THE POLARIZED RAMSEY'S THEOREM

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Abstract. We study the effective and proof-theoretic content of the polarized Ramsey's theorem, a variant of Ramsey's theorem obtained by relaxing the definition of homogeneous set. Our investigation yields a new characterization of Ramsey's theorem in all exponents, and produces several combinatorial principles which, modulo bounding for Σ^0_2 formulas, lie (possibly not strictly) between Ramsey's theorem for pairs and the stable Ramsey's theorem for pairs.

1. Introduction

In this article, we will investigate several variants of Ramsey's theorem from the point of view of computability theory and reverse mathematics. The standard version of Ramsey's theorem, stated following Definition 1.1 below, has been the subject of many such investigations; the interested reader may wish to consult Mileti [12, Sections 1 and 2] for a partial survey of previous work. For background material in computability theory and reverse mathematics, see, respectively, Soare [15] and Simpson [14].

To begin, we recall some standard terminology.

Definition 1.1. Fix an infinite set X and $n, k \geq 1$.

(1) $[X]^n$ denotes the set $\{Y \subset X : |Y| = n\}$.
(2) A *k-coloring on X of exponent n* is a function $f : [X]^n \to k$, where k is identified with the set $\{0, \ldots, k-1\}$ of its predecessors in ω. When $X = \omega$ and $n = 2$, we refer to f as a coloring of *pairs*.
(3) A set $H \subseteq X$ is *homogeneous* for f if H is infinite and $f \upharpoonright [H]^n$ is constant.
(4) For $n = 2$, f is *stable* if $\lim_s f(\{x, s\})$ exists for every $x \in X$.

Ramsey's theorem (RT). *For every $n, k \geq 1$, every $f : [\omega]^n \to k$ has a homogeneous set.*

The statement of Ramsey's theorem can be easily formalized in the language of second order arithmetic. The next definition lists several well-known related principles.

Definition 1.2. Fix $n, k \geq 1$. The following definitions are made in second order arithmetic.

(1) RT^n_k is the statement that every $f : [\mathbb{N}]^n \to k$ has a homogeneous set.
(2) SRT^2_k is the statement that every stable $f : [\mathbb{N}]^2 \to k$ has a homogeneous set.

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(3) RT^n is the statement that for all $j \in \mathbb{N}$, RT^n_j.
(4) SRT^2 is the statement that for all $j \in \mathbb{N}$, SRT^2_j.

Call a tuple $\langle x_1, \ldots, x_n \rangle \in \omega^n$ (by which we always we mean one with $x_i \neq x_j$ whenever $i \neq j$) *increasing* if $x_1 < \cdots < x_n$. When dealing with a coloring $f : [\omega]^n \to k$, It is convenient to write $f(x_1, \ldots, x_n)$ in place of $f(\{x_1, \ldots, x_n\})$ whenever $\langle x_1, \ldots, x_n \rangle$ is an increasing tuple. Indeed, one could easily regard f as being defined on increasing tuples only, as doing so would not affect which sets are homogeneous for it. Our investigation, however, will turn out to be more sensitive to this distinction as a consequence of involving the following two variations on the notion of homogeneous set.

Definition 1.3. Fix $n, k \geq 1$ and $f : [\omega]^n \to k$.

(1) A *p-homogeneous* set for f is a sequence $\langle H_1, \ldots, H_n \rangle$ of infinite sets such that for some $c < 2$, called the *color* of this sequence, $f(\{x_1, \ldots, x_n\}) = c$ for every tuple $\langle x_1, \ldots, x_n \rangle \in H_1 \times \cdots \times H_n$.
(2) If (1) holds just for increasing tuples, we call $\langle H_1, \ldots, H_n \rangle$ an *increasing* p-homogeneous set.

We will study the logical strength of the following "polarized" versions of Ramsey's theorem. The name comes from a similar combinatorial principle first studied by Erdős and Rado in [3].

Polarized theorem (PT). *For every $n, k \geq 1$, every $f : [\omega]^n \to k$ has a p-homogeneous set.*

Increasing polarized theorem (IPT). *For every $n, k \geq 1$, every $f : [\omega]^n \to k$ has an increasing p-homogeneous set.*

Remark 1.4. Every homogeneous set computes a p-homogeneous one. For if $f : [\omega]^n \to k$ is a coloring and H is homogeneous for f, then clearly $\langle H_1, \ldots, H_n \rangle$, where $H_1 = \cdots = H_n = H$, is p-homogeneous for f.

One encounters a striking dissimilarity between Ramsey's theorem for pairs and the polarized theorem for pairs by considering the following example. Define $f : [\omega]^2 \to 2$ by letting $f(\{x, y\})$ equal 0 if x and y have like parity, and 1 otherwise. Let H_1 consist of the even numbers, H_2 of the odds, and notice that $\langle H_1, H_2 \rangle$ is p-homogeneous for f with color 1. Yet f obviously admits no homogeneous set with this color.

Analyzing the computational complexity of p-homogeneous sets and increasing p-homogeneous sets will be the focus of Section 2 below. We will show there that the major complexity bounds established by Jockusch [9] for homogeneous sets of computable colorings hold also for p-homogeneous and increasing p-homogeneous sets.

By analogy with Definition 1.2 we define the following weaker forms of PT and IPT, whose proof-theoretic strength we will study in Sections 3 and 4.

Definition 1.5. Fix $n, k \geq 1$. The following definitions are made in second order arithmetic.

(1) PT^n_k is the statement that every $f : [\mathbb{N}]^n \to k$ has a p-homogeneous set.
(2) IPT^n_k is the statement that every $f : [\mathbb{N}]^n \to k$ has an increasing p-homogeneous set.

(3) SPT^2_k is the statement that every stable $f : [\mathbb{N}]^2 \to k$ has a p-homogeneous set.
(4) SIPT^2_k is the statement that every stable $f : [\mathbb{N}]^2 \to k$ has an increasing p-homogeneous set.
(5) PT^n is the statement that for all $j \in \mathbb{N}$, PT^n_j.
(6) IPT^n is the statement that for all $j \in \mathbb{N}$, IPT^n_j.
(7) SPT^2 is the statement that for all $j \in \mathbb{N}$, SPT^2_j.
(8) SIPT^2 is the statement that for all $j \in \mathbb{N}$, SIPT^2_j.

Note that we could easily define what it means for a tuple to be *decreasing* rather than increasing, and call a tuple *monotone* if it is either increasing or decreasing. A *monotone* p-homogeneous set for $f : [\omega]^n \to k$ could then be defined in the obvious way. But since every 2-tuple is monotone, it follows that for all $k \in \omega$, PT^2_k coincides with the statement that every $f : [\mathbb{N}]^2 \to k$ has a monotone p-homogeneous set. We will see in Theorem 4.1 that, modulo provability in RCA_0, the same is true in higher exponents, so we omit the latter statement from the preceding definition.

Our results about the above principles occupy Sections 3 and 4 below. Chief among these is that PT^n_k is equivalent to RT^n_k over RCA_0 for all n and k, which, at least for $n = 2$, may be surprising given the example following Remark 1.4. In Section 5, we conclude with some questions and problems.

