounded versions of some Weihrauch principles.

Definition 16. We define the following Weihrauch principles. In the first three principles, the output can be viewed either as a canonical code for a finite set, or equivalently as a set together with the integer corresponding to its cardinality.

– $\mathsf{EMB}_{<\omega}$: the bounded e-matroid basis problem.

$$\begin{aligned}\mathsf{EMB}_{<\omega} = \{(n, M, B) :\ & n \in \mathbb{N},\ M \text{ is an e-matroid}, \operatorname{rank}(M) \leq n,\\ & \text{and } B \text{ is a basis for } M\}\end{aligned}$$

– $\mathsf{GAC}_{<\omega}$: The bounded graph antichain problem. Letting $\operatorname{Max}(G)$ be the set of maximal antichains of G, we have

$$\begin{aligned}\mathsf{GAC}_{<\omega} = \{(n, G, A)\ :\ & n \in \mathbb{N},\ G \text{ is a graph},\\ & \text{each set of } n \text{ vertices has a path connected pair},\\ & \text{and } A \in \operatorname{Max}(G)\}\end{aligned}$$

– $\mathsf{C}^{\subset}_{\max}$: Picking a maximal element (relative to the containment partial ordering) in the complement of an enumeration of finite nonempty sets whose range includes all sets larger than some bound.

$$\begin{aligned}\mathsf{C}^{\subset}_{\max} = \{(n, f, X) \ :\ & n \in \mathbb{N},\ f\colon \mathbb{N} \to [\mathbb{N}]^{<\mathbb{N}}_{\neq\emptyset},\ X \in [\mathbb{N}]^{<\mathbb{N}}, \\ & \text{range}(f) \text{ includes all sets of cardinality} \geq n, \\ & X \notin \text{range}(f),\ \text{and} \\ & (\forall Y \in [\mathbb{N}]^{<\mathbb{N}})[Y \supsetneq X \to Y \in \text{range}(f)]\}\end{aligned}$$

– $\mathsf{C}^{\#}_{\max}$: Picking an element of maximal cardinality in the complement of an enumeration of finite nonempty sets whose range includes all sets larger than some bound.

$$\begin{aligned}\mathsf{C}^{\#}_{\max} = \{(n, f, X) \ :\ & n \in \mathbb{N},\ f\colon \mathbb{N} \to [\mathbb{N}]^{<\mathbb{N}}_{\neq\emptyset},\ X \in [\mathbb{N}]^{<\mathbb{N}}, \\ & \text{range}(f) \text{ includes all sets of cardinality} \geq n, \\ & X \notin \text{range}(f),\ \text{and} \\ & (\forall Y \in [\mathbb{N}]^{<\mathbb{N}})[|Y| > |X| \to Y \in \text{range}(f)]\}\end{aligned}$$

Theorem 17. *The following strong Weihrauch equivalences hold:*

$$\mathsf{EMB}_{<\omega} \equiv_{\mathrm{sW}} \mathsf{GAC}_{<\omega} \equiv_{\mathrm{sW}} \mathsf{C}^{\subset}_{max} \equiv_{\mathrm{sW}} \mathsf{C}^{\#}_{max}.$$

Proof. We will prove each of the following reductions, proceeding from right to left:

$$\mathsf{C}^{\subset}_{\max} \leq_{\mathrm{sW}} \mathsf{C}^{\#}_{\max} \leq_{\mathrm{sW}} \mathsf{GAC}_{<\omega} \leq_{\mathrm{sW}} \mathsf{EMB}_{<\omega} \leq_{\mathrm{sW}} \mathsf{C}^{\subset}_{\max}.$$

To prove $\mathsf{EMB}_{<\omega} \leq_{\mathrm{sW}} \mathsf{C}^{\subset}_{\max}$, suppose (M, e) is an e-matroid such that every subset of M of size at least n is in the range of e. Let $\{X_j : j \in \mathbb{N}\}$ be an enumeration of $[\mathbb{N}]^{<\mathbb{N}}$ and let (i, j) denote the output of a bijective pairing function. Note that every $m \in \mathbb{N}$ has a unique representation of the form $2(i, j) + \varepsilon$ where $i, j \in \mathbb{N}$ and $\varepsilon \in \{0, 1\}$. Define $f\colon \mathbb{N} \to [\mathbb{N}]^{<\mathbb{N}}$ by

$$f(2(i,j)+\varepsilon) = \begin{cases} X_j & \text{if } \varepsilon = 0 \wedge i \notin M \wedge i \in X_j, \\ e((i,j)) & \text{otherwise.} \end{cases}$$

