A SURVEY OF THE REVERSE MATHEMATICS OF ORDINAL ARITHMETIC

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Abstract. This article surveys theorems of reverse mathematics concerning the comparability, addition, multiplication and exponentiation of countable well orderings.

In [13], Simpson points out that ATR_0 is "strong enough to accommodate a good theory of countable ordinal numbers, encoded by countable well orderings." This paper provides a substantial body of empirical evidence supporting Simpson's claim. With a few very interesting exceptions, most theorems of ordinal arithmetic are provable in RCA_0 or are equivalent to ATR_0. Consequently, up to equivalence over RCA_0, Friedman's early result [2] on the equivalence of ATR_0 and comparability of well orderings encapsulates most of countable ordinal arithmetic.

This paper is divided into sections on definitions and alternative definitions of well orderings, comparability and upper bounds, addition, multiplication, exponentiation, and other topics. The last section addresses Cantor's normal form theorem, transfinite induction schemes, and indecomposable well orderings, and concludes with a list of some omitted topics. Whenever possible, references to proofs are provided, rather than the actual proofs. Throughout, arbitrary sets are denoted by capital roman letters, but well ordered sets are denoted by lower-case greek letters. This notation emphasizes the parallels between the encoded theory and the usual development of ordinal arithmetic.

§1. Definitions of well orderings.

First, we will define linear orderings and well orderings.

Definition. (RCA_0) Let X be a set of pairs. We will write $x \le y$ if $(x, y) \in X$. We say that X is a (countable) *linear ordering*, denoted $\mathsf{LO}(X)$, if

1. $x \le y \to (x \le x \wedge y \le y)$,
2. $(x \le y \wedge y \le z) \to x \le z$,
3. $(x \le y \wedge y \le x) \to x = y$, and
4. $(x \le x \wedge y \le y) \to (x \le y \vee y \le x)$.

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The *field* of X is the set $\{x \in \mathbb{N} \mid x \le x\}$. We say that X is a (countable) *well ordering*, denoted $\mathsf{WO}(X)$, if for every nonempty $Y \subset \text{field}(X)$ there is an element $y_0 \in Y$ such that $y \in Y$ implies $y_0 \le y$. That is, $\mathsf{WO}(X)$ if every nonempty subset of X has a least element. A well ordered set with a largest element is called a *successor*. A well ordered set with no largest element is called a *limit*.

Some papers (for example [3]) define well orderings as linear orderings with no infinite descending sequences. This definition is equivalent to the preceding one, and the equivalence is provable in RCA_0.

THEOREM 1. (RCA_0) *Let X be a linear ordering. The following are equivalent:*

1. *X is well ordered. That is, every nonempty subset of X has a least element.*
2. *X contains no infinite descending sequences.*

PROOF. Theorem 2 of [5]. ⊣

Cantor's original definition of "well ordered aggregate" was closer to clause 3 of the following theorem. His definition is equivalent to the usual one, but the proof of the equivalence requires ACA_0.

THEOREM 2. (RCA_0) *The following are equivalent:*

1. ACA_0.
2. *If α is a well ordering, then every subset of α with an upper bound has a least upper bound.*
3. *Suppose X is a linear ordering. Then X is well ordered if and only if every subset of X with a strict upper bound has a least strict upper bound.*

PROOF. The equivalence of clause 2 and ACA_0 is Theorem 3 of [5]. The equivalence of clause 3 and ACA_0 is shown in Corollary 6 of [5]. ⊣

§2. Comparability and upper bounds. In this section, after presenting a long list of equivalent versions of comparability, we will look at descending sequences, strict inequality, comparisons to ω, suprema, and bounds for Σ^1_1 classes. We begin by defining two forms of comparability of well orderings.

DEFINITION. (RCA_0) If α and β are well orderings, then α is *strongly less than or equal to* β, denoted $\alpha \le_s \beta$, if there is an order preserving map of α onto an initial segment of β. If the initial segment is β itself, we also write $\alpha \equiv_s \beta$. If $\alpha + 1 \le_s \beta$, then we write $\alpha <_s \beta$.

DEFINITION. (RCA_0) If α and β are well orderings, then α is *weakly less than or equal to* β, denoted $\alpha \le_w \beta$, if there is an order preserving map of α into β. If $\alpha \le_w \beta$ and $\beta \le_w \alpha$, then we write $\alpha \equiv_w \beta$. If $\alpha + 1 \le_w \beta$, then we write $\alpha <_w \beta$.

Sometimes, the same result will hold for both $\leq_s$ and $\leq_w$. We will drop the subscripts and write $\leq$ and $<$ whenever statements hold for both forms of comparability.