2. Computability Theory

It is clear from Remark 1.4 that many theorems about the complexity of homogeneous sets carry over trivially to p-homogeneous sets. We list some of these.

Theorem 2.1. *Fix $n, k \geq 2$.*
(1) *Every computable $f : [\omega]^n \to k$ has a Π^0_n p-homogeneous set.*
(2) *Every computable $f : [\omega]^n \to k$ has a p-homogeneous set whose jump is computable in $0^{(n)}$.*
(3) *Every computable stable $f : [\omega]^2 \to k$ has a Δ^0_2 p-homogeneous set.*
(4) *For any sequence of noncomputable sets, every computable $f : [\omega]^2 \to k$ admits a p-homogeneous set not computing any member of this sequence.*
(5) *Every computable $f : [\omega]^2 \to k$ admits a low$_2$ p-homogeneous set.*

Proof. This follows at once from Remark 1.4 and (1) Theorem 5.5 of Jockusch [9]; (2) Theorem 5.6 of [9]; (3) Lemma 3.10 of Cholak, Jockusch, and Slaman [1]; (4) Seetapun's theorem (cf. [13, Theorem 2.1]); (5) Theorem 3.1 of [1]. □

The next proposition shows that for stable colorings, the converse to Remark 1.4 holds. Thus, up to degree, homogeneous sets for stable colorings are the same as p-homogeneous sets, which in turn are the same as increasing p-homogeneous sets.

Proposition 2.2. *For every $k \geq 2$ and every stable $f : [\omega]^2 \to k$, every increasing p-homogeneous set for f computes a homogeneous set.*

Proof. Let $f : [\omega]^2 \to k$ be a stable coloring and assume that $\langle H_1, H_2 \rangle$ is an increasing p-homogeneous set for f, say with color $c < k$. We construct a homogeneous set for f computably from $\langle H_1, H_2 \rangle$ as follows. Let $a_0 = \min(H_1)$, and suppose that $a_0 < \cdots < a_n$ have been defined for some $n \geq 0$. Since H_2 is infinite and, for each $i \leq n$, $f(a_i, x) = c$ for every $x > a_i$ in H_2, it follows by stability of f that $\lim_s f(a_i, s) = c$. Hence, there exists $x > a_n$ in H_1 such that $f(a_i, x) = c$

for all $i \leq n$, and we let a_{n+1} be the least such x. By induction we get a set $\{a_0, a_1, \ldots\} \subseteq H_1$ such that $f(a_m, a_n) = c$ whenever $m < n$, and so this set is homogeneous for f. □

It follows, for example, that there exists a computable stable coloring with no low increasing p-homogeneous set (cf. Downey, Hirschfeldt, Lempp, and Solomon [2]).

The degrees of homogeneous and p-homogeneous sets for general colorings are harder to compare. One point of similarity is the next theorem, which is the analog of a well-known result, Lemma 5.9 of [9], due to Jockusch. In that lemma, Jockusch realized that non-trivial information could be coded into homogeneous sets by coloring 3-tuples instead of 2-tuples. Our theorem basically says that this information is not lost by passing to increasing p-homogeneous sets. The proof closely follows that of the original, and we include it here only to highlight a minor necessary adjustment.

Theorem 2.3. *For every $n \geq 1$, there exists a computable $f : [\omega]^{n+1} \to 2$ every increasing p-homogeneous set of which computes $0^{(n-1)}$.*

We first need a lemma.

Lemma 2.4. *Fix $n \geq 1$. If $f : [\omega]^n \to 2$ is Δ^0_2, there exists a computable $g : [\omega]^{n+1} \to 2$ such that whenever $\langle H_1, \ldots, H_{n+1}\rangle$ is an increasing p-homogeneous set for g, $\langle H_1, \ldots, H_n\rangle$ is an increasing p-homogeneous set for f.*

Proof. We let g be any computable function such that $f(\overline{x}) = \lim_s g(\overline{x}, s)$ for all increasing tuples $\overline{x} \in \omega^n$, which exists since f is Δ^0_2. Let $\langle H_1, \ldots, H_{n+1}\rangle$ be any increasing p-homogeneous set for g, say with color $c < 2$, and suppose $\langle x_1, \ldots, x_n\rangle$ is an increasing tuple in $H_1 \times \cdots \times H_n$. We have that $g(x_1, \ldots, x_n, x) = c$ for all sufficiently large $x \in H_{n+1}$, which implies, since H_{n+1} is infinite and $\lim_s g(\overline{x}, s)$ exists for all $\overline{x}$, that $\lim_s g(x_1, \ldots, x_n, s) = c$. It follows that $f(x_1, \ldots, x_n) = c$ by definition of g, and hence that $\langle H_1, \ldots, H_n\rangle$ is an increasing p-homogeneous set for f, as claimed. □

Proof of Theorem 2.3. Jockusch and McLaughlin [10, Theorem 4.13] proved the existence of an increasing 0-majorreducible function of degree $\mathbf{0}^{(n-1)}$, i.e. an increasing function from ω to ω Turing equivalent to $0^{(n-1)}$ and computable from every function which dominates it. Let g be such a function and let $f_0 : [\omega]^2 \to 2$ be the Δ^0_n coloring defined by

$$f_0(x, y) = \begin{cases} 0 & \text{if } y > g(x) \\ 1 & \text{otherwise} \end{cases}$$

for all numbers $x < y$.

Let $\langle H_1, H_2\rangle$ be an increasing p-homogeneous set for f_0, noting that it must have color 0 since for any $x \in H_1$ there is certainly an element $y \in H_2$ with $y > g(x)$ and hence $f_0(x, y) = 0$. Define a sequence $\langle a_0, b_0, a_1, b_1, \ldots\rangle$ inductively by letting a_n be the least element of H_1 greater than a_i and b_i for all $i < n$, and letting b_n be the least element of H_2 greater than a_n. The function $m(n) = b_n$ is then computable from $\langle H_1, H_2\rangle$ and, since g is increasing and $f_0(a_n, b_n) = 0$ and $a_n \geq n$ for all n, we have $m(n) > g(a_n) \geq g(n)$. It follows that m dominates g and hence that $0^{(n-1)} \equiv_T g \leq_T m \leq_T \langle H_1, H_2\rangle$.

Thus f_0 can serve as the base case of a finite induction with which we complete the proof. Assume that for some $m \geq 0$ there exists a Δ^0_{n-m} coloring

$f_m : [\omega]^{m+2} \to 2$ every increasing p-homogeneous set of which computes $0^{(n-1)}$. Applying Lemma 2.4, relativized to $0^{(n-m-2)}$, to f_m yields a Δ^0_{n-m-1} coloring $f_{m+1} : [\omega]^{m+3} \to 2$ every increasing p-homogeneous set of which computes an increasing p-homogeneous set for f_m and so computes $0^{(n-1)}$. By induction, $f = f_{n-1}$ is the desired computable 2-coloring of $[\omega]^{n+1}$. □

Part (1) of Theorem 2.1 gives an upper bound on the complexity of p-homogeneous sets with respect to the arithmetical hierarchy. A lower bound on increasing p-homogeneous sets can be obtained by virtually the same argument as that used to prove Theorem 5.1 of [9], thus establishing that the arithmetical bounds on homogeneous, p-homogeneous, and increasing p-homogeneous sets agree. For the sake of completeness, we reproduce the proof here.