The range of f consists of the range of e plus all finite sets containing any elements of the complement of M. Apply $\mathsf{C}^{\subset}_{\max}$ to f to obtain a finite set $B \subseteq \mathbb{N}$ in the complement of the range of f that is maximal with respect to the containment partial ordering. The range of f includes all finite sets containing elements of the complement of M, so $B \subseteq M$. Furthermore, the range of f includes the range of e, so B is independent in (M, e). By maximality, B spans (M, e), so B is a basis for (M, e).

To prove $\mathsf{GAC}_{<\omega} \leq_{\mathrm{sW}} \mathsf{EMB}_{<\omega}$, emulate the reduction of GAC to EMB from the proof of Theorem 12. Because G has at most n connected components, every set of $n+1$ elements in the related matroid is dependent and so appears in the range of the enumeration.

To prove $\mathsf{C}^{\#}_{\max} \leq_{\mathrm{sW}} \mathsf{GAC}_{<\omega}$, suppose $f\colon \mathbb{N} \to [\mathbb{N}]^{<\mathbb{N}}_{\neq\emptyset}$ and the range of f includes all finite subsets of cardinality at least n. For each b with $1 \leq b < n$, let $g_b \colon \mathbb{N} \to [\mathbb{N}]^{<\mathbb{N}}$ be an enumeration of all subsets of $\mathbb{N}$ of cardinality exactly b. We will construct a graph G consisting of $n-1$ subgraphs each with one or two connected components. The vertices of G are $\{u^b_j, v^b_j : 1 \leq b < n \wedge j \in \mathbb{N}\}$. For each b with $1 \leq b < n$ and each $j \in \mathbb{N}$, add the edge (u^b_j, u^b_{j+1}) to the edge set E of G. For each b with $1 \leq b < n$, define $k^b_0 = 0$. Suppose k^b_j is defined. If $(\exists t \leq j)[f(t) = g_b(k^b_j)]$, add (v^b_j, u^b_j) to E and set $k^b_{j+1} = k^b_j + 1$. Otherwise, if $(\forall t \leq j)[f(t) \neq g_b(k^b_j)]$, add (v^b_j, v^b_{j+1}) to E and set $k^b_{j+1} = k^b_j$. Note that the graph G is uniformly computable from f.

Apply $\mathsf{GAC}_{<\omega}$ to find a maximal (finite) antichain D in G. Let b_0 be the largest number less than n such that D contains two vertices with superscript b_0. (If no such b_0 exists, $\emptyset$ is the largest set in the complement of the range of f.) At least one of these vertices must be $v^{b_0}_j$ for some j. Let j_0 be the largest value such that $v^{b_0}_{j_0} \in D$. Then $g_{b_0}(k^{b_0}_{j_0})$ is a set of maximal cardinality in the complement of the range of f.

To conclude the proof, we need only show that $\mathsf{C}^{\subset}_{\max} \leq_{\mathrm{sW}} \mathsf{C}^{\#}_{\max}$. Any f and n satisfying the hypotheses of $\mathsf{C}^{\subset}_{\max}$ also satisfy those of $\mathsf{C}^{\#}_{\max}$. Any subset in the complement of the range of f that is maximal in cardinality is also maximal with respect to the containment partial ordering, so the identity functionals witness the desired reduction.

We close our discussion of Weihrauch reducibility with the following theorem that adds $\mathsf{VSB}_{<\omega}$ to the equivalences of Theorem 17. Here $\mathsf{VSB}_{<\omega}$ is the problem which, given an input of $n \in \mathbb{N}$ and a vector space in which every set of n vectors is linearly dependent, returns a basis for the vector space.

Theorem 18. $\mathsf{VSB}_{<\omega} \equiv_{\mathrm{sW}} \mathsf{C}^{\subset}_{max}$.

