REVERSE MATHEMATICS AND ORDINAL SUPREMA

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Abstract. Suprema of well orderings appear in two forms. The first is the supremum of a bounded subset of elements of a well ordering. The second is the supremum of a collection of well orderings. This article explores the reverse mathematics of each of these forms and their applications to ordinal arithmetic.

Using suitable coding, the language of second order arithmetic can express a wide variety of statements pertaining to countable well orderings and the functions on and between them. Once formalized, the logical strength of these statements can be analyzed using the tools of reverse mathematics. A survey of results of this sort appears in this volume [4].

In this article, we will use lower-case greek letters to denote countable well orderings. We will denote the least element of any well ordering by 0, and use 1 to denote the next smallest element. Often, we also use $k \in \mathbb{N}$ to denote a k-element well ordering, and ω for the natural numbers with their usual ordering. The operator $+$ is treated as a concatenation operation on orderings. As a consequence of these notational conventions, the theorems presented here bear a strong resemblance to set theoretic ordinal arithmetic. However, here α represents some countable well ordering, not an ordinal satisfying the traditional set theoretic definition.

Suppose that α and β are countable well orderings. If there is an order preserving bijection between α and an initial segment of β, we say that α is *strongly less than or equal to* β and write $\alpha \leq_s \beta$. We write $\alpha \equiv_s \beta$ if $\alpha \leq_s \beta$ and $\beta \leq_s \alpha$. Similarly, $\alpha \leq_w \beta$ denotes *weak comparability*, asserting the existence of an order preserving injection of α into β. One of the earliest results of reverse mathematics is Friedman's theorem showing the equivalence of ATR_0 to the assertion that any two countable well orderings are strongly comparable [2]. Proofs can be found in [3] and [9].

In the next section, we explore suprema within well orderings and their connection to various ways of defining well orderings. Then we examine comparability and suprema of collections of well orderings. The last section contains an analysis of an exercise from Sierpiński and its recent generalization.

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Meeting

§1. Suprema within well orderings. We say that a linearly ordered set α is *well ordered* if every nonempty subset of α has a least element. In practice it is often convenient to use the equivalent definition that well orderings are devoid of infinite descending sequences. The equivalence of these definitions is easily provable in RCA_0 using the following technical lemma.

LEMMA 1. (RCA_0) *Let X be a linear ordering. For any sequence S of elements of X, there is a set Y of elements in S such that S and Y have the same upper bounds. This also holds for lower bounds.*

PROOF. Working in RCA_0, suppose that X is a linear ordering and that $S = \langle s_i \mid i \in \mathbb{N}\rangle$ is a sequence of elements of X. If S has a maximum element, then the set containing only that element satisfies the requirements for Y. If S has no largest element, then by Δ^0_1 comprehension, we may construct a subsequence of S as follows. Let $x_0 = s_0$. Given $\{x_0, \dots, x_k\}$ let x_{k+1} be the element x of least index in S such that $x > \max\{x_0, \dots, x_k, s_0, \dots, s_k\}$ with respect to ordering on X and with respect to the usual ordering on the integers. Since S has no maximum, such an x always exists. Because $s_k < x_{k+1}$ for all k, S and $\langle x_i \mid i \in \mathbb{N}\rangle$ have the same upper bounds in X. Because $\langle x_i \mid i \in \mathbb{N}\rangle$ is an increasing sequence of integers, by Δ^0_1 comprehension the set $\{x_i \mid i \in \mathbb{N}\}$ exists. The argument for lower bounds is identical except that the sequence $\langle x_i \mid i \in \mathbb{N}\rangle$ is constructed so that it is descending with respect to the ordering on X. ⊣

We can now prove the equivalence of two familiar definitions of a well ordering.

THEOREM 2. (RCA_0) *Let X be a linear ordering. The following are equivalent:*

1. *X contains no infinite descending sequences.*
2. *Every nonempty subset of X has a least element.*

PROOF. To show that clause 1 implies clause 2, suppose that X contains a nonempty subset with no least element. Using Δ^0_1 comprehension we can construct an infinite descending sequence from such a set. To prove the converse, suppose X contains an infinite descending sequence. By Lemma 1, X contains a set with exactly the same lower bounds. This set has no least element. ⊣

In [1], Cantor uses a different definition of well ordering. He defines a well ordered set as a set with a least element such that every subset with a strict upper bound has a least strict upper bound. He then shows that this definition is equivalent to those in the preceding theorem. We will show that ACA_0 is required to carry out Cantor's proof. As a first step, we examine the existence of general least upper bounds.