Lemma 2.5. *There exists a computable $f : [\omega]^2 \to 2$ with no Δ^0_2 increasing p-homogeneous set.*

Proof. Let $\{p_e : e \in \omega\}$ be a listing of all the primitive recursive functions in two variables. For each e, let D_e consist of the least $2e + 2$ elements x such that $\lim_s p_e(x, s) = 1$ if there exist such elements, and let $D_e = \varnothing$ if there do not. For $e, s > 0$, let $D_{e,s}$ consist of the least $2e + 2$ elements $x < s$ such that $p_e(x, s) = 1$ provided such elements exist, and let $D_{e,s} = \varnothing$ otherwise. We construct $f : [\omega]^2 \to 2$ by stages as follows.

Stage $s \in \mathbb{N}$: We define f on $[0, s) \times \{s\}$. To this end, we consider substages $e \leq s$ such that at each substage $e < s$ we define f on at most two new elements.

Substage $e < s$: If $D_{e,s} = \varnothing$, go to substage $e + 1$. Otherwise, $|D_{e,s}| = 2e + 2$ by definition. At each previous substage we added at most two elements of $\omega \times \{s\}$ to the domain of f, so $\text{dom}(f)$ contains no more than $2e$ elements from $[0, s) \times \{s\}$. Thus there are elements $x < y$ in $D_{e,s}$ such that f_s is not defined on $\langle x, s\rangle$ and $\langle y, s\rangle$, and we pick the least such elements and let $f_s(x, s) = 0$ and $f_s(y, s) = 1$.

Substage s: For all $x < s$ with $\langle x, s\rangle$ not yet in $\text{dom}(f)$, define $f(x, s) = 0$.

It is clear that f is a computable coloring $[\omega]^2 \to 2$. Seeking a contradiction, suppose that $\langle H_1, H_2\rangle$ is a Δ^0_2 increasing p-homogeneous set for f. Then in particular $H_1(x) = \lim_s p_e(x, s)$ for some e. Since H_1 is infinite, it follows that D_e is nonempty and hence that $D_{e,s} = D_e$ for all large enough s. Pick such an s to also be large enough that $D_{e,t} = D_e$ for all $t \geq s$. By construction, for each $t \geq s$ there exist $x, y \in D_{e,t} = D_e \subset H_1$ with $f(x, t) \neq f(y, t)$, so choosing $t > s, \max(D_e)$ in H_2 now contradicts increasing p-homogeneity of $\langle H_1, H_2\rangle$. □

Theorem 2.6. *For every $n \geq 2$, there exists a computable $f : [\omega]^n \to 2$ with no Δ^0_n increasing p-homogeneous set.*

Proof. We proceed by induction on n, the base case $n = 2$ being Lemma 2.5. Since that proof obviously relativizes, we may assume the present result and all its relativizations for some $n \geq 2$, and prove it in relativized form for $n + 1$. Fixing an arbitrary set X and relativizing the induction hypothesis to X' yields a $\Delta^{0,X}_2$ coloring $f : [\omega]^n \to 2$ with no $\Delta^{0,X'}_n = \Delta^{0,X}_{n+1}$ increasing p-homogeneous set. By Lemma 2.4 relative to X, there exists an X-computable coloring $g : [\omega]^{n+1} \to 2$ such that whenever $\langle H_1, \ldots, H_{n+1}\rangle$ is increasing p-homogeneous for g, $\langle H_1, \ldots, H_n\rangle$ is increasing p-homogeneous for f. So in particular, any $\Delta^{0,X}_{n+1}$ increasing p-homogeneous set for g would yield such a set for f, which cannot be. This completes the proof. □

Corollary 2.7. *For every $n \geq 2$, there exists a computable $f : [\omega]^n \to 2$ with no Σ^0_n increasing p-homogeneous set.*

Proof. Fix $f : [\omega]^n \to 2$ and assume $\langle H_1, H_2 \rangle$ is a Σ^0_n increasing p-homogeneous set for it. Then H_1 and H_2 are both infinite and Σ^0_n and so contain infinite Δ^0_n subsets, $\widetilde{H}_1$ and $\widetilde{H}_2$. The pair $\langle \widetilde{H}_1, \widetilde{H}_2 \rangle$ is consequently a Δ^0_n increasing p-homogeneous set for f. So by taking f as in the statement Theorem 2.6, the corollary now follows. □

3. Reverse Mathematics of Exponent $n = 2$

We begin this section by summarizing the most obvious relationships between the principles stated in Definitions 1.2 and 1.5. The proofs are immediate from Remark 1.4 and the relevant definitions.

Proposition 3.1 (RCA_0)**.** *For every $n, k \geq 2$,*

(1) $\mathsf{RT}^n_k \to \mathsf{PT}^n_k \to \mathsf{IPT}^n_k$.
(2) $\mathsf{PT}^2_k \to \mathsf{SPT}^2_k$.
(3) $\mathsf{IPT}^2_k \to \mathsf{SIPT}^2_k$.
(4) $\mathsf{SRT}^2_k \to \mathsf{SPT}^2_k \to \mathsf{SIPT}^2_k$.

To better gauge the proof-theoretic strength of our principles, we briefly review the statements of some which have already been studied in the literature.

Definition 3.2. The following definitions are made in second order arithmetic.

(1) CAC is the statement that for every partial ordering $\preceq$ on $\mathbb{N}$ there exists an infinite set X which, under $\preceq$, is either a chain, i.e. for any $x, y \in X$ either $x \preceq y$ or $y \preceq x$, or else an antichain, i.e. for any two $x, y \in X$, $x \not\preceq_P y$ and $y \not\preceq_P x$.
(2) ADS is the statement that for every linear order $\preceq$ on $\mathbb{N}$ there exists an infinite set $X \subseteq L$ which, under $\preceq$, is either an ascending sequence, i.e. $x \preceq y$ if and only if $x \leq y$ for all $x, y \in X$, or else a descending sequence, i.e. $x \preceq y$ if and only if $x \geq y$ for all $x, y \in X$.
(3) SADS is the statement that every linear order of type $\omega + \omega^*$ has a subset of type ω or ω^*.
(4) D^2_k is the statement that for every stable $f : [\mathbb{N}]^2 \to k$ there exists an infinite set X and $c < k$ such that $\lim_s f(x, s) = c$ for all $x \in X$.
(5) D^2 is the statement that for all $j \geq 1$, D^2_j.
(6) DNR is the statement that for every set X there exists a function f such that for all $e \in \mathbb{N}$, $f(e) \neq \Phi^X_e(e)$.
(7) COH is the statement that for every sequence $\langle X_i : i \in \mathbb{N} \rangle$ of sets, there exists an infinite set X such that for every $i \in \mathbb{N}$, either $X \subseteq^* X_i$ or $X \subseteq^* \overline{X_i}$.
(8) $\mathsf{B}\Gamma$, where Γ is a set of formulas in two free number variables, is the collection of all statements

$$\forall n[(\forall x < n)(\exists y)\varphi(x, y) \to (\exists m)(\forall x < n)(\exists y < m)\varphi(x, y)]$$

for $\varphi \in \Gamma$.