Proof. By Theorem 17, $\mathsf{EMB}_{<\omega} \leq_{\mathrm{sW}} \mathsf{C}^{\subset}_{\max}$. The proof of $\mathsf{VSB} \leq_{\mathrm{sW}} \mathsf{EMB}$ in Theorem 12 preserves dimension, so that argument also witnesses that $\mathsf{VSB}_{<\omega} \leq_{\mathrm{sW}} \mathsf{EMB}_{<\omega}$. By transitivity, $\mathsf{VSB}_{<\omega} \leq_{\mathrm{sW}} \mathsf{C}^{\subset}_{\max}$.

Next we will adapt arguments from the proofs of Lemma 14 and Theorem 15 to show that $\mathsf{C}^{\#}_{\max} \leq_{\mathrm{sW}} \mathsf{VSB}_{<\omega}$. Fix n and $f \colon \mathbb{N} \to [\mathbb{N}]^{<\mathbb{N}}$ such that the range of f includes all sets of cardinality $\geq n$.For each $j < n$, let h_j be a bijective enumeration of $\{X : X \subset \mathbb{N} \wedge j \leq |X| < n\} \times \mathbb{N}$. Emulating the moving marker construction of Lemma 14, for each $j < n$ define g_j such that either the range of f includes all sets of cardinality k for $j \leq k < n$ and g_j is surjective or the unique value not in the range of g_j is some m such that $h_j(m) = (X_0, m_0)$ where $j \leq |X_0| < n$ and X_0 is in the complement of the range of f. (For use in the proof of Theorem 19, note that the convergence of the moving marker construction can be formally proved using the collection principle $\mathsf{B}\Sigma^0_1$, which is provable in RCA_0.)

Now we carry out an n-fold analog of the vector space construction in the proof of Theorem 15. The goal of the construction is to form a space V as a direct sum of subspaces W_i, $i < n$, such that if j_0 is the largest size of a set omitted from the range of f, then the dimension of W_i is 1 for $i > j_0$ and the dimension is 2 for $i \leq j_0$. This will ensure that the dimension of V is finite, and moreover will allow us to compute the value of j_0 if we know the exact dimension of V.

Let V_0 be the set of formal sums $\sum_{(i,k)\in I_k\times[0,n)} q_{(i,k)}x_{(i,k)}$ where, for each $k < n$, each I_k is finite and $0 \neq q_{(i,k)} \in \mathbb{Q}$. Identifying $h_j(m) = (X_0, m_0)$ with the integer code for the pair, for each $k < n$ and each m, let $x'_{(m,k)} = x_{(2h_k(m),k)} + (m+1)x_{(2h_k(m)+1,k)}$ and $X' = \{x'_{(m,k)} : m \in \mathbb{N} \wedge k < n\}$. Let U_0 be the linear span of X' and set $V_1 = V_0/U_0$. Let U_1 be the linear span in V_1 of $\{x_{(2m,k)} : m \in \mathbb{N} \wedge k < n\}$ and let $V = V_1/U_1$. Then V has a two dimensional subspace corresponding to each $j < n$ such that the range of f omits a set of cardinality k with $j \leq k < n$, and a one dimensional subspace corresponding to each $j < n$ such that f maps $\mathbb{N}$ onto the sets of cardinality k with $j \leq k < n$. Thus the dimension of V is between n and $2n$, and any set of $2n + 1$ vectors is linearly dependent.

(For use in the proof of Theorem 19, note that the claim that any collection of $2n+1$ vectors of V is linearly dependent can be proved in RCA_0 as follows. Fix a set of $2n + 1$ nonzero vectors, $S = \{u_0, \ldots, u_{2n}\}$. Let B_0 be the finite set of those vectors of the form $x_{(i,k)}$ that appear in the sums representing each u_i. Because S is finite, Σ^0_1 induction suffices to find the smallest subset of B_0 that spans S. Call this set B_1. By minimality, B_1 is linearly independent. For each $k < n$, the function g_k omits at most one value, so B_1 contains at most two vectors of the form $x_{(i,k)}$. Thus $|B_1| \leq 2n$. Let $B_1 = \{v_0, \ldots, v_j\}$ where $j < 2n$. The vectors of B_1 span S, so $u_0 = \sum_{i\leq j} c_i v_i$, with some $c_{i_0} \neq 0$. Solving for v_{i_0}, we see that v_{i_0} is in the span of $B_2 = \{u_0\} \cup B_1 \setminus \{v_{i_0}\}$. Thus B_2 is a linearly independent set spanning S. Iterating this process by primitive recursion, we eventually find a $u_m \in S$ which is a linear combination of $\{u_i : i < m\}$. Thus S is linearly dependent.)