THEOREM 3. (RCA_0) *The following are equivalent:*

1. ACA_0.

2. *If α is a well ordering, then every subset of α with an upper bound has a least upper bound.*

PROOF. First assume ACA_0 and suppose that α is well ordered. Let S be a subset of α with an upper bound. By arithmetical comprehension, the set of all upper bounds for S exists. Since α is well ordered, this set of upper bounds has a least element.

To prove the reversal, assume that ACA_0 fails. By the proof of Theorem 2 of [6], there is a well ordering β such that $\omega \leq_s \beta$, $\beta \not\leq_s \omega$ and $\omega + 1 \not\leq_s \beta$. Suppose that f is an order preserving map of ω onto an initial segment of β. Since $\beta \not\leq_s \omega$, the range of f is bounded above in β. Because the domain of f is ω, we may view its range as a sequence of elements of β. By Lemma 1, there is a subset Y of β such that Y has exactly the same upper bounds as the range of f. If Y had a least upper bound, we could extend f to witness that $\omega + 1 \leq_s \beta$. Thus Y is a subset of β with an upper bound, but no least upper bound. ⊣

The relationship between upper bounds and strict upper bounds is shown by the following theorem.

THEOREM 4. (RCA_0) *Suppose X is a linear ordering. The following are equivalent:*

1. *Every subset of X with an upper bound has a least upper bound, and every nonempty final segment $\{x \in X \mid x > a\}$ has a least element.*
2. *Every subset of X with a strict upper bound has a least strict upper bound.*

PROOF. Suppose X is a linear ordering and clause 1 holds. Let Y be a subset of X with a strict upper bound. By clause 1, Y has a least upper bound; call it b. If $b \notin Y$, then b is the desired strict upper bound. Otherwise, the least element of the final segment $\{x \in X \mid x > b\}$ is the least strict upper bound.

Now suppose that clause 2 holds. Let Y be a subset of X with an upper bound b. If Y has a maximum element, then that element is the least upper bound of Y. If Y has no maximum element, then b is a strict upper bound, and the least strict upper bound provided by clause 2 is the least upper bound of Y. Finally, the least element of the final segment $\{x \in X \mid x > a\}$ is the least strict upper bound of $\{a\}$. ⊣

In light of the preceding two theorems, ACA_0 is necessary and sufficient to show that well ordered sets satisfy Cantor's definition. (This is summarized below in Corollary 6.) The next theorem shows that ACA_0 is necessary and sufficient to prove the converse.

THEOREM 5. (RCA_0) *The following are equivalent:*

1. ACA_0.
2. *Let X be a linear ordering. If every subset of X with a strict upper bound has a least strict upper bound, then X is well ordered.*

Proof. First, assume ACA_0 and let X be a linear ordering satisfying the hypothesis of clause 2. Let Y be a nonempty subset of X. By arithmetical comprehension, the set $Z = \{x \in X \mid \forall y \in Y (x < y)\}$ exists. Each element of Y is a strict upper bound of Z. By our hypothesis, Z has a least strict upper bound; call it b. If $b \notin Y$, then for every $y \in Y$ we have $b < y$, and so $b \in Z$, contradicting the construction of b. Thus, b is an element of Y, and can easily be shown to be the least element of Y. Since Y was an arbitrary nonempty subset, X is well ordered.

To prove the reversal, assume that ACA_0 fails. By the proof of Theorem 3.1 of [3], there is a well ordering β such that $\omega \leq_w \beta$, $\beta \not\leq_w \omega$, and $\omega + 1 \not\leq_w \beta$. Invert the order on β; call the resulting linear ordering B. Since β is well ordered, each subset of B has a largest element which is its least upper bound. Given any final segment of B with no least element, it is possible to construct an embedding of $\omega + 1$ into β. Since this would contradict $\omega + 1 \not\leq_w \beta$, every final segment of B has a least element. By Theorem 4, every subset of B with a strict upper bound has a least strict upper bound. However, since $\omega \leq_w \beta$, B contains an infinite descending sequence. By an application of Theorem 2, B is not well ordered. Thus clause 2 fails, as desired. ⊣

The preceding analysis of Cantor's definition of well ordering is summarized in the following corollary.