The principles CAC and ADS were introduced by Hirschfeldt and Shore [7, p. 178], DNR by Giusto and Simpson [4, p. 1478], and COH, D^2_k, and D^2 by Cholak, Jockusch, and Slaman [1, Statements 7.7, 7.8, and 7.9, resp.]. We will mention established relations among these principles as we need them.

3.1. **The stable case.** A relatively weak yet ubiquitous bounding principle in investigations such as ours is $\mathsf{B}\Sigma^0_2$. It is known to be equivalent to $\mathsf{B}\Pi^0_1$ (cf. [5, Lemma 2.10]), and by Theorem 6.4 of Hirst [8] also to RT^1. We can exploit these equivalences to to establish a close connection between the stable versions of our principles and the stable version of Ramsey's theorem for pairs.

Theorem 3.3 (RCA_0)**.** *For every $k \geq 2$, the following are equivalent:*

(1) $\mathsf{D}^2_k + \mathsf{B}\Sigma^0_2$.
(2) SRT^2_k.
(3) $\mathsf{SPT}^2_k + \mathsf{B}\Sigma^0_2$.
(4) $\mathsf{SIPT}^2_k + \mathsf{B}\Sigma^0_2$.

Moreover, $\mathsf{D}^2 \leftrightarrow \mathsf{SRT}^2 \leftrightarrow \mathsf{SPT}^2 \leftrightarrow \mathsf{SIPT}^2$.

Proof. In Lemma 7.10 of [1], Cholak, Jockusch, and Slaman claimed that D^2_2 implies SRT^2_2 and conversely, but their proof of the first implication appears to require the use of $\mathsf{B}\Sigma^0_2$, as pointed out in Section 2.2 of [6]. Since SRT^2_2 implies $\mathsf{B}\Sigma^0_2$ by [1, Lemma 10.6], this consequently establishes the equivalence of (1) and (2) for $k = 2$, and the argument can be easily generalized to arbitrary k. The equivalence of D^2 with SRT^2 is by Lemma 7.12 of [1], and the implications from (2) to (3) and from (3) to (4) are immediate by Proposition 3.1(4) above.

For the implication from (4) to (1), we formalize our proof above of Proposition 2.2 above. The only thing nontrivial there was the inductive step in the construction of the sequence $\langle a_n : n \in \mathbb{N}\rangle$. Note that if $a_0 < \cdots < a_n$ are defined then for every $i \leq n$ there exists $s \in \mathbb{N}$ such that for all $x > s$, $f(a_i, x) = c$. By $\mathsf{B}\Pi^0_1$ there exists $t \in \mathbb{N}$ such that for every $i \leq n$, there exists $s \leq t$ such that for all $x > s$, $f(a_i, x) = c$. Hence, for every $i \leq n$ and any $x > t$, $f(a_i, x) = c$, and we can let a_{n+1} be the least element of H_1 that is greater than t. Thus the construction of the sequence can be carried out using $\mathsf{B}\Sigma^0_2$, as desired.

To prove the "moreover" part of the theorem it now clearly suffices to show that SIPT^2 implies $\mathsf{B}\Sigma^0_2$, or equivalently RT^1. To this end, let $k \in \mathbb{N}$ and $g : \mathbb{N} \to k$ be given and define $f : [\mathbb{N}]^2 \to k$ by $f(x, y) = g(x)$ for all $x < y$ in $\mathbb{N}$. Since $f(x, y) = f(x, z)$ for all $x < y < z$ in $\mathbb{N}$, f is trivially stable. Thus we can use SIPT^2 to fix an increasing p-homogeneous set $\langle H_1, H_2\rangle$ for f, and H_1 is clearly homogeneous for g. □

Since $\mathsf{B}\Sigma^0_2$ holds in every ω-model of RCA_0, it follows that every ω-model of $\mathsf{RCA}_0 + \mathsf{SIPT}^2_k$ is a model also of SRT^2_k. Hence, any proof via ω-models that a given principle does not imply SRT^2_k shows also that it does not imply SIPT^2_k (the same observation applies of course also to D^2_k).

Proposition 3.4. *With the possible exception of D^2_2, none of the principles in Definition 3.2 imply SIPT^2_2 over RCA_0.*

Proof. Hirschfeldt and Shore [7, Corollaries 3.11 and 3.12] exhibited an ω-model of $\mathsf{RCA}_0 + \mathsf{CAC}$ in which SRT^2_2 fails. On the other hand, they showed that ADS, SADS, and COH all follow from CAC (cf. [7], Propositions 3.1, 2.7 and 4.5, resp.), so none of these can imply SIPT^2_2. As for DNR and $\mathsf{B}\Sigma^0_2$, both hold in any ω-model of WKL_0 consisting entirely of low sets, as $\mathsf{WKL}_0 \vdash \mathsf{DNR}$ by Lemma 6.18 of [4]. But by the main result of [2], SRT^2_2 does not hold in any such model. □

It follows from Theorem 3.3 that every consequence of SRT^2_k is a consequence of $\mathsf{SIPT}^2_k + \mathsf{B}\Sigma^0_2$. Among the strongest such consequences which have been studied are D^2_k, DNR, and SADS, and we now show that $\mathsf{B}\Sigma^0_2$ is not needed to obtain these from SIPT^2_k. See Lemma 7.10 of [1], Theorem 2.4 of [6], and Proposition 3.3 of [7] for proofs that SRT^2_k implies, respectively, D^2_k, DNR, and SADS over RCA_0.

Proposition 3.5 (RCA_0)**.** *For every $k \geq 2$, $\mathsf{SIPT}^2_k \to \mathsf{D}^2_k \to \mathsf{DNR}$.*

Proof. Let a stable $f : [\mathbb{N}]^2 \to k$ be given, and by SIPT^2_k choose an increasing p-homogeneous set $\langle H_1, H_2\rangle$ for f, say with color $c < k$. Fix $x \in H_1$. Since $f(x,y) = c$ for every $y \in H_2$, and since H_2 is infinite and $\lim_s f(x,s)$ exists, it must be that $\lim_s f(x,s) = c$. This establishes the first implication. The second follows by Theorem 2.4 of Hirschfeldt, Jockusch, Kjos-Hanssen, Lempp, and Slaman [6] that $\mathsf{SRT}^2_2 \to \mathsf{DNR}$, as their proof, on closer inspection, actually shows that $\mathsf{D}^2_2 \to \mathsf{DNR}$. □

Proposition 3.6 (RCA_0)**.** *For every $k \geq 2$, $\mathsf{SIPT}^2_k \to \mathsf{SADS}$.*

Proof. Let $\preceq$ be a linear ordering on $\mathbb{N}$ of type $\omega + \omega^*$ and define $f : [\mathbb{N}]^2 \to 2$ by

$$f(x,y) = \begin{cases} 0 & \text{if } x \preceq y \\ 1 & \text{otherwise} \end{cases} .$$

for all elements $x < y$ of $\mathbb{N}$. Fix any $x \in \mathbb{N}$, noticing that x either has finitely many predecessors under $\preceq$, or else finitely many successors. In the former case, $x \preceq y$ for cofinitely many $y \in \mathbb{N}$ and so $\lim_s f(x,s) = 0$, and in the latter case, symmetrically, $\lim_s f(x,s) = 1$. Thus f is stable, and we can apply SIPT^2_k to obtain an increasing p-homogeneous set $\langle H_1, H_2\rangle$. For every $x \in H_1$ we clearly have $\lim_s f(x,s) = c$ where c is the color of $\langle H_1, H_2\rangle$. In other words, either every $x \in H_1$ has finitely many predecessors under $\preceq$, in which case $(H_1, \preceq)$ is of type ω, or every $x \in H_2$ has finitely many successors, and then $(H_1, \preceq)$ is of type ω^*. □

3.2. **The general case.** The following theorem is an immediate consequence of Theorem 3.3.