Apply $\mathsf{VSB}_{<\omega}$ to find a basis B for V. Then $k = |B| - n - 1$ is the cardinality of the largest set omitted from the range of f. Let P be the finite collection of odd numbers m such that (m, k) appears as an index in a formal sum for an element of B. Let $R = \{m \mid 2m + 1 \in P\}$. Exactly one m in R does not appear in the range of g_k. Thus for exactly one m in R, $\{x_{(0,k)}, x_{(2m+1,k)}\}$ is linearly independent. Sequentially examine linear combinations of the form $q_0x_{(0,k)} + q_1x_{(2m+1,k)}$, ejecting values from R corresponding to vectors equal to 0 in V, until only one is left. Viewed as a code for a pair, the first component of that value is a code for a set of maximum cardinality in the complement of the range of f. Thus $\mathsf{C}^{\#}_{\max} \leq_{\mathrm{sW}} \mathsf{VSB}_{<\omega}$. By Theorem 17, $\mathsf{C}^{\subseteq}_{\max} \leq_{\mathrm{sW}} \mathsf{VSB}_{<\omega}$.

4 Reducibility and Reverse Mathematics

We conclude by extracting a final reverse mathematics result from the proofs of Theorem 17 and Theorem 18, extending the list of equivalences in Theorem 5.

Theorem 19. (RCA_0) *The following are equivalent:*

(1) $\mathsf{I}\Sigma^0_2$*, the induction scheme for* Σ^0_2 *formulas with set parameters.*

(2) *Let V be a countable vector space such that for some n, every subset of n vectors is linearly dependent. Then V has a basis.*
(3) *A formalized version of $\mathsf{C}^{\#}_{max}$. Suppose $f\colon \mathbb{N} \to [\mathbb{N}]^{<\mathbb{N}}_{\neq\emptyset}$ and there is an n such that for all $X \in [\mathbb{N}]^{<\mathbb{N}}$, $[|X| \geq n \to \exists t(f(t) = X)]$. Then there is an $X \in [\mathbb{N}]^{<\mathbb{N}}$ such that $(\forall t)[f(t) \neq X$ and for all $Y \in [\mathbb{N}]^{<\mathbb{N}}$, $[|X| < |Y| \to \exists t(f(t) = Y)]$.*
(4) *A formalized version of $\mathsf{C}^{\subset}_{max}$. Suppose $f\colon \mathbb{N} \to [\mathbb{N}]^{<\mathbb{N}}_{\neq\emptyset}$ and there is an n such that for all $X \in [\mathbb{N}]^{<\mathbb{N}}$, $(|X| \geq n \to \exists t(f(t) = X))$. Then there is an $X \in [\mathbb{N}]^{<\mathbb{N}}$ such that $(\forall t)[f(t) \neq X]$ and for all $Y \in [\mathbb{N}]^{<\mathbb{N}}$, $[X \subsetneq Y \to \exists t(f(t) = Y)]$.*

Proof. First, we use (1) to prove (2). If V is a vector space and every set of n vectors is linearly dependent, the construction from the proof of Theorem 12 can be formalized to yield an e-matroid of rank no more than n. By Theorem 5, $\mathsf{I}\Sigma^0_2$ implies that this matroid has a basis which is also a basis of V.

To show that (2) implies (3), formalize the argument form the proof of Theorem 18 showing that $\mathsf{C}^{\#}_{\mathrm{max}} \leq_{\mathrm{sW}} \mathsf{VSB}_{<\omega}$, using the parenthetical comments. As noted, the convergence of the moving marker construction is provable in RCA_0, as is the claim that every set of $2n+1$ vectors is linearly dependent.

The proof that (3) implies (4) follows immediately from the fact that any set that is maximal in the sense of (3) is automatically maximal in the sense of (4).

The proof that $\mathsf{EMB}_{<\omega} \leq_{\mathrm{sW}} \mathsf{C}^{\subset}_{\mathrm{max}}$ from Theorem 17 can be formalized in RCA_0 to show that (4) implies item (1) of Theorem 5. By Theorem 5, this implies $\mathsf{I}\Sigma^0_2$, completing the proof.

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