A few properties of $\leq_s$ and $\leq_w$ are provable in RCA_0. For example, RCA_0 can prove that $\leq_s$ and $\leq_w$ are both transitive relations. Here is another example.

THEOREM 3. (RCA_0) *If β is a proper initial segment of a well ordering α, then $\alpha \not\leq_w \beta$.*

PROOF. Lemma 2.3 of [3]. ⊣

Additionally, the fact that $\alpha \leq_s \beta$ implies $\alpha \leq_w \beta$ is provable in RCA_0, although the converse requires ATR_0, as shown in the next theorem. The next theorem also shows the equivalence of a wide variety of statements on comparability. The terminology has all been defined with one exception. A well ordering α is *indecomposable* if for every final segment $\beta = \{a \in \alpha \mid b < a\}$ we have $\alpha \leq_w \beta$. More results about indecomposable well orderings appear in Section 6.

THEOREM 4. (RCA_0) *Suppose that α and β denote well orderings and that $\langle \alpha_i | i \in \mathbb{N} \rangle$ denotes a sequence of well orderings. Then (the universal closures of) the following are equivalent:*

1. ATR_0.
2. (Strong comparability of well orderings.) $\alpha \leq_s \beta$ *or* $\beta \leq_s \alpha$.
3. (Weak comparability of well orderings.) $\alpha \leq_w \beta$ *or* $\beta \leq_w \alpha$.
4. (Weak comparability of indecomposable well orderings.) *If α and β are indecomposable, then $\alpha \leq_w \beta$ or $\beta \leq_w \alpha$.*
5. (The class of well orderings has no infinite antichains.) *For some distinct i and j, $\alpha_i \leq_w \alpha_j$.*
6. *For some distinct i and j, $\alpha_i \leq_s \alpha_j$.*
7. (WO is wqo: the class of well orderings is well-quasi-ordered.) *For some $i < j$, $\alpha_i \leq_w \alpha_j$.*
8. *For some $i < j$, $\alpha_i \leq_s \alpha_j$.*
9. *$\alpha \leq_w \beta$ implies $\alpha \leq_s \beta$.*
10. *If $\alpha \leq_w \beta$ and $\beta \leq_w \alpha$, then $\alpha \equiv_s \beta$.*
11. *If $X \subset \alpha$, then $X \leq_s \alpha$.*

PROOF. The equivalence of ATR_0 and Clause 2 was announced by Friedman in [2]. The result appears as Theorem V.6.8 in Simpson's book [13]. Clause 3 follows from Theorem 3.21 of [3] and clause 4 follows from Theorem 4.4 of [6]. Clauses 5 and 7 are due to Shore [11]. The much easier proofs for 6 and 8 appear as Theorem 5.4 of [3]. The equivalence of clause 9 is Theorem 2 of [7]. Clause 10 follows from Theorem 5.2 of [3] and clause 11 follows from Theorem 2.9 of [3]. ⊣

Using clause 7 of the preceding theorem, given any sequence of well orderings (comparable or not) one can locate a pair that is ordered in accordance with their subscripts. This is strictly stronger than asserting that there is no infinite strictly descending sequence of well orderings, as shown by the following theorem.

THEOREM 5. (ACA_0) *The following are equivalent:*
1. $\Sigma^1_1 - \mathsf{AC}_0$.
2. *There is no sequence* $\langle \alpha_i | i \in \mathbb{N} \rangle$ *of well orderings such that* $\alpha_{i+1} <_w \alpha_i$ *for every* i *in* $\mathbb{N}$.
3. *There is no sequence* $\langle \alpha_i | i \in \mathbb{N} \rangle$ *of indecomposable well orderings such that* $\alpha_{i+1} <_w \alpha_i$ *for every* i *in* $\mathbb{N}$.

PROOF. Theorem 4.2 of [3]. ⊣

In the finite case, the preceding theorem becomes an extended version of transitivity. In the following, $(Y)_k = \{j \mid (j,k) \in Y\}$.

THEOREM 6. (ACA_0) *The following are equivalent:*
1. (Bounded $\Sigma^1_1 - \mathsf{AC}_0$) *For any* Σ^1_1 *formula* ψ *and any* b,
$$(\forall k < b \exists X\ \psi(k, X)) \rightarrow (\exists Y \forall k < b\ \psi(k, (Y)_k)).$$
2. *Let* $\langle \alpha_i | i \leq b \rangle$ *be a sequence of well orderings such that for all* $i < b$, $\alpha_i <_w \alpha_{i+1}$. *Then* $\alpha_0 <_w \alpha_b$.
3. *Let* $\langle \alpha_i | i \leq b \rangle$ *be a sequence of indecomposable well orderings such that for all* $i < b$, $\alpha_i <_w \alpha_{i+1}$. *Then* $\alpha_0 <_w \alpha_b$.

