Reverse Mathematics and Rank Functions for Directed Graphs

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Abstract

A rank function for a directed graph G assigns elements of a well ordering to the vertices of G in a fashion that preserves the order induced by the edges. While topological sortings require a one-to-one matching of vertices and elements of the ordering, rank functions frequently must assign several vertices the same value. Theorems stating basic properties of rank functions vary significantly in logical strength. Using the techniques of reverse mathematics, we present results that require the subsystems $\mathbf{RCA}_0$, $\mathbf{ACA}_0$, $\mathbf{ATR}_0$, and $\mathbf{\Pi^1_1-CA}_0$.

A great deal of graph theory can be formalized in the language of second order arithmetic. For example, it is possible to describe a countable directed graph G by specifying a set $V \subseteq \mathbb{N}$ of (codes for) vertices and a set $E \subseteq V \times V$ of directed edges. The reader may wish to formalize some of the following notions. A path in G from u_0 to u_n is a finite sequence of vertices $u_0, u_1, \ldots, u_n$ such that for each $j < n$, $(u_j, u_{j+1}) \in E$. G is acyclic if it contains no paths with repeated vertices. A vertex v is an initial node for G if it has no incoming edges, that is if for every vertex u, the edge (u, v) is not in G. In this paper, we suppress most of the encoding of graph theoretic concepts. For more detailed expositions of encoding mathematics in second order arithmetic, see [4] or [8]. Also, only brief indications of the axioms of the subsystems are given. For detailed descriptions of the axioms systems, refer to [7] or [8].

In [1], a number of theorems of algorithmic graph theory are analyzed in the subsystems of second order arithmetic associated with reverse mathematics. In particular, topological sortings of graphs are studied. In a topological sorting, the vertices of a directed graph are matched with the elements of an initial segment of $\mathbb{N}$. Rank functions allow the use of well orderings other than $\mathbb{N}$ and assignments of values that are not injective. The formal definition follows.

Definition. Suppose G is a directed acyclic graph with vertex set V and edge set E. The pair $\langle f, \alpha \rangle$ is a *rank function* for G if α is a well ordering, f maps V onto α, and the following two properties hold:

1) $(u, v) \in E$ implies $f(u) < f(v)$, and

2) for any $v \in V$ and $x \in \alpha$, if $x < f(v)$, then there is a $u \in V$ such that $(u, v) \in E$ and $x \leq f(u)$.

The definition of rank function parallels the formalization of derived sequences used in [3]. The next three sections contain results concerning the existence of rank functions, uniqueness of the functions, and some results on bounded ranks.

Existence of rank functions

We begin by working in the weak base system $\mathbf{RCA}_0$. This axiom system includes basic arithmetic axioms, induction restricted to Σ^0_1 formulas, and a set existence axiom asserting the existence of Δ^0_1 definable sets. $\mathbf{RCA}_0$ can prove the existence of rank functions for finite graphs. The proof is motivated by the observation that for finite graphs, the rank function measures the maximum distance of each vertex from initial nodes.

Theorem 1 ($\mathbf{RCA}_0$)**.** *Every finite directed acyclic graph has a rank function.*

Proof. Working in $\mathbf{RCA}_0$, suppose G is a directed acyclic graph with n vertices. We construct a function from the vertices to an initial segment of $n = \{0, 1, \ldots, n-1\}$ as follows. Let $v_0, ..., v_k$ be a list of all initial nodes of G. Since G is finite, this list is nonempty. For each $i \leq k$, set $f(v_i) = 0$. For every other vertex v of G, list all the paths through G from each v_i to v. Because G is acyclic, there are at most $n!$ such paths, each of which can be coded by an integer less than a predetermined bound. Suppose $v_i, u_1, u_2, \ldots, u_m, v$

is a path of maximum length and, to insure determinacy of the algorithm, minimum code number. Set $f(v) = m+1$. Because the domain of f is finite, $\mathbf{RCA}_0$ can prove the existence of the range of f. Let α denote the range of f with the standard ordering. Routine arguments verify that $\langle f, \alpha\rangle$ is a rank function for G. □

Note that the preceding theorem can also be proved by induction on the size of the graph. However, such a proof requires special care to avoid exceeding the restricted induction scheme available in $\mathbf{RCA}_0$.