Proposition 3.7 (RCA_0)**.** *For every $k \geq 2$, $\mathsf{IPT}^2_k + \mathsf{B}\Sigma^0_2 \to \mathsf{SRT}^2_k$ and $\mathsf{IPT}^2 \to \mathsf{SRT}^2$.*

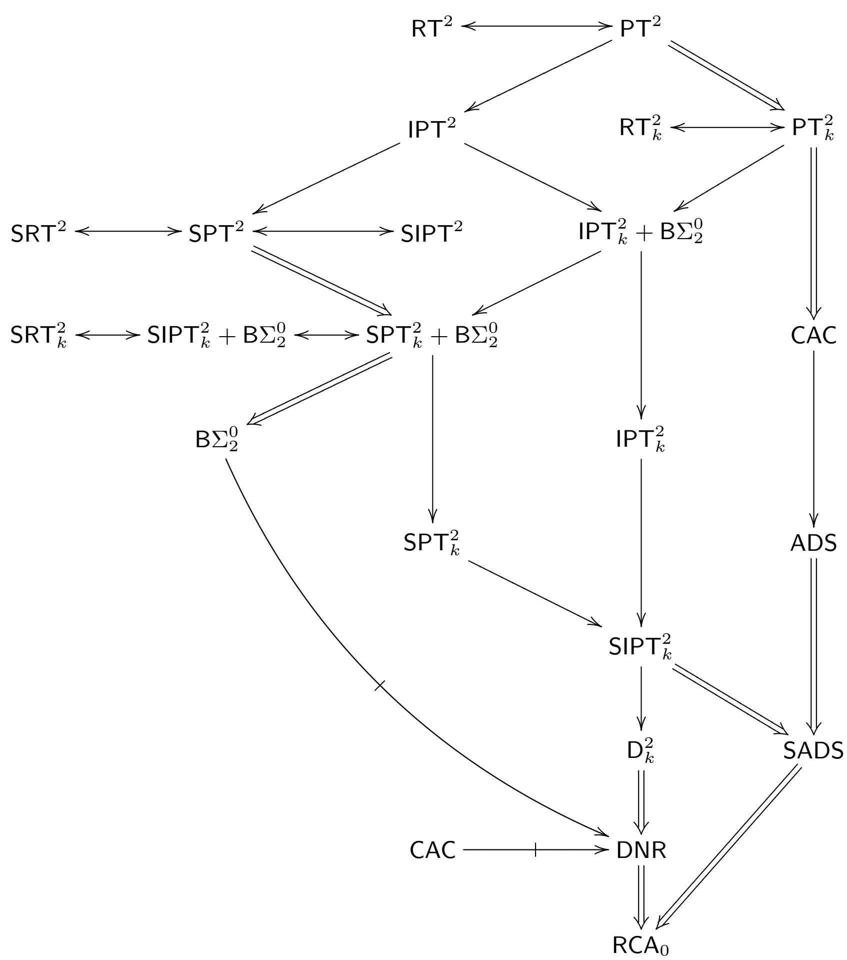
We do not know if it is possible to strengthen or sharpen the above result, such as by showing that IPT^2_k does or does not imply $\mathsf{B}\Sigma^0_2$. But we can do considerably better in the case of PT^2_k. This suggests that the difference between parts (1) and (2) of Definition 1.3 is more significant than it may have seemed.

Theorem 3.8. *For every $k \geq 2$, $\mathsf{RCA}_0 \vdash \mathsf{RT}^2_k \leftrightarrow \mathsf{PT}^2_k$, and $\mathsf{RCA}_0 \vdash \mathsf{RT}^2 \leftrightarrow \mathsf{PT}^2$.*

Proof. Fix $k \geq 2$. That RT^2_k implies PT^2_k over RCA_0 is by (1) of Proposition 3.1 above, so we need only establish the converse. Cholak, Jockusch, and Slaman [1, Lemmas 7.11 and 7.13] showed that over RCA_0, $\mathsf{RT}^2_2 \leftrightarrow \mathsf{SRT}^2_2 + \mathsf{COH}$ and $\mathsf{RT}^2 \leftrightarrow \mathsf{SRT}^2 + \mathsf{COH}$ (cf. Section A.1 of Mileti [11] for a correction to the proof of the former), and the first result easily generalizes to k colors. In view of Proposition 3.7 above it thus suffices to show that PT^2_k implies $\mathsf{B}\Sigma^0_2$ and COH over RCA_0. Both of these principles follow from ADS by Propositions 2.10 and 4.5 of Hirschfeldt and Shore [7], so it is enough to show that $\mathsf{RCA}_0 \vdash \mathsf{PT}^2_k \to \mathsf{ADS}$.

Arguing in RCA_0, let $\preceq$ be a linear ordering on $\mathbb{N}$, and define $f : [\mathbb{N}]^2 \to 2$ as in the proof of Proposition 3.6. Let $\langle H_1, H_2\rangle$ be a p-homogeneous set for f obtained by applying PT^2_k. Set $a_0 = \min(H_1)$ and for $n \geq 0$, let a_{n+1} to be the least element $> a_n$ of H_2 if n is even and of H_1 if n is odd. By Δ^0_1 comprehension we obtain an increasing sequence $\langle a_n : n \in \mathbb{N}\rangle$ such that $a_{2n} \in H_1$ and $a_{2n+1} \in H_2$ for all n. Since for all n, $f(a_n, a_{n+1}) = c$ where c is the color of $\langle H_1, H_2\rangle$, it follows that either $a_n \preceq a_{n+1}$ for all n or $a_{n+1} \preceq a_n$ for all n. The range of $\langle a_n : n \in \mathbb{N}\rangle$ is therefore an ascending or descending sequence under $\preceq$. □

The results of this section are summarized in the following diagram (arrows denote implications provable in RCA_0, double arrows denote implications which are known to be strict, and negated arrows indicate nonimplications). For the sake of clarity, we include only the most relevant relations from previous investigations; cf. [7, p. 199] for a complete summary of these. Let k be any number ≥ 2.



All the implications have been explained or attributed above. That SADS is not provable in RCA_0 and does not imply ADS is by Corollaries 2.6 and 2.16 of [7], while that CAC does not imply DNR is [7, Corollary 3.11]. That SRT^2 is strictly stronger than SRT^2_k and RT^2 than RT^2_k follows for $k = 2$ by a remark following Corollary 11.5 of [1], but of course SRT^2_k is equivalent to SRT^2_2 and RT^2_k to RT^2_2 for any standard

k. And to see that DNR is not provable in RCA_0 (or even in $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$) observe that no diagonally noncomputable function can be computable, so $\mathsf{RCA}_0 + \neg\mathsf{DNR}$ must hold in the Turing ideal of the computable sets.