PROOF. Theorem 4.1 of [3]. ⊣

In the preceding, we used $\alpha < \beta$ to denote $\alpha + 1 \leq \beta$. Proving that $\alpha \leq \beta \wedge \alpha \not\equiv \beta$ implies this form of inequality requires either ACA_0 (for strong comparability) or ATR_0 (for weak comparability). In both cases, the converse implication is provable in RCA_0 using Theorem 3.

THEOREM 7. (RCA_0) *The following are equivalent:*
1. ACA_0.
2. *If* α *and* β *are well orderings with* $\alpha \leq_s \beta$ *and* $\beta \not\leq_s \alpha$, *then* $\alpha <_s \beta$.

PROOF. Theorem 2 of [8]. ⊣

THEOREM 8. (RCA_0) *The following are equivalent:*
1. ATR_0.
2. *If* α *and* β *are well orderings such that* $\alpha \leq_w \beta$ *and* $\beta \not\leq_w \alpha$, *then* $\alpha <_w \beta$.

PROOF. Theorem 5 of [8]. ⊣

We will use ω to denote $\mathbb{N}$ with the usual ordering. Restricting statements about comparability to a special case for ω often reduces the strength of the statement from ATR_0 to ACA_0. In the following theorem, compare clause 2 to clause 10 of Theorem 4 and clause 3 to clause 2 of Theorem 8.

THEOREM 9. (RCA_0) *The following are equivalent:*
1. ACA_0.

2. *If α is a well ordering such that $\omega \leq_w \alpha$ and $\alpha \leq_w \omega$, then $\alpha \equiv_s \omega$.*
3. *If β is a well ordering such that $\omega \leq_w \beta$ and $\beta \not\leq_w \omega$, then $\omega <_w \beta$.*

PROOF. Clause 2 is Theorem 5.3 of [3]. Clause 3 is Theorem 6 of [8]. ⊣

We conclude the section with two results on upper bounds for collections of ordinals. The first result supplies a frequently useful upper bound on Σ^1_1 classes of well orderings.

THEOREM 10. (RCA_0) *The following are equivalent:*

1. ATR_0.
2. *For any Σ^1_1 formula $\psi(X)$ we have*
   $\forall X\,(\psi(X) \to \mathsf{WO}(X)) \to \exists\alpha\,(\mathsf{WO}(\alpha) \wedge \forall X(\psi(X) \to X \leq_s \alpha))\,.$

PROOF. Theorem V.6.9 of [13]. ⊣

ATR_0 is necessary and sufficient to prove the existence of unique suprema of sequences of ordinals. The following theorem holds for both $\leq_s$ and $\leq_w$.

THEOREM 11. (RCA_0) *The following are equivalent:*

1. ATR_0.
2. *Suppose $\langle \alpha_x \mid x \in \beta\rangle$ is a well ordered sequence of well orderings. Then $\sup\langle \alpha_x \mid x \in \beta\rangle$ exists. That is, there is a well ordering α unique up to order isomorphism satisfying*
   - $\forall x \in \beta(\alpha_x \leq \alpha)$, *and*
   - $\forall \gamma(\gamma < \alpha \to \exists x \in \beta(\alpha_x \not\leq \gamma))$.

PROOF. Theorem 7 of [5]. ⊣

§3. Addition. This section explores the properties of ordinal addition. The section begins with a definition of addition and a verification that addition preserves well orderings. Then two theorems describing the interaction of addition and comparability are presented. The section concludes with two theorems on triangular numbers.

DEFINITION. Let $\langle \alpha_b \mid b \in \beta\rangle$ be a well ordered sequence of well orderings. The notation $\sum_{b\in\beta} \alpha_b$ denotes the set $\{(b,a) \mid b \in \beta \wedge a \in \alpha_b\}$ ordered by the relation $(b_0, a_0) < (b_1, a_1)$ if and only if $b_0 < b_1$ or both $b_0 = b_1$ and $a_0 < a_1$. Finite sums defined in this fashion may be denoted by $\alpha_0 + \cdots + \alpha_k$.

THEOREM 12. (RCA_0) *If $\langle \alpha_b \mid b \in \beta\rangle$ is a well ordered sequence of well orderings, then $\sum_{b\in\beta} \alpha_b$ is well ordered.*

PROOF. Let $\langle \alpha_b \mid b \in \beta\rangle$ be a well ordered sequence of well orderings, and construct $\sum_{b\in\beta} \alpha_b$ as in the definition. Suppose that $\sum_{b\in\beta} \alpha_b$ is not well ordered. By Theorem 1, we can find an infinite descending sequence $\langle x_i \mid i \in \mathbb{N}\rangle$ in the sum. For each i, let $x_i = (b_i, a_i)$. If the sequence $\langle b_i \mid i \in \mathbb{N}\rangle$ has no least element, then β is not well ordered, yielding a contradiction. Thus, we may select an i such that for all $j > i$, $b_j = b_i$. In this case, $\langle a_j \mid j \geq i\rangle$

contains an infinite descending sequence in α_{b_i}, yielding another contradiction and completing the proof. ⊣

Weak comparability and strong comparability behave differently with respect to addition, as shown by the next two theorems.