Corollary 6. (RCA_0) *The following are equivalent:*

1. ACA_0.
2. *Suppose that X is a linear ordering. If every nonempty subset of X has a least element, then every subset of X with a strict upper bound has a least strict upper bound.*
3. *Suppose that X is a linear ordering. If every subset of X with a strict upper bound has a least strict upper bound, then every nonempty subset of X has a least element.*

Proof. To see that clause 1 implies clause 2, assume ACA_0 and let X be a linear ordering in which every nonempty subset has a least element. By Theorem 3, every subset of X with an upper bound has a least upper bound. Every nonempty final segment is a set, and therefore has a least element. By Theorem 4, every subset of X with a strict upper bound has a least strict upper bound.

To prove that clause 2 implies clause ACA_0, assume RCA_0 and clause 2. Let α be a well ordering, and let Y be a subset of α with an upper bound b. If Y has a maximal element, then that element is the least upper bound of Y. If Y has no maximal element, then b is a strict upper bound. By clause 2, Y has a least strict upper bound, which must be the least upper bound of Y. Thus, every subset of α with an upper bound has a least upper bound. By Theorem 3, ACA_0 holds.

Theorem 5 shows the equivalence of clauses 1 and 3. ⊣

§2. Suprema of collections of well orderings. In the preceding section we considered suprema of collections of elements within a well ordering. Now we turn our attention to suprema of collections of well orderings. Clearly, finding a supremum requires some comparability, so it is not surprising that ATR_0 is needed to prove the existence of suprema in this setting. The following theorem holds for both weak and strong comparability, so the s and w subscripts on $\leq$ are suppressed in the statement. The summation notation in the proof is used to indicate the result of concatenating a well ordered sequence of well orderings. RCA_0 suffices to prove that such constructs are well ordered. (See section 3 of [4]).

THEOREM 7. (RCA_0) *The following are equivalent:*

1. ATR_0
2. *Suppose* $\langle \alpha_x \mid x \in \beta \rangle$ *is a well ordered sequence of well orderings. Then* $\sup\langle \alpha_x \mid x \in \beta \rangle$ *exists. That is, there is a well ordering* α *unique up to order isomorphism satisfying*
   - $\forall x \in \beta(\alpha_x \leq \alpha)$, *and*
   - $\forall \gamma(\gamma + 1 \leq \alpha \to \exists x \in \beta(\alpha_x \not\leq \gamma))$.

PROOF. First, we will prove that ATR_0 implies clause 2 for strong comparability. Assume that ATR_0 holds and fix $\langle \alpha_x \mid x \in \beta \rangle$. By ATR_0, for each $x \in \beta$,

$$\alpha_x \leq_s \left(\sum_{y \in \beta} \alpha_y \right) + 1.$$

Furthermore, using ATR_0 we can construct a set A so that for every $x \in \beta$, A contains the least upper bound of the initial segment of $\sum_{y \in \beta} \alpha_y$ which is order isomorphic to α_x. The initial segment of $\sum_{y \in \beta} \alpha_y$ lying below the least strict upper bound of A is the desired α. Uniqueness follows easily from strong comparability of well orderings. The weak comparability version of clause 2 follows from the strong comparability version.

To prove the reversal in the strong comparability case, select two well orderings α_1 and α_2. By clause 2, there is an α such that $\alpha_1 \leq_s \alpha$ and $\alpha_2 \leq_s \alpha$. Since α_1 and α_2 are both order isomorphic to initial segments of α, they must be comparable. By Friedman's theorem on strong comparability of well orderings [2], this implies ATR_0.

For the weak comparability case of the reversal, suppose that ATR_0 fails. We will show that clause 2 fails also. We say that a well ordering α is *indecomposable* if for each final segment $\beta = \{x \in \alpha \mid x > b\}$ we have $\alpha \leq_w \beta$. As an example, ω is the least indecomposable well ordering. By Theorem 4.4 of [5], we can find indecomposable well orderings α_1 and α_2 such that $\alpha_1 \not\leq_w \alpha_2$ and $\alpha_2 \not\leq_w \alpha_1$. Using the fact that α_2 is indecomposable and $\alpha_2 \not\leq_w \alpha_1$, we can show that $\alpha_1 \leq_w \alpha_1 + \alpha_2$, $\alpha_2 \leq_w \alpha_1 + \alpha_2$, and $\gamma + 1 \leq_w \alpha_1 + \alpha_2$ implies $\alpha_2 \not\leq_w \gamma$. Thus $\alpha_1 + \alpha_2$ satisfies the two conditions given in clause 2. Similarly, $\alpha_2 + \alpha_1$ satisfies these conditions. If we assume that clause 2 is true, then by the uniqueness of the supremum, $\alpha_1 + \alpha_2 \equiv_w \alpha_2 + \alpha_1$. Because of the indecomposability of α_1, this implies $\alpha_1 \leq_w \alpha_2$, yielding a contradiction. Thus clause 2 fails, as desired. ⊣

It would be interesting to know if the weak comparability version of clause 2 of Theorem 7 with the uniqueness condition omitted implies ATR_0.