4. Reverse Mathematics of Exponent $n \geq 3$

In this section, we show that in higher exponents the polarized and increasing polarized theorems are provably equivalent to Ramsey's theorem. Drawing on results of Jockusch [9], Simpson (cf. [14, Theorem III.7.6]) was able to show that for $n \geq 3$, RT^n is equivalent over RCA_0 to ACA_0. We can similarly draw on Theorem 2.3 above to obtain the following result. Here we recall that a monotone p-homogeneous set, as discussed at the end of Section 1, is defined as in Definition 1.3, but restricting to only monotone tuples.

Theorem 4.1. *For every $n \geq 3$ and $k \geq 2$, the following are equivalent over RCA_0:*

(1) ACA_0.
(2) PT^n.
(3) PT^n_k.
(4) *For all $j \in \mathbb{N}$, every $f : [\mathbb{N}]^n \to j$ has a monotone p-homogeneous set.*
(5) *Every $f : [\mathbb{N}]^n \to k$ has a monotone p-homogeneous set.*
(6) IPT^n.
(7) IPT^n_k.

Proof. We fix $n \geq 3, k \geq 2$, and argue in RCA_0. That (1) implies (2) follows from Remark 1.4 and the fact, mentioned above, that $\mathsf{ACA}_0 \vdash \mathsf{RT}^n$.

The implications from (2) to (3) to (5) to (7) and from (2) to (4) to (6) to (7) are trivial.

It remains to show that (7) implies (1). To this end, first notice that IPT^n_k implies IPT^3_2. Indeed, given $f : [\mathbb{N}]^3 \to 2$, define $g : [\mathbb{N}]^n \to 2$ by $g(x_1, x_2, x_3, \ldots, x_n) = f(x_1, x_2, x_3)$, let $\langle H_1, \ldots, H_n\rangle$ be any p-homogeneous set for g given by IPT^n_k, and notice that $\langle H_1, H_2, H_3\rangle$ is necessarily p-homogeneous for f. Thus to complete the proof it suffices to show that IPT^3_2 implies arithmetical comprehension, or equivalently, that IPT^3_2 implies the existence of the range of a given injective function $F : \mathbb{N} \to \mathbb{N}$. Define $f : [\mathbb{N}]^3 \to 2$ by

$$f(x_1, x_2, x_3) = \begin{cases} 1 & \text{if } \neg\exists t \in [x_2, x_3](F(t) \leq x_1) \\ 0 & \text{otherwise} \end{cases},$$

and let $\langle H_1, H_2, H_3\rangle$ be p-homogeneous for f. Notice that the color of $\langle H_1, H_2, H_3\rangle$ must be 1, since otherwise we could pick $x_1 + 2$ disjoint increasing pairs from $H_2 \times H_3$, two of which must contain elements mapping to the same value below x_1, and so witnessing that F is not injective.. Now if $y \in \mathbb{N}$ is given, choose any increasing sequence $\langle x_1, x_2, x_3\rangle \in H_1 \times H_2 \times H_3$ with $y < x_1$. Then $y \in \operatorname{ran}(F)$ if and only if there is a $t < x_2$ with $F(t) = y$, since if $y = F(t)$ for some $t \geq x_2$ we can choose $x_4 > t$ in H_3 to witness that $f(x_1, x_2, x_4) = 0$, a contradiction. ☐

The preceding theorem and Theorem 3.8 now immediately yield the following new characterization of Ramsey's theorem in a given exponent.

Corollary 4.2. *For every $n, k \geq 1$, $\mathsf{RCA}_0 \vdash \mathsf{RT}^n_k \leftrightarrow \mathsf{PT}^n_k$.*

For the usual version of Ramsey's theorem, every infinite subset of a homogeneous set is also homogeneous. Furthermore, the ordering of the infinite homogeneous subset is not mentioned in the statement, so it is easy to see that repeated applications of Ramsey's theorem can yield sets that are simultaneously homogeneous for a number of colorings. With a little effort, a similar result can be proved in the increasing polarized case. This is formulated in the next lemma and applied in the subsequent result.

Lemma 4.3 (RCA_0)**.** *Fix $n, k \in \mathbb{N}$ and let $\langle S_1, \dots, S_n \rangle$ be a sequence of infinite sets. Then IPT^n_k implies that for any $f : [\mathbb{N}]^n \to k$ there is an increasing p-homogeneous set $\langle H_1, \dots, H_n \rangle$ for f such that $H_i \subseteq S_i$ for all i.*

Proof. Working in RCA_0, assume IPT^n_k and fix $\langle S_1, \dots, S_n \rangle$ as above. We define infinite subsets of each S_i as follows. Let $s_{n,0}$ be the minimum element of S_n. Having defined $s_{i+1,j}$ for some i with $1 \le i < n$, let $s_{i,j}$ be the least element of S_i greater than $s_{i+1,j}$. Finally, for $j \ge 0$, let $s_{n,j+1}$ be the least element of S_n greater than $s_{1,j}$. Then for all i, $1 \le i \le n$, the sequence $\langle s_{i,j} \rangle_{j \in \mathbb{N}}$ exists by primitive recursion and is an infinite increasing subsequence of S_i. Furthermore, if $\langle s_{1,j_1}, \dots, s_{n,j_n} \rangle \in S_1 \times \cdots \times S_n$ is an increasing tuple, then so is $\langle j_1, \dots, j_n \rangle \in \mathbb{N} \times \cdots \times \mathbb{N}$.

Now fix $f : [\omega]^n \to k$, and define the coloring $g : [\omega]^n \to k$ by $g(j_1, \dots, j_n) = f(s_{1,j_1}, \dots, s_{n,j_n})$. Apply IPT^n_k to g to obtain a fixed color c and a c-colored increasing p-homogeneous set $\langle G_1, \dots, G_n \rangle$. Define $\langle H_1, \dots, H_n \rangle$ by setting $H_i = \{s_{i,j} : j \in G_i\}$. Then for each i such that $1 \le i \le n$, H_i is an infinite subset of S_i.

We claim that $\langle H_1, \dots, H_n \rangle$ is a c-colored increasing p-homogeneous set for f. Choose any increasing tuple in $H_1 \times \cdots \times H_n$. By definition of the H_i, we can write this tuple as $\langle s_{1,j_1}, \dots, s_{n,j_n} \rangle$ where for each i we have $j_i \in G_i$. Hence, as remarked above, $\langle j_1, \dots, j_n \rangle$ is an increasing tuple in $G_1 \times \cdots \times G_n$. By p-homogeneity of $\langle G_1, \dots, G_n \rangle$, $g(j_1, \dots, j_n) = c$, and so by construction $f(s_{1,j_1}, \dots, s_{n,j_n}) = c$ also. □

For our last result, we turn to ACA_0', the subsystem of second order arithmetic obtained by adding to the axioms of ACA_0 the statement that for all sets X and all $n \in \mathbb{N}$, the nth Turing jump of X exists. (It is known that ACA_0' is strictly stronger than ACA_0 and strictly weaker than ACA_0^+, the system consisting of ACA_0 together with the assertion of the existence for every set X of its ωth Turing jump.) Since the set universe of any ω-model of ACA_0 is closed under any standard number of jumps, it follows from Lemma 5.9 of Jockusch [9] that every ω-model of $\mathsf{RCA}_0 + \mathsf{RT}$ is also a model of ACA_0'. Mileti [11, Proposition 7.1.4] established the stronger result that $\mathsf{RCA}_0 \vdash \mathsf{ACA}_0' \leftrightarrow \mathsf{RT}$, and we now show that this equivalence extends to IPT. Our argument is different from that sketched by Mileti in that the reversal from IPT to ACA_0' is obtained not by formalizing the proof of Lemma 5.9 of [9] but by directly appealing to the definition of the jump.