THEOREM 13. ($\mathsf{RCA_0}$) *If $\langle \alpha_b \mid b \in \beta \rangle$ and $\langle \gamma_b \mid b \in \beta \rangle$ are well ordered sequences of well orderings, and $\langle f_b \mid b \in \beta \rangle$ is a sequence of functions witnessing that $\alpha_b \le_w \gamma_b$ for each $b \in \beta$, then $\sum_{b\in\beta} \alpha_b \le_w \sum_{b\in\beta} \gamma_b$. In particular, if $\alpha_0 \le_w \gamma_0$ and $\alpha_1 \le_w \gamma_1$ are well orderings, then $\alpha_0 + \alpha_1 \le_w \gamma_0 + \gamma_1$.*

PROOF. The order preserving injection witnessing $\sum_{b\in\beta} \alpha_b \le_w \sum_{b\in\beta} \gamma_b$ can be directly constructed from those witnessing $\alpha_b \le_w \gamma_b$ for each b. ⊣

THEOREM 14. ($\mathsf{RCA_0}$) *The following are equivalent:*

1. $\mathsf{ATR_0}$.
2. *If $\langle \alpha_b \mid b \in \beta \rangle$ and $\langle \gamma_b \mid b \in \beta \rangle$ are well ordered sequences of well orderings such that $\alpha_b \le_s \gamma_b$ for each $b \in \beta$, then $\sum_{b\in\beta} \alpha_b \le_s \sum_{b\in\beta} \gamma_b$.*
3. *If $\alpha_0 \le_s \gamma_0$ and $\alpha_1 \le_s \gamma_1$ are well orderings, then $\alpha_0 + \alpha_1 \le_s \gamma_0 + \gamma_1$.*

PROOF. First we will prove clause 2 using $\mathsf{ATR_0}$. Let $\langle \alpha_b \mid b \in \beta \rangle$ and $\langle \gamma_b \mid b \in \beta \rangle$ be well ordered sequences of well orderings such that $\alpha_b \le_s \gamma_b$ for each $b \in \beta$. As a notational convenience, let $\alpha = \sum_{b\in\beta} \alpha_b$ and $\gamma = \sum_{b\in\beta} \gamma_b$. By Theorem 12, both α and γ are well ordered. By strong comparability of well orderings (Theorem 4 clause 2), either $\alpha \le_s \gamma$ or $\gamma \le_s \alpha$. If $\alpha \le_s \gamma$, then we are done. Suppose that f witnesses that $\gamma \le_s \alpha$. If f maps γ onto a proper initial segment of α, then there must be a least $b \in \beta$ such that f maps the least element of γ_b into α_c where $c < b$. This implies that $\gamma_c + 1 \le_w \alpha_c$, contradicting the claim $\alpha_c \le_s \gamma_c$. Thus f must map γ onto all of α, witnessing $\alpha \equiv_s \gamma$ and therefore $\alpha \le_s \gamma$.

Since clause 3 is a special case of clause 2, we can complete the proof of the theorem by using clause 3 to derive $\mathsf{ATR_0}$. Toward this end, let β and δ be well orderings. $\mathsf{RCA_0}$ proves that $\beta \le_s \beta + \delta$ and $\beta \le_s \beta$. By clause 3, $\beta + \beta \le_s \beta + \delta + \beta$. The map witnessing this relationship must either map the second copy of β in $\beta + \beta$ onto an initial segment of δ, or can be inverted to map δ onto an initial segment of a copy of β. Thus we have $\beta \le_s \delta$ or $\delta \le_s \beta$, and by clause 2 of Theorem 4, $\mathsf{ATR_0}$ follows. ⊣

We conclude this section with two theorems from [5] on transfinite triangular numbers and their generalizations. The statements of these theorems make use of ordinal exponentiation, which is defined in section 5.