§3. The γ-lemma and Sierpiński's problem. An interesting statement of ordinal arithmetic called the γ-lemma is proved in [7]. This lemma, which is stated in terms of ordinal suprema, considerably extends a result that appears as Exercise 6 in Section 8 of Chapter XIV of Sierpiński's *Cardinal and Ordinal Numbers* [8]. (The page number varies with the printing.) In this section, we will solve Sierpiński's exercise in RCA_0, and show that the γ-lemma is equivalent to ATR_0, providing a proof theoretic verification of the added strength of the γ-lemma.

THEOREM 8. (Sierpiński's exercise) *For each positive natural number n,* RCA_0 *proves*

$$\sum_{\alpha<\omega^n} \alpha \equiv_s \omega^{2n-1}.$$

PROOF. For $n = 1$, the result says that $0+1+2+\cdots \equiv_s \omega$, where the sum on the left represents the result of concatenating a sequence of finite orderings of increasing size. Formally, we want to show that ω is order isomorphic to $S = \{\langle i,j\rangle \mid i \geq 1 \wedge j < i\}$ ordered by the relation

$$\langle i_1, j_1\rangle < \langle i_2, j_2\rangle \;\leftrightarrow\; \big(i_1 < i_2 \vee (i_1 = i_2 \wedge j_1 < j_2)\big).$$

Using RCA_0, it is possible to define a function $h : S \to \omega$ by $h(j,k) = j(j-1)/2 + k$ and prove that it is an order preserving bijection between S and ω.

For $n > 1$, the goal is to construct an order preserving bijection between $\sum_{\alpha<\omega^n} \alpha$ and ω^{2n-1}. This is most easily done by composing a sequence of bijections between successive terms in the following equality.

$$\begin{aligned}\sum_{\alpha<\omega^n} \alpha &= \sum_{j<\omega} \sum_{\alpha<\omega^{n-1}} (\omega^{n-1}\cdot j + \alpha)\\ &= \sum_{j<\omega} \sum_{\alpha<\omega^{n-1}} \omega^{n-1}\cdot j\\ &= \sum_{j<\omega} (\omega^{n-1})^2\\ &= \omega^{2n-2}\cdot\omega = \omega^{2n-1}.\end{aligned}$$

The construction of these bijections depends heavily on the indecomposability of ω^{n-1}, which is provable in RCA_0 [5]. ⊣

For each $n \in \omega$, RCA_0 proves that ω^n is well ordered. However, RCA_0 cannot prove that for all $n \in \omega$, ω^n is well ordered. Adding induction for Σ^0_2 formulas to RCA_0 yields an axiom system that is strong enough to prove the universally quantified statement. In this (slightly) stronger system, it is also possible to prove the universally quantified version of Sierpiński's exercise.

The authors of [7] note that Sierpiński's exercise follows easily from the following statement:

> **γ-lemma**: Suppose γ is an ordinal and f is a non-decreasing function from ω^γ into the ordinals. Then
>
> $$\sum_{\alpha<\omega^\gamma} f(\alpha) = \sup\{f(\alpha)\cdot\omega^\gamma \mid \alpha<\omega^\gamma\}.$$

If we formalize the γ-lemma without reference to the uniqueness of the supremum, and merely assert that the summation acts as a supremum for the set $\{f(\alpha)\cdot\omega^\gamma \mid \alpha<\omega^\gamma\}$, the resulting statement is equivalent to ATR_0.