Not every infinite graph has a rank function. However, the following definition gives a necessary and sufficient condition for the existence of a rank function.

Definition. A directed graph is well founded if for every infinite sequence of vertices $\langle v_i | i \in \mathbb{N}\rangle$ there is some j such that (v_{j+1}, v_j) is not an edge in the graph.

The necessity of the condition is provable in $\mathbf{RCA}_0$.

Theorem 2 ($\mathbf{RCA}_0$)**.** *Suppose G is a countable directed graph. If G has a rank function, then G is well founded.*

Proof. Suppose $\langle f, \alpha\rangle$ is a rank function for G. Also suppose, by way of contradiction, that $\langle v_i | i \in \mathbb{N}\rangle$ is a sequence of vertices in G such that for every $j \in \mathbb{N}$, (v_{j+1}, v_j) is an edge in G. Then $\langle f(v_i) | i \in \mathbb{N}\rangle$ is an infinite descending sequence in α, contradicting the fact that α is well ordered. □

To prove that well foundedness is sufficient to insure the existence of a rank function, we must introduce some additional terminology and two technical lemmas.

Definition. Suppose G is a directed graph with subgraph $\tilde{G}$. The pair $\langle f, A\rangle$ is a *partial rank function* for G if A is a linear ordering, f maps the vertices of $\tilde{G}$ onto A, and the two properties from the definition of rank function hold.

Lemma 3 ($\mathbf{RCA}_0$)**.** *Let G be a countable directed graph. If G is well founded and $\langle f, A\rangle$ is a partial rank function for G, then A is well ordered.*

Proof. Suppose G and $\langle f, A\rangle$ are as in the hypothesis. By way of contradiction, suppose $\langle a_i | i \in \mathbb{N}\rangle$ is an infinite descending sequence in A. Since G is countable, each vertex can be assigned a natural number code. Since f maps a subset of the vertices of G onto A, there is a vertex $u_0 \in G$ with minimal code number satisfying $f(u_0) = a_0$. Suppose that we have defined a vertex u_n such that $f(u_n) \geq a_n$. Since $a_{n+1} < a_n \leq f(u_n)$, applying the second property of the definition of partial rank function yields a vertex u_{n+1} with minimal code such that $a_{n+1} \leq f(u_{n+1}) < f(u_n)$ and (u_{n+1}, u_n) is an edge of G. $\mathbf{RCA}_0$ suffices to prove that the sequence of vertices $\langle u_n | n \in \mathbb{N} \rangle$ exists, contradicting the hypothesis that G is well founded. □

The next lemma and the following theorem use the subsystem $\mathbf{ATR}_0$. This axiom system extends $\mathbf{RCA}_0$ by adding a set comprehension axiom that allows iterations of arithmetical comprehension along well orderings. The proof of Lemma 4 is given in [3], where it appears as Lemma 4.3.

Lemma 4 ($\mathbf{ATR}_0$). *Let $\psi(X)$ be a Σ^1_1 formula such that for all X if $\psi(X)$, then X is well ordered. Then there is a well ordering β such that for all X, if $\psi(X)$, then $X < \beta$. (Here $X < \beta$ means that X is order isomorphic to a proper initial segment of β.)*

Now we have the machinery needed to prove the existence of rank functions.

Theorem 5 ($\mathbf{ATR}_0$). *Every countable well founded directed graph has a rank function.*

Proof. Let G be a countable well founded directed graph with vertex set V and edge set E. We will construct a rank function $\langle f, \alpha \rangle$ by applying arithmetical transfinite recursion. This proof is very similar to the three step proof in $\mathbf{ATR}_0$ of the existence of derived sequences for closed sets. (See Lemma 4.4 of [3].) First we will find an upper bound for α. Then we will construct the rank function. Finally, we will verify the properties of the function.

The bound on α is found by applying Lemma 4. Let $\psi(X)$ be the Σ^1_1 formula stating that X is a linear ordering and that for some function g, $\langle g, X \rangle$ is a partial rank function for G. By Lemma 3, if $\psi(X)$ holds, then X is well ordered. By Lemma 4, there is a well ordering β such that for all X, $\psi(X)$ implies $X < \beta$. β will serve as the upper bound for our construction.