THEOREM 15. *For each positive natural number n, $\mathsf{RCA_0}$ proves*

$$\sum_{\alpha<\omega^n} \alpha \equiv_s \omega^{2n-1}.$$

PROOF. Theorem 8 of [5]. ⊣

THEOREM 16. ($\mathsf{RCA_0}$) *The following are equivalent:*

1. ATR_0.
2. (γ-lemma) *Suppose that* $\langle \alpha_b \mid b \in \omega^\gamma \rangle$ *is a well ordered sequence of well orderings such that* $b < b'$ *implies* $\alpha_{b'} + 1 \not\leq \alpha_b$. *Then*
   - *For all* $b \in \omega^\gamma$, $\alpha_b \omega^\gamma \leq \sum_{b \in \omega^\gamma} \alpha_b$, *and*
   - *If* $\delta < \sum_{b \in \omega^\gamma} \alpha_b$, *then there is an* $b \in \omega^\gamma$ *such that* $\alpha_b \omega^\gamma \not\leq \delta$.

PROOF. Theorem 9 of [5]. $\dashv$

The statement of the preceding theorem is somewhat complicated by the fact that RCA_0 is not sufficiently strong to prove that ω^γ is well ordered when γ is well ordered. For more on this, see Theorem 31.

§4. Multiplication. This section begins with a definition of ordinal multiplication and verification that products are well ordered and multiplication is associative. This is followed by results on comparability and multiplication, distributive laws, division algorithms, right factors and prime factors.

DEFINITION. If α and β are well orderings, the product $\alpha\beta$ is the set $\{(a, b) \mid a \in \alpha \wedge b \in \beta\}$, ordered by the relation $(a_1, b_1) < (a_2, b_2)$ if and only if $(b_1 < b_2) \vee (b_1 = b_2 \wedge a_1 < a_2)$.

THEOREM 17. (RCA_0) *Suppose that* α, β *and* γ *are well ordered. Then* $\alpha\beta$ *is well ordered, and* $\alpha(\beta\gamma) \equiv_s (\alpha\beta)\gamma$.

PROOF. Working in RCA_0, one can show that if $\alpha\beta$ contains an infinite descending sequence, then either α or β must contain an infinite descending sequence. The associative law is proved by direct construction of a bijection between the orderings. $\dashv$

As with addition, weak comparability and strong comparability interact in different fashions with multiplication.

THEOREM 18. (RCA_0) *If* α_0, α_1, β_0, *and* β_1 *are well orderings such that* $\alpha_0 \leq_w \alpha_1$ *and* $\beta_0 \leq_w \beta_1$, *then* $\alpha_0\beta_0 \leq_w \alpha_1\beta_1$.

PROOF. Lemma 6 of [7]. $\dashv$

THEOREM 19. (RCA_0) *The following are equivalent:*

1. ATR_0.
2. *If* α_0, α_1, β_0, *and* β_1 *are well orderings such that* $\alpha_0 \leq_s \alpha_1$ *and* $\beta_0 \leq_s \beta_1$, *then* $\alpha_0\beta_0 \leq_s \alpha_1\beta_1$.

PROOF. Theorem 7 of [7]. $\dashv$

In ordinal arithmetic, multiplication on the right by a successor ordinal preserves strict inequalities. This statement varies in strength depending on the form of comparability used. Recall that a well ordering with a largest element is called a *successor*.

THEOREM 20. (RCA_0) *If* α_0, α_1 *and* β *are wellorderings,* β *is a successor, and* $\alpha_0 <_w \alpha_1$, *then* $(\alpha_0\beta) <_w \alpha_1\beta$.

PROOF. Lemma 11 of [7]. ⊣

THEOREM 21. ($\mathsf{RCA_0}$) *The following are equivalent:*

1. $\mathsf{ATR_0}$.
2. *If α_0, α_1 and β are well orderings, β is a successor, and $\alpha_0 <_s \alpha_1$, then $(\alpha_0\beta) <_s \alpha_1\beta$.*

PROOF. Theorem 12 of [7]. ⊣

The statements in the following omnibus theorem can be proved in $\mathsf{RCA_0}$ for both types of comparability.

THEOREM 22. ($\mathsf{RCA_0}$) *The following statements hold for all well orderings.*

1. *If $\beta \neq 0$, then $\alpha \leq \alpha\beta$. Furthermore, $(\alpha\beta)+1 \nleq \alpha$.*
2. *If $\alpha \neq 0$ and $\alpha\beta_0 \equiv \alpha\beta_1$, then $\beta_0 \equiv \beta_1$.*
3. *If $\beta_0 < \beta_1$, then $\alpha\beta_0 \leq \alpha\beta_1$.*
4. *If $(\alpha\beta_0) < \alpha\beta_1$, then $\beta_0 < \beta_1$.*
5. *If $(\alpha_0\beta) < \alpha_1\beta$, then $\alpha_0 < \alpha_1$.*
6. *If β is a successor and $\alpha_0\beta \equiv \alpha_1\beta$, then $\alpha_0 \equiv \alpha_1$.*

The left distributive law for ordinal multiplication over ordinal addition is provable in $\mathsf{RCA_0}$. The corresponding right distributive law fails. Sherman's inequality, which is a weak version of the right distributive law is equivalent to $\mathsf{ATR_0}$. These results hold for both forms of comparability and are stated as the next two theorems.