Thus we begin by looking at precisely how the jump is formalized. In the language of second order arithmetic, we define the following convenient abbreviations. Given any set X, we write $Y = X'$ precisely when

$$\forall \langle m, e \rangle\, [\langle m, e \rangle \in Y \leftrightarrow (\exists t)\, \Phi^X_{e,t}(m)\downarrow].$$

Here $\Phi^X_{e,t}(m)\downarrow$ is a fixed formalization of the assertion that the Turing machine with code number e, using an oracle for X, halts on input m with the entire computation (including the use) bounded by t. For the nth jump, $n \geq 1$, we write $Y = X^{(n)}$ if there is a finite sequence $\langle X_0, \ldots, X_n \rangle$ such that $X_0 = X$, $X_n = Y$, and for every $i < n$, $X_{i+1} = X_i'$. Thus $Y = X'$ if and only if $Y = X^{(1)}$. For a given $n \in \mathbb{N}$, we say that $X^{(n)}$ *exists* provided there exists a set Y with $Y = X^{(n)}$. In what follows, we will also need a notation for finite approximations to jumps. For any set X and integer s define

$$X_s' = \{\langle m, e \rangle : (\exists t < s)\ \Phi_{e,t}^X(m) \downarrow\},$$

and for integers $u_1, \ldots, u_n$ define

$$X_{u_n,\ldots,u_1,s}^{(n+1)} = (X_{u_n,\ldots,u_1}^{(n)})_s'.$$

Using this notation, we can state and prove our proposition addressing polarized versions of Ramsey's theorem and ACA_0'.

Proposition 4.4. *The following are equivalent over* RCA_0*:*

(1) ACA_0'.
(2) RT.
(3) PT.
(4) IPT.

Proof. In Proposition 7.1.4 of [11], Mileti gives a proof of the implication from (1) to (2). Remark 1.4 can be used to prove that (2) implies (3) implies (4), so only (4) implies (1) remains to be proved. As the current literature contains very few examples of detailed proofs involving ACA_0', we include the following somewhat technical proof of this final implication.

Assume RCA_0 and (4). Given X and n, we wish to show that $X^{(n)}$ exists. Define the function $f : [\mathbb{N}]^{2n+1} \to n+1$ by setting $f(s_0, s_1, \ldots, s_n, u_1, \ldots, u_n)$ equal to the least positive $i \leq n$ such that

$$\exists \langle m, e \rangle < s_{n-i}\ [\langle m, e \rangle \in X_{s_n,\ldots,s_{n-i+1}}^{(i)} \not\leftrightarrow \langle m, e \rangle \in X_{u_n,\ldots,u_{n-i+1}}^{(i)}]$$

if such an i exists, and 0 otherwise. Apply IPT to f to get an increasing p-homogeneous set $\langle H_0, \ldots, H_{2n} \rangle$ of color c. The following argument shows that $c = 0$.

By way of contradiction, suppose $c = i > 0$. By removing elements from $H_1, \ldots, H_{2n}$ if necessary, we can arrange for $\min(H_0) < \min(H_1) < \cdots < \min(H_{2n})$. Define a coloring g by letting $g(s_0, s_1, \ldots, s_n, u_1, \ldots, u_n)$ be the least $\langle m, e \rangle < \min(H_{n-i})$ with $\langle m, e \rangle \in X_{s_n,\ldots,s_{n-i+1}}^{(i)} \not\leftrightarrow \langle m, e \rangle \in X_{u_n,\ldots,u_{n-i+1}}^{(i)}$ if such exists, and $\min(H_{n-i})$ otherwise. Applying Lemma 4.3 to g, we can find a p-homogeneous set for g contained in $H_0, \ldots, H_{2n}$, and clearly its color must be less than $\min(H_{n-i})$. Consequently, without loss of generality, we may assume that there is a fixed $\langle m_0, e_0 \rangle < \min(H_{n-i})$ such that for all increasing tuples $\langle s_0, \ldots, s_n, u_1, \ldots, u_n \rangle$ in $H_0 \times \cdots \times H_{2n}$,

$$\langle m_0, e_0 \rangle \in X_{s_n,\ldots,s_{n-i+1}}^{(i)} \not\leftrightarrow \langle m_0, e_0 \rangle \in X_{u_n,\ldots,u_{n-i+1}}^{(i)}. \tag{4.4.1}$$

Fixing any such increasing tuple, notice that the minimality of i forces $X_{s_n,\ldots,s_{n-i+2}}^{(i-1)}$ to agree with $X_{u_n,\ldots,u_{n-i+2}}^{(i-1)}$ on all values less than s_{n-i+1}. Thus, we have that

$$(\exists t < s_{n-i+1})\Phi_{e_0,t}^{X_{s_n,\ldots,s_{n-i+2}}^{(i-1)}}(m_0) \downarrow \text{ implies } (\exists t < u_{n-i+1})\Phi_{e_0,t}^{X_{u_n,\ldots,u_{n-i+2}}^{(i-1)}}(m_0) \downarrow .$$

By (4.4.1) and the definition of approximations to the jump, the converse of this implication must fail, so it must be that

$$(\exists t < u_{n-i+1})\Phi_{e_0,t}^{X^{(i-1)}_{u_n,\dots,u_{n-i+2}}}(m_0)\downarrow .$$

Now choose an increasing tuple $\langle s_0^*, \dots, s_n^*, u_1^*, \dots, u_n^*\rangle$ in $H_0 \times \cdots \times H_{2n}$ with $s^*_{n-i+1} > u_{n-i+1}$ and $u^*_{n-i+2} \geq u_{n-i+2}$. Since the argument just given applies to any increasing tuple, and in particular to

$$\langle s_0, \dots, s_n, u_1, \dots, u_{n-i+1}, u^*_{n-i+2}, \dots, u^*_n\rangle,$$

we have

$$(\exists t < u_{n-i+1})\Phi_{e_0,t}^{X^{(i-1)}_{u^*_n,\dots,u^*_{n-i+2}}}(m_0)\downarrow . \tag{4.4.2}$$

But $X^{(i-1)}_{s^*_n \dots s^*_{n-i+2}}$ and $X^{(i-1)}_{u^*_n \dots u^*_{n-i+2}}$ must agree on all elements below s^*_{n-i+1} and hence below u_{n-i+1}. And since u_{n-i+1} bounds the use of the computation in (4.4.2) and $u_{n-i+1} < s^*_{n+i-1}$, we have

$$(\exists t < s^*_{n-i+1})\Phi_{e_0,t}^{X^{(i-1)}_{s^*_n,\dots,s^*_{n-i+2}}}(m_0)\downarrow \wedge (\exists t < u^*_{n-i+1})\Phi_{e_0,t}^{X^{(i-1)}_{u^*_n,\dots,u^*_{n-i+2}}}(m_0)\downarrow,$$

which contradicts (4.4.1). This completes the proof of our claim that $c = 0$.