We will use $\mathbf{ATR}_0$ to construct the rank function for G. $\mathbf{ATR}_0$ asserts that we can construct a sequence of sets $\langle Y^\lambda | \lambda < \beta \rangle$, where for each λ, Y^λ is uniformly arithmetically definable from the sets $\{Y^\gamma | \gamma < \lambda\}$. To define the rank function, let Y^0 be the set of initial nodes. Given Y^γ for all $\gamma < \lambda$, let Y^λ consist of those vertices $v \in V$ satisfying the following two properties:

1. $\forall\gamma < \lambda(v \notin Y^\gamma)$, and
2. $\forall u \in V((u,v) \in E \to \exists\gamma < \lambda(u \in Y^\gamma))$.

Given the sequence of sets $\langle Y^\lambda | \lambda < \beta\rangle$, define $f : V \to \beta$ by setting $f(v)$ equal to the (unique least) λ such that $v \in Y^\lambda$. Let α be the range of f. The remainder of the proof verifies that $\langle f, \alpha\rangle$ is a rank function for G.

First, we verify that $\langle f, \alpha\rangle$ is a partial rank function for G. If $v \in \text{domain}(f)$, then by clause 2 of the definition of Y^λ, $\{u|(u,v) \in E\} \subset \text{domain}(f)$. Consequently, the restriction of G to the vertices in domain(f) is a subgraph of G. We must verify each of the properties of a partial rank function. First, if $u, v \in \text{domain}(f)$, $(u,v) \in E$, and $f(v) = \lambda$, then by clause 2 of the definition of Y^λ, there is a $\gamma < \lambda$ such that $f(u) = \gamma$. Second, suppose $v \in \text{domain}(f)$ and $x \in \alpha$ such that $x < f(v)$. As noted before, $\{u|(u,v) \in E\} \subset \text{domain}(f)$. Suppose, by way of contradiction, that for every u such that $(u,v) \in E$, $f(u) < x$. Let γ be the least element of α greater than every element of $\{f(u)|(u,v) \in E\}$. Then $\gamma \leq x$, and v satisfies both clauses of the definition of Y^γ. Thus $f(v) = \gamma \leq x$, contradicting our assumption that $x < f(v)$. Summarizing, we have shown that $\langle f, \alpha\rangle$ is a partial rank function for G.

Finally, we must show that $\text{domain}(f) \supseteq V$. Suppose otherwise. If every vertex not in the domain of f has a predecessor not in the domain of f, then G is not well founded. Thus, there is a vertex $v \notin \text{domain}(f)$ with the property that for every u such that $(u,v) \in E$, $u \in \text{domain}(f)$. Let $S = \{f(u)|(u,v) \in E\}$. Since S is a subset of α, and α is a proper initial segment of β, there is a least $\lambda \in \beta$ which is greater than each element of S. The vertex v satisfies both clauses of the definition of Y^λ, so $\lambda \in \alpha$ and $v \in \text{domain}(f)$, yielding a contradiction. Thus $\text{domain}(f) = V$, completing the proof. □

Our proof of the reversal of the preceding theorem uses transfinite versions of some restricted induction and least element schemes. For a more extended analysis of schemes of this sort, see [5].

Lemma 6. $\mathbf{RCA}_0$ *can prove the* Σ^0_1 *transfinite least element principle, and the* Π^0_1 *transfinite induction scheme.*

Proof. We will work in $\mathbf{RCA}_0$. To prove the Σ^0_1 transfinite least element principle, suppose that α is a countable well ordering, θ is a quantifier free formula, and that for some $a_0 \in \alpha$, $\exists t\theta(a_0, t)$ holds. By way of contradiction, suppose that for each $a \in \alpha$, if $\exists t\theta(a,t)$ holds then there is an $\tilde{a} \in \alpha$ such that $\tilde{a} < a$ and $\exists t\theta(\tilde{a},t)$. Enumerate all pairs of the form (a,t) where $a \in \alpha$ and $t \in \mathbb{N}$. Given a_n, let a_{n+1} be the first coordinate of the first pair (a,t) such that $\theta(a,t)$ holds and $a < a_n$. $\mathbf{RCA}_0$ proves that the descending sequence $\langle a_j | j \in \mathbb{N}\rangle$ exists, contradicting the claim that α is well founded. Thus the Σ^0_1 transfinite least element principle holds.