THEOREM 23. ($\mathsf{RCA_0}$) *For well orderings α, β, and γ, $\alpha(\beta+\gamma) \equiv \alpha\beta+\alpha\gamma$.*

PROOF. The bijection can be constructed using only recursive comprehension and the definitions of the arithmetical operations. ⊣

THEOREM 24. ($\mathsf{RCA_0}$) *The following are equivalent:*

1. $\mathsf{ATR_0}$.
2. (Sherman's Inequality) *If α, β, and γ are well orderings, then*

$$(\alpha+\beta)\gamma \leq \alpha\gamma+\beta\gamma.$$

PROOF. Theorem 5.4 of [6]. ⊣

Some special instances of the right distributive law do hold. The following example is useful in manipulating ordinals expressed in Cantor normal form. Ordinal exponentiation is defined in the next section.

THEOREM 25. ($\mathsf{RCA_0}$) *If $m_0, m_1, ...m_k > 0$, $\beta > 0$ is a well ordering, and $\alpha_0, \alpha_1, ...\alpha_k$ are initial segments of a well ordering α that satisfy $\alpha_{i+1} <_s \alpha_i$ for each $i < k$, then*

$$(\omega^{\alpha_0}m_0 + \omega^{\alpha_1}m_1 + \cdots + \omega^{\alpha_k}m_k)\omega^\beta \equiv_s \omega^{\alpha_0+\beta}.$$

PROOF. Lemma 3 of [9]. ⊣

The next two theorems give three versions of the division algorithm. The version in the first theorem is a strong comparability analog of clause 2 in the second theorem.

THEOREM 26. ($\mathsf{RCA_0}$) *If α, β and γ are well orderings satisfying $\gamma <_s \alpha\beta$, then there are well orderings α_1 and β_1 such that $\alpha_1 <_s \alpha$, $\beta_1 \leq_s \beta$, and $\gamma \equiv_s \alpha\beta_1 + \alpha_1$.*

PROOF. Lemma 3 of [7]. ⊣

The next theorem holds with either form of comparability in the third clause.

THEOREM 27. ($\mathsf{RCA_0}$) *The following are equivalent:*

1. $\mathsf{ATR_0}$.
2. *If α, β and γ are well orderings satisfying $\gamma <_w \alpha\beta$, then there are well orderings α_1 and β_1 satisfying $\alpha_1 <_w \alpha$, $\beta_1 \leq_w \beta$, and $\gamma \equiv_w \alpha\beta_1 + \alpha_1$.*
3. *If α and γ are well orderings, then there are well orderings α_1 and β such that $\alpha_1 < \alpha$ and $\gamma \equiv \alpha\beta + \alpha_1$.*

PROOF. Theorems 4 and 5 of [7]. ⊣

This section concludes with some material on right factors. First we will consider the strength of the assertion that a right factor of a product is no larger than the product. The strength here depends on the form of comparability used in the statement.

THEOREM 28. ($\mathsf{RCA_0}$) *If $\alpha \neq 0$ and $\beta \neq 0$ are well orderings, then $\beta \leq_w \alpha\beta$.*

PROOF. Lemma 8 of [7]. ⊣

THEOREM 29. ($\mathsf{RCA_0}$) *The following are equivalent:*

1. $\mathsf{ATR_0}$.
2. *If $\alpha \neq 0$ and $\beta \neq 0$ are well orderings, then $\beta \leq_s \alpha\beta$.*

PROOF. Theorem 9 of [7]. ⊣

A ordinal α is said to be *prime* if whenever $\alpha = \lambda\rho$, either $\rho = 1$ or $\rho = \alpha$. The notion of prime can be formalized using either weak or strong comparability. Using either formalization, the following theorem holds.

THEOREM 30. ($\mathsf{RCA_0}$) *The following are equivalent:*

1. $\mathsf{ATR_0}$.
2. *Every well ordering has a prime right factor. That is, if α is a well ordering, then there are well orderings λ and ρ such that $\alpha \equiv \lambda\rho$ and ρ is prime.*
3. *If α is a well ordering, then for some $k \in \mathbb{N}$, there are prime well orderings $\rho_1, \rho_2 ... \rho_k$ such that $\alpha \equiv \rho_k\rho_{k-1} \cdots \rho_1$.*

PROOF. Theorem 6 and Corollary 7 of [9]. ⊣

§5. Exponentiation. This section begins with the definition of ordinal exponentiation, discusses closure and basic properties of exponentiation, and concludes with a theorem on the existence of ordinal logarithms.