Next, we use $\langle H_0, \dots, H_{2n}\rangle$ to define a new finite sequence of sets $\langle X_0, \dots, X_n\rangle$. Let $X_0 = X$ and for each i with $1 \leq i \leq n$, let $\langle m, e\rangle \in X_i$ if and only if $\langle m, e\rangle \in X^{(i)}_{s_n,\dots,s_{n-i+1}}$, where $\langle s_0, \dots s_n, u_1 \dots u_n\rangle$ is the lexicographically least increasing tuple in $H_0 \times \cdots \times H_{2n}$ such that $\langle m, e\rangle < s_{n-i}$. Note that the X_i are defined simultaneously rather than inductively, so by recursive comprehension the entire sequence $\langle X_0, \dots, X_n\rangle$ exists.

We claim that for each $i < n$, $X_{i+1} = X_i'$. Fixing i, we prove containment in both directions. This will obviously complete the proof, since then $\langle X_0, \dots, X_n\rangle$ will be a sequence witnessing that $X_n = X^{(n)}$ and hence that $X^{(n)}$ exists.

First, suppose $\langle m, e\rangle \in X_{i+1}$, and let $\langle s_0, \dots s_n, u_1 \dots u_n\rangle$ be the lexicographically least increasing tuple with $\langle m, e\rangle < s_{n-i-1}$, so we have $\langle m, e\rangle \in X^{(i+1)}_{s_n,\dots,s_{n-i}}$. Applying the definition of approximations of jumps, $\langle m, e\rangle \in (X^{(i)}_{s_n,\dots,s_{n-i+1}})'_{s_{n-i}}$, and so

$$(\exists t < s_{n-i})\Phi_{e,t}^{X^{(i)}_{s_n,\dots,s_{n-i+1}}}(m)\downarrow .$$

Since s_{n-i} bounds the use of this computation, homogeneity of $\langle H_0, \dots, H_{2n}\rangle$ implies that $X^{(i)}_{s_n,\dots,s_{n-i+1}}$ agrees with X_i below this use. It consequently follows that $(\exists t < s_{n-i})\Phi^{X_i}_{e,t}(m)\downarrow$, so $\langle m, e\rangle \in X_i'$, as wanted.

Now suppose $\langle m, e\rangle \in X_i'$. By definition of the jump, we can find a t such that $\Phi^{X_i}_{e,t}(m)\downarrow$. Let $\langle s_0, \dots, s_n, u_1, \dots, u_n\rangle$ be the lexicographically least increasing tuple in $H_0 \times \cdots \times H_{2n}$ such that $\langle m, e\rangle < s_{n-i-1}$, and choose $v_{n-i} \in H_{n-i}$ such that $v_{n-i} > \max\{t, s_{n-i-1}\}$. Choose an increasing tuple $\langle v_{n-i+1}, \dots, v_n\rangle$ in $H_{n-i+1} \times \cdots \times H_n$ with $v_{n-i} < v_{n-i+1}$. By homogeneity of $\langle H_0, \dots, H_{2n}\rangle$ and the definition of X_i, the sets X_i and $X^{(i)}_{v_n,\dots,v_{n-i+1}}$ agree on elements below v_{n-i}. Thus

$$(\exists w < v_{n-i})\Phi_{e,w}^{X^{(i)}_{v_n,\dots,v_{n-i+1}}}(m)\downarrow,$$

or more succinctly, $\langle m, e\rangle \in (X^{(i)}_{v_n,\dots,v_{n-i+1}})'_{v_{n-i}} = X^{(i+1)}_{v_n,\dots,v_{n-i}}$. Homogeneity of $\langle H_0, \dots, H_{2n}\rangle$ now implies that $\langle m, e\rangle \in X^{(i+1)}_{s_n,\dots,s_{n-i}}$ and hence that $\langle m, e\rangle \in X_{i+1}$, which is what was to be shown. □

5. Questions

The main questions pertaining to Section 3 are about which of the implications established there can be reversed. Of particular note are the following, stated here for simplicity for two colors.

Question 5.1. Does $\mathsf{IPT}^2_2 + \mathsf{B}\Sigma^0_2$ imply RT^2_2 over RCA_0? Does IPT^2_2 imply $\mathsf{B}\Sigma^0_2$?

Of course, the equivalence of $\mathsf{IPT}^2_2 + \mathsf{B}\Sigma^0_2$ with RT^2_2 would follow from an affirmative answer to the long-standing open question of whether SRT^2_2 implies RT^2_2 (cf. [1, Question 13.6]). Separating RT^2_2 and $\mathsf{IPT}^2_2 + \mathsf{B}\Sigma^0_2$, therefore, is likely to be at least as hard as obtaining a negative answer to that question.

Question 5.2. Does SIPT^2_2 or SPT^2_2 imply SRT^2_2? Equivalently (by Theorem 3.3), does either imply $\mathsf{B}\Sigma^0_2$?

A related question is whether SIPT^2_2 implies the stable version SCAC of CAC, studied by Hirschfeldt and Shore in [7, Section 3] (c.f. [7, Definition 3.2] for the statement of this principle). It is known that SCAC implies SADS and $\mathsf{B}\Sigma^0_2$ (cf. [7, Propositions 3.3 and 4.1, resp.]), so an affirmative answer to this question would similarly settle the preceding one, and would extend Proposition 3.6 above.

On the computability-theoretic side of our investigation, we do not have a precise characterization of the relationship between the Turing degrees of homogeneous, p-homogeneous, and increasing p-homogeneous sets. A strong connection could be inferred from an affirmative answer to the following question.

Question 5.3. Given a computable coloring $f : [\omega]^2 \to 2$, does there exists a computable coloring $g : [\omega]^2 \to 2$ such that every p-homogeneous (respectively, increasing p-homogeneous) set for g computes a homogeneous (respectively, p-homogeneous) set for f?

For an alternative approach, recall the following definition due to Mileti.

Definition 5.4 (Mileti, [11, Definition 5.12])**.** A degree $\mathbf{d}$ is *Ramsey*, respectively *s-Ramsey*, if every computable coloring $f : [\omega]^2 \to k$, respectively every stable such coloring, has a homogeneous set of degree at most $\mathbf{d}$.

By analogy, we can define a degree $\mathbf{d}$ to be *p-Ramsey*, respectively *ip-Ramsey*, if every computable coloring of pairs has a p-homogeneous, respectively increasing p-homogeneous, set of degree at most $\mathbf{d}$. That every Ramsey degree is p-Ramsey is immediate by Remark 1.4, and every p-Ramsey degree is obviously ip-Ramsey.

Question 5.5. Is every ip-Ramsey degree a p-Ramsey degree? Is every p-Ramsey degree a Ramsey degree?

(Notice that while we could also formulate the polarized and increasing polarized analogs of s-Ramsey degrees, these would all coincide by Proposition 2.2.)

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