To show that the Π^0_1 transfinite induction scheme holds, suppose not and use the Σ^0_1 transfinite least element principle to derive a contradiction. □

Although $\mathbf{ATR}_0$ is a relatively strong axiom system, its use in Theorem 5 is provably unavoidable, as shown by the following theorem.

Theorem 7 ($\mathbf{RCA}_0$)**.** *The following are equivalent:*

1) $\mathbf{ATR}_0$*.*

2) Every countable well founded directed graph has a rank function.

Proof. Theorem 5 states that *1)* implies *2)*. To prove the reversal, we will use *2)* to show that every pair of countable well orderings is strongly comparable. Proofs that comparability of well orderings is equivalent to $\mathbf{ATR}_0$ can be found in [2] and [8]. Let α and β be well orderings. Let G_α be the graph which has the elements of α as its vertices and includes all edges of the form (x,y) where $x < y$ in the ordering on α. Define G_β similarly. Let G be the graph consisting of G_α and G_β. Since G is a well founded directed graph, we may apply *2)*. Let $\langle f, \gamma\rangle$ be the rank function for G.

We claim that α is order isomorphic to an initial segment of γ, that is, $\alpha \le \gamma$. Since the elements of α are vertices of G_α and hence of G, f maps α into γ. By the definitions of G_α and rank functions, f is order preserving, and hence one to one. To see that f maps α onto an initial segment of γ, suppose otherwise. Suppose that v is a vertex of G_α, $x \in \gamma$, $x \notin \{f(u) | u \in \alpha\}$, and $x < f(v)$. Applying Lemma 6 in the form of the Σ^0_1 transfinite least element principle, we can find the least vertex w of G_α such that $f(w) > x$. By the definition of rank function, there is a vertex y of G such that $x \le f(y) < f(w)$ and (y,w) is an edge of G. By the choice of w, $y \notin G_\alpha$. But by the construction of G, since there are no edges from vertices of G_β to vertices of G_α, $y \notin G_\beta$. Thus, no such y exists, and f witnesses that $\alpha \le \gamma$.

The preceding paragraph can be repeated with α replaced by β, yielding $\alpha \le \gamma$ and $\beta \le \gamma$. Since both α and β are order isomorphic to initial segments of γ, they are comparable. □

Imitating Cantor and Bendixson, we can extend rank functions to graphs that are not well founded. The resulting statement is equivalent to $\mathbf{\Pi^1_1 - CA}_0$, which is substantially stronger than $\mathbf{ATR}_0$.

Theorem 8 ($\mathbf{RCA}_0$)**.** *The following are equivalent:*

1) $\mathbf{\Pi^1_1 - CA}_0$.

2) If G is a directed graph with vertex set V, then there is a pair $\langle f, \alpha\rangle$ such that α is a well ordering, $f : V \to \alpha \cup \{\infty\}$, and the following properties hold:

- *f is a rank function on the preimage of α, and*
- *if $f(u) = \infty$, then there is a $v \in V$ such that (v, u) is an edge in G and $f(v) = \infty$.*

Proof. To prove that *1)* implies *2)*, assume $\mathbf{\Pi^1_1 - CA}_0$ and let G be a directed graph with vertex set V. Let V_∞ be the set of non-well-founded vertices of G. That is, put v in V_∞ if there is an infinite sequence $\langle v_i | i \in \mathbb{N}\rangle$ such that (v_0, v) is an edge and for each i, (v_{i+1}, v_i) is an edge. Since V_∞ is the complement of a Π^1_1 definable set, $\mathbf{\Pi^1_1 - CA}_0$ proves its existence. The subgraph of G with vertices in $V - V_\infty$ is well founded and has a rank function f by Theorem 5. Extending the domain of f to V by setting $f(v) = \infty$ for each $v \in V_\infty$ yields a function satisfying the requirements of *2)*.