THEOREM 9. (RCA_0) *The following are equivalent:*

1. ATR_0.
2. *(γ-lemma) Suppose that ω^γ is well ordered and f assigns a well ordered set to each $\alpha<\omega^\gamma$ in such a way that if $\alpha<\beta<\omega^\gamma$ then $f(\beta)+1 \not\leq f(\alpha)$. Then*
   - *For all $\alpha<\omega^\gamma$, $f(\alpha)\cdot\omega^\gamma \leq \sum_{\alpha<\omega^\gamma} f(\alpha)$, and*
   - *If $\delta < \sum_{\alpha<\omega^\gamma} f(\alpha)$, then there is an $\alpha<\omega^\gamma$ such that $f(\alpha)\cdot\omega^\gamma \not\leq \delta$.*

PROOF. We will prove that clause 1 implies the strong comparability version of clause 2. (The weak comparability version follows immediately from the strong version.) Assume ATR_0 and suppose that γ and f satisfy the hypotheses of clause 2. Applying Theorem 7, let $\lambda = \sup\langle f(\alpha)\cdot\omega^\gamma \mid \alpha<\omega^\gamma\rangle$. By the definition of the supremum, $f(\alpha)\cdot\omega^\gamma \leq_s \lambda$ for all $\alpha<\omega^\gamma$, and if $\delta<\lambda$ then for some $\alpha<\omega^\gamma$, $f(\alpha)\cdot\omega^\gamma \not\leq_s \delta$. Consequently, we can complete the proof by showing that $\lambda \equiv \sum_{\alpha<\omega^\gamma} f(\alpha)$. This can be done by the following straightforward imitation of the proof of the γ-lemma from [7]. First we note that ATR_0 suffices to prove

$$\begin{aligned}\sum_{\alpha<\omega^\gamma} f(\alpha) &\equiv_s \sup\{\sum_{\beta<\alpha} f(\beta) \mid \alpha<\omega^\gamma\}\\ &\leq_s \sup\{\sum_{\beta<\alpha} f(\alpha) \mid \alpha<\omega^\gamma\}\\ &\equiv_s \sup\{f(\alpha)\cdot\alpha \mid \alpha<\omega^\gamma\}\\ &\leq_s \sup\{f(\alpha)\cdot\omega^\gamma \mid \alpha<\omega^\gamma\} = \lambda.\end{aligned}$$

Also, for any $\beta < \omega^\gamma$, ATR_0 can prove

$$\begin{aligned} f(\beta)\cdot\omega^\gamma &\leq_s \sum_{\alpha<\beta} f(\alpha) + f(\beta)\cdot\omega^\gamma \\ &\equiv_s \sum_{\alpha<\beta} f(\alpha) + \sum_{\alpha<\omega^\gamma} f(\beta) \\ &\leq_s \sum_{\alpha<\beta} f(\beta) + \sum_{\alpha<\omega^\gamma} f(\beta+\alpha) \\ &\equiv_s \sum_{\alpha<\beta+\omega^\gamma} f(\alpha) \\ &\equiv_s \sum_{\alpha<\omega^\gamma} f(\alpha). \end{aligned}$$

The last step uses the fact that ATR_0 can prove that ω^γ is indecomposable. (See [5].) The second string of inequalities shows that

$$\sum_{\alpha<\omega^\gamma} f(\alpha) \geq \sup\{f(\alpha)\cdot\omega^\gamma \mid \alpha<\omega^\gamma\} = \lambda$$

which combined with the first string of inequalities yields $\lambda \equiv \sum_{\alpha<\omega^\gamma} f(\alpha)$, as desired.

To prove the reversal, assume RCA_0 and $\neg\mathsf{ATR}_0$. Let α_0 and α_1 be indecomposable well orderings such that $\alpha_0 \not\leq_w \alpha_1$ and $\alpha_1 \not\leq_w \alpha_0$. Let $\gamma = 1$, and note that RCA_0 proves that $\omega = \omega^1$ is well ordered. Define f by setting $f(0) = \alpha_0$ and $f(n) = \alpha_1$ for $n \geq 1$. If $m < n < \omega$, then the relation $f(n)+1 \not\leq f(m)$ is satisfied for all n and either form of comparability. However, by the weak incomparability of α_0 and α_1, for either version of comparability we have

$$f(0)\cdot\omega = \alpha_0\cdot\omega \not\leq \alpha_0 + \alpha_1\cdot\omega = \sum_{\alpha<\omega} f(\alpha).$$

Thus the first item in clause 2 fails, completing the proof. ⊣

The preceding theorem shows that the combinatorial content of the γ-lemma is significantly stronger that the statement from Sierpiński's exercise. Using the result of the preceding section, ATR_0 suffices to prove the existence of the supremum in the informal version of the γ-lemma, so a straightforward formalization of that version is also equivalent to ATR_0.

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