Definition. Let α and β be well orderings. The set $\exp(\alpha, \beta)$ is the collection of all finite sequences of the form

$$\langle (b_0, a_0), (b_1, a_1), \ldots, (b_n, a_n) \rangle$$

such that (1) for all $i \leq n$, $b_i \in \beta$ and $0 \neq a_i \in \alpha$, and (2) whenever $i < j \leq n$, we have $b_i > b_j$ in the order on β. We define α^β as the ordering with field $\exp(\alpha, \beta)$, ordered lexicographically. In particular, suppose that σ and τ are distinct elements of $\exp(\alpha, \beta)$. If σ extends τ, then $\sigma > \tau$. If j is the least integer such that $(b_j, a_j) = \sigma(j) \neq \tau(j) = (b'_j, a'_j)$ and either $b_j > b'_j$ or both $b_j = b'_j$ and $a_j > a'_j$, then $\sigma > \tau$. Otherwise $\tau > \sigma$.

Intuitively, if we identify each element of α and β with the initial segment lying below it, the element $\langle (b_0, a_0), (b_1, a_1), \ldots, (b_n, a_n) \rangle$ corresponds to the ordinal $\alpha^{b_0} a_0 + \cdots + \alpha^{b_n} a_n$. The ordering on the sequences is the same as the ordering on the corresponding ordinals.

Unlike the case with ordinal addition and multiplication, RCA_0 does not suffice to prove that α^β is well ordered if α and β are.

Theorem 31. (RCA_0) *The following are equivalent:*

1. ACA_0.
2. *If α and β are well ordered, then so is α^β.*
3. *If α is well ordered, then so is 2^α.*

Proof. This result is included in a larger equivalence theorem proved by Girard in [4]. For a direct proof, see Theorem 2.6 of [6]. ⊣

RCA_0 suffices to prove numerous basic properties of exponentiation. The next theorem holds with either strong or weak of comparability.

Theorem 32. (RCA_0) *Suppose that α, β and γ are well orderings. The following hold:*

1. $\alpha^{\beta+\gamma} \equiv \alpha^\beta \alpha^\gamma$.
2. $(\alpha^\beta)^\gamma \equiv \alpha^{(\beta\gamma)}$.
3. $\alpha \leq \beta$ *implies* $\alpha^\gamma \leq \beta^\gamma$.
4. $\alpha \leq \beta$ *implies* $\gamma^\alpha \leq \gamma^\beta$.
5. $2^\omega \equiv \omega$.
6. $\omega^\alpha \equiv 2^{(\omega\alpha)}$.

Proof. Theorems 2.3, 2.4, and 2.5 of [6] ⊣

The following useful partial converse to clause 4 of Theorem 32 is provable in ACA_0.

Theorem 33. (ACA_0) *If α and β are well orderings and $\omega^\alpha \leq_w \omega^\beta$, then $\alpha \leq_w \beta$.*

PROOF. Lemma 4.3 of [6] ⊣

To conclude this section, we list two results on the existence and uniqueness of logarithms. The next result holds with both strong and weak comparability.

THEOREM 34. ($\mathsf{RCA_0}$) *The following are equivalent:*

1. $\mathsf{ATR_0}$.
2. (Existence of logarithms) *If $\alpha > 1$ and β are well orderings, then there is a well ordering γ such that $\alpha^\gamma \le \beta < \alpha^{\gamma+1}$.*

PROOF. Theorem 2.7 of [6]. ⊣

THEOREM 35. ($\mathsf{ATR_0}$) (Uniqueness of logarithms) *If α, β, γ and δ are well orderings such that $\alpha^\gamma \le_w \beta <_w \alpha^{\gamma+1}$, $\alpha^\delta \le_w \beta <_w \alpha^{\delta+1}$, then $\gamma \equiv_s \delta$.*

PROOF. Theorem 2.8 of [6]. ⊣

§6. Other topics. This section contains formalized versions of Cantor's normal form theorem, several results about indecomposable well orderings, and some transfinite induction schemes. In the last paragraph, we list some related topics in reverse mathematics which were not included in this survey. We will begin with two normal form results.

THEOREM 36. ($\mathsf{RCA_0}$) *Let $\alpha > 1$ and β be well orderings. Fix an element of α^β, $x_0 = \langle (b_0, a_0), \dots, (b_n, a_n)\rangle$. Let $\mu = \{x \in \alpha^\beta \mid x < x_0\}$. For each $i \le n$, let $\beta_i = \{b \in \beta \mid b < b_i\}$ and $\alpha_i = \{a \in \alpha \mid a < a_i\}$. Then*

$$\mu \equiv_s \alpha^{\beta_0}\alpha_0 + \alpha^{\beta_1}\alpha_1 + \cdots + \alpha^{\beta_n}\alpha_n.$$

PROOF. Lemma 5.1 of [6]. ⊣

The next theorem holds with both forms of comparability. Recall that the notation $\alpha < \beta$ is used to abbreviate $\alpha + 1 \le \beta$.