To prove the reversal, it suffices to find a function that selects well founded trees from a sequence of trees. Given a tree T, construct G_T as follows. The vertices of G_T will be the nodes of T. If w is an immediate extension of u, include the edge (w, u). Informally, the edges of G_T are directed toward the root of T. $\mathbf{RCA}_0$ can prove that G_T is well founded if and only if T is well founded. Suppose $\langle T_n | n \in \mathbb{N}\rangle$ is a sequence of trees. Using only $\mathbf{RCA}_0$, we can construct the sequence $\langle G_{T_n} | n \in \mathbb{N}\rangle$ and take its disjoint union to create a graph G. Let ρ_n denote the vertex corresponding to the root of T_n for each n. Applying *2)*, we obtain a function f such that $f(\rho_n) = \infty$ if and only if T_n is not well founded. □

Uniqueness of rank functions

Although $\mathbf{ATR}_0$ is required to prove that rank functions exist, $\mathbf{RCA}_0$ can prove that they are unique. First we need some technical results.

Lemma 9 (**$\mathbf{RCA}_0$**). *Suppose $\langle f, \alpha \rangle$ and $\langle g, \beta \rangle$ are rank functions for the same countable graph. Then for every pair of vertices u and v, $f(u) < f(v)$ if and only if $g(u) < g(v)$.*

Proof. Suppose $\langle f, \alpha \rangle$ and $\langle g, \beta \rangle$ are rank functions for the graph G which has vertex set V and edge set E. Working in $\mathbf{RCA}_0$, by Lemma 6 we may apply Π^0_1 transfinite induction on α. Let $\psi(a)$ denote the formula

$$\forall u \forall v((f(u) < a \wedge f(v) < a) \rightarrow (f(u) < f(v) \leftrightarrow g(u) < g(v))).$$

For the base case, suppose 0 denotes the least element of α. Since $\forall u(f(u) \not< 0)$, $\psi(0)$ is vacuously true.

Now suppose that $\forall x < a\ \psi(x)$ holds. We will prove $\psi(a)$. Fix u and v so that $f(u) < a$ and $f(v) < a$.

First suppose that $f(u) < f(v)$. By the definition of a rank function, $\exists w \in V(f(u) \leq f(w) \wedge (w, v) \in E)$. Note that $f(w) < f(v) < a$. By the induction hypothesis, specifically by $\psi(f(v))$, we have $f(w) < f(u)$ if and only if $g(w) < g(u)$. Since $f(u) \leq f(w)$, $f(w) \not< f(u)$, and so $g(w) \not< g(u)$, which in turn implies that $g(u) \leq g(w)$. Because $(w, v) \in E$, the definition of rank function yields $g(w) < g(v)$. Summarizing, $g(u) \leq g(w) < g(v)$, so we have shown that if $f(u) < f(v)$ then $g(u) < g(v)$. To be very specific, we have shown that

$$\forall u \forall v((f(u) < a \wedge f(v) < a) \rightarrow (f(u) < f(v) \rightarrow g(u) < g(v))).$$

To complete the proof of the biconditional, we will assume $f(u) \not< f(v)$ and prove $g(u) \not< g(v)$. By trichotomy in α, $f(v) < f(u)$ or $f(v) = f(u)$. If $f(v) < f(u)$, then by imitating the preceding paragraph we can show that $g(v) < g(u)$, which by trichotomy in β implies that $g(u) \not< g(v)$ as desired. Now suppose that $f(u) = f(v)$. By way of contradiction, suppose that $g(u) < g(v)$. Then by the definition of rank functions, we can choose a $w \in V$ so that $(w, v) \in E$ and $g(u) \leq g(w) < g(v)$. Because $(w, v) \in E$, $f(w) < f(v)$. Since $f(u) = f(v)$, $f(w) < f(u)$. Since $f(u) < a$, we have $f(w) < a$, $f(u) < a$, and $f(w) < f(u)$, so by the observation at the end of the preceding paragraph we immediately obtain $g(w) < g(u)$. But this contradicts our previous conclusion that $g(u) \leq g(w)$. Thus $g(u) \not< g(v)$. Summarizing, we have shown that if $f(u) \not< f(v)$, then $g(u) \not< g(v)$, completing the proof of the biconditional and the induction step. The lemma follows by Π^0_1 transfinite induction. □

Applying trichotomy and the preceding lemma yields the following corollary.

Corollary 10 ($\mathbf{RCA_0}$)**.** *Suppose $\langle f, \alpha \rangle$ and $\langle g, \beta \rangle$ are rank functions for the same countable graph. Then for every pair of vertices u and v, $f(u) = f(v)$ if and only if $g(u) = g(v)$.*

The next two theorems assert the uniqueness of derived sequences.

Theorem 11 ($\mathbf{RCA_0}$)**.** *Suppose that $\langle f, \alpha \rangle$ and $\langle g, \beta \rangle$ are rank functions for the same countable graph. Then α is order isomorphic to β.*

Proof. Let $\langle f, \alpha \rangle$ and $\langle g, \beta \rangle$ be rank functions for G. Let V denote the vertex set of G. Note that the elements of V are actually integer codes for the vertices of G. Since f maps V onto α, for each $y \in \alpha$ there is a least (code for a) vertex $v \in V$ such that $f(v) = y$; we will write $v = \tilde{f}(y)$ for such a vertex. Define the function $h : \alpha \to \beta$ by setting $h(y) = g(\tilde{f}(y))$ for each $y \in \alpha$. By Δ^0_1 comprehension, the function h exists. Routine arguments show that h is well defined and has domain α.

We will show that h is order preserving. Suppose that $a_0, a_1 \in \alpha$, and $a_0 < a_1$. Let $u = \tilde{f}(a_0)$ and $v = \tilde{f}(a_1)$. Since $f(u) = a_0 < a_1 = f(v)$, by Lemma 9 we have $g(u) < g(v)$. Since $g(u) = g(\tilde{f}(a_0)) = h(a_0)$ and $g(v) = g(\tilde{f}(a_1)) = h(a_1)$, we have $h(a_0) < h(a_1)$ as desired. Thus, h is order preserving. As an immediate consequence, we know that h is one to one.

To show that h is onto, pick $b \in \beta$. Since g maps V onto β, there is a vertex v such that $g(v) = b$. Let $a = f(v)$, and let $w = \tilde{f}(a)$. Note that $f(v) = a = f(w)$. By Corollary 10, $g(w) = g(v)$. Thus, $h(a) = g(\tilde{f}(a)) = g(w) = g(v) = b$. Summarizing, we have shown that h is an order preserving bijection between α and β. □

Theorem 12 ($\mathbf{RCA_0}$)**.** *Suppose that $\langle f, \alpha \rangle$ and $\langle g, \beta \rangle$ are rank functions for the countable graph G, and h is an order preserving bijection from α to β. Then for all vertices v of G, $h(f(v)) = g(v)$.*

Proof. Let G, f, α, g, β, and h be as in the hypothesis of the theorem. Let V denote the vertex set of G, and let E denote the edge set. In order to prove that $h(f(v)) = g(v)$, we wish to eliminate the possibilities that $h(f(v)) < g(v)$ or that $h(f(v)) > g(v)$.

First suppose that there is a vertex v such that $h(f(v)) < g(v)$. By Lemma 6 we can apply the transfinite Σ^0_1 least element principle and select the least $b \in \beta$ such that there is a vertex v such that $h(f(v)) < g(v) = b$. By the definition of rank function, there is a $w \in V$ such that $(w, v) \in E$ and $h(f(v)) \leq g(w) < b = g(v)$. Since $(w, v) \in E$, $f(w) < f(v)$, and since h is order preserving, $h(f(w)) < h(f(v))$. Thus, $h(f(w)) < g(w) < b$, contradicting the minimality of b. Thus, for every vertex v, $h(f(v)) \not< g(v)$.

Now suppose that there is a vertex v such that $h(f(v)) > g(v)$. Because h is an order preserving bijection, this implies that there is a vertex v such that $h^{-1}(g(v)) < f(v)$. Interchanging the roles of f and g in the preceding paragraph and replacing h by h^{-1} yields another contradiction. Thus, for every vertex v, $h(f(v)) \not> g(v)$.

Since β is well ordered, it satisfies trichotomy. In light of the previous two paragraphs, for every vertex v, we have $h(f(v)) = g(v)$. □

Bounded ranks

By bounding the size of the well ordering, the strength of the existence theorems for rank functions can be reduced. The main theorem in this section uses $\mathbf{ACA}_0$, which allows single applications of arithmetical comprehension. $\mathbf{ACA}_0$ is substantially weaker than $\mathbf{ATR}_0$.

We will need some terminology. A rank function $\langle f, \alpha \rangle$ is *bounded by* β if α is (order isomorphic to) an initial segment of β. We say that a directed graph G is *path bounded* if for every pair of vertices v and u of G there is an integer n such that every path from v to u has length at most n. We say that G is *uniformly path bounded* if there is an integer n such that for every pair v and u, the length of each path from v to u is at most n.

The proofs of the statements in Lemma 13 are routine, and have been omitted.

Lemma 13 ($\mathbf{RCA}_0$). *Suppose that G is a countable directed graph.*

1) If G is uniformly path bounded, then G is path bounded.

2) If G is path bounded, then G is acyclic.

3) If G has a rank function bounded by $\mathbb{N}$, then G is path bounded.

4) If G has a rank function bounded by $k \in \mathbb{N}$, then G is uniformly path bounded.

Using Lemma 13, we can prove our final theorem. Aficionados of reverse mathematics might initially guess that the bounds in part *3)* of the theorem should result in a statement provable in $\mathbf{WKL}_0$. Clearly, this is not the case.

Theorem 14 ($\mathbf{RCA}_0$). *The following are equivalent.*

1) $\mathbf{ACA}_0$.

2) If G is a countable directed graph, then G is path bounded if and only if G has a rank function bounded by $\mathbb{N}$.

3) If G is a countable directed graph, then G is uniformly path bounded if and only if G has a rank function bounded by some $n \in \mathbb{N}$.

Proof. To prove that *1)* implies *2)*, assume $\mathbf{ACA}_0$ and suppose G is a path bounded directed graph. Define the function f by setting $f(v) = 0$ for each initial node v. For any other vertex u, set $f(u)$ to the length of the longest path from an initial node to u. By arithmetical comprehension, both f and $\text{range}(f)$ exist. Let $\alpha = \text{range}(f)$. The reader may wish to verify that $\langle f, \alpha \rangle$ is a rank function for G bounded by $\mathbb{N}$. This proves one implication of *2)*; the converse follows from Lemma 13.

To prove that *2)* implies *3)*, assume $\mathbf{RCA}_0$ and *2)*. Suppose that G is a directed graph uniformly path bounded by n. By Lemma 13, G is path bounded, so by *2)*, G has a rank function $\langle f, \alpha \rangle$ bounded by $\mathbb{N}$. If there is an integer $m \in \alpha$ such that $m > n$, then there is a vertex u_0 of G such that $f(u_0) = m$. A path leading to u_0 can be defined as follows. Given any vertex u_j such that $f(u_j) = m - j$, by the definition of rank function there is a (unique least) vertex u_{j+1} such that (u_{j+1}, u_j) is an edge and $f(u_{j+1}) = m - j - 1$. Consequently, G contains a path $u_m, u_{m-1}, \ldots, u_0$ of length m, contradicting the claim that G is uniformly path bounded by

$n < m$. Summarizing, $\langle f, \alpha \rangle$ is a rank function for G which is bounded by $n+1$. This completes the proof of one implication of *3)*; the converse follows from Lemma 13.

To prove that *3)* implies *1)*, it suffices to use $\mathbf{RCA}_0$ and *3)* to prove the existence of the range of an arbitrary injection. (For a proof of this, see Lemma 2.3 of [6].) Let $h : \mathbb{N} \to \mathbb{N}$ denote the injection. Construct a graph G as follows. The vertices of G are v_0 (the source node), $\{x_n | n \in \mathbb{N}\}$ and $\{y_n | n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, include the edges (v_0, y_n) and (v_0, x_n) in G. Additionally, if $h(j) = k$, include the edge (x_j, y_k). $\mathbf{RCA}_0$ suffices to prove that G exists and that it is uniformly path bounded by 2. By *3)*, there is a rank function $\langle f, \alpha \rangle$ for G. Since G contains paths of length 2 but no paths of length 3, $\alpha = \{0, 1, 2\}$. Furthermore, $k \in \mathbb{N}$ is in range(h) if and only if $f(y_k) = 2$. By Δ^0_1 comprehension, the range of h exists, completing the proof of the reversal. □

Theorems 7 and 14 both continue to hold if G is required to have a single source node from which every other vertex can be reached.

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