THEOREM 37. ($\mathsf{RCA_0}$) *The following are equivalent:*

1. $\mathsf{ATR_0}$.
2. *If $\alpha > 1$ and β are well orderings then there are finite sequences of well orderings $\gamma_0 > \gamma_1 > \cdots > \gamma_n$ and $\delta_0, \delta_1, \dots, \delta_n$ such that $0 < \delta_i < \alpha$ for each $i \le n$ and*

   $$\beta \equiv \alpha^{\gamma_0}\delta_0 + \alpha^{\gamma_1}\delta_1 + \cdots + \alpha^{\gamma_n}\delta_n.$$

   Furthermore, this representation is unique in the following sense. If $\alpha^{\gamma'_0}\delta'_0 + \alpha^{\gamma'_1}\delta'_1 + \cdots + \alpha^{\gamma'_m}\delta'_m$ is a similar representation of β, then $m = n$ and for every i, $\gamma'_i \equiv \gamma_i$ and $\delta'_i \equiv \delta_i$.
3. (Cantor's normal form theorem) *If β is a well ordering then there is a finite sequence $\gamma_0 > \gamma_1 > \cdots > \gamma_n$ of well orderings and a finite collection $d_0, \dots d_n$ of positive integers such that*

   $$\beta \equiv \omega^{\gamma_0}d_0 + \cdots + \omega^{\gamma_n}d_n.$$

PROOF. Theorem 5.2 of [6]. ⊣

Recall that a well ordering α is *indecomposable* if for every final segment β of α, we have $\alpha \leq_w \beta$. The next theorem lists useful properties of indecomposable well orderings that are provable in RCA_0. Clause 1 is a particularly handy fact. Note that in clause 5, the hypothesis must include the requirement that ω^α is well ordered. For some well ordered sets α, ACA_0 may be required to prove that ω^α is well ordered, as shown in Theorem 31.

THEOREM 38. (RCA_0) *Suppose that α, β, and γ are well orderings.*

1. *If α is indecomposable and $\alpha <_w \beta + \gamma$, then $\alpha \leq_w \beta$ or $\alpha <_w \gamma$.*
2. *If $\alpha \equiv_w \beta$, then α is indecomposable if and only if β is indecomposable.*
3. *If $\alpha\beta \equiv_w \gamma$ then γ is indecomposable if and only if β is indecomposable.*
4. *If β is indecomposable and $\alpha <_w \beta$, then $\alpha\omega \leq_w \beta$.*
5. *If ω^α is well ordered, then it is indecomposable.*

PROOF. Clause 1 is Lemma 3.3 of [3]. Clauses 2 through 5 are Theorems 3.2 through 3.5 of [6]. ⊣

Finally, we turn our attention to a few transfinite induction schemes. Let ATI_0 denote the scheme

$$[\forall x \in \alpha(\forall y \in \alpha(y < x \to \psi(y)) \to \psi(x))] \to \forall x \in \alpha\ \psi(x),$$

where $\psi(x)$ is an arithmetical formula and α is a well ordering. This scheme can be modified by restricting the complexity of $\psi(x)$. For example, $\Sigma^0_1 - \mathsf{TI}_0$ denotes ATI_0 with $\psi(x)$ restricted to Σ^0_1 formulas. Similarly, let $\Pi^0_1 - \mathsf{TLE}_0$ denote the transfinite least element scheme

$$[\exists x \in \alpha\ \psi(x)] \to \exists x \in \alpha(\psi(x) \land \forall y \in \alpha(y < x \to \neg\psi(y)))$$

where α is a well ordering and $\psi(x)$ is Π^0_1. Schema for $\Pi^0_j - \mathsf{TLE}_0$ and $\Sigma^0_j - \mathsf{TLE}_0$ are defined similarly. We have two theorems relating these schemes.

THEOREM 39. RCA_0 *proves* $\Pi^0_1 - \mathsf{TI}_0$ *and* $\Sigma^0_1 - \mathsf{TLE}_0$.

PROOF. Lemma 6 of [10]. ⊣

THEOREM 40. *For $j \geq 2$ and $k \geq 1$,* RCA_0 *proves that the following are equivalent:*

1. ACA_0.
2. ATI_0.
3. $\Sigma^0_k - \mathsf{TI}_0$.
4. $\Pi^0_k - \mathsf{TLE}_0$.
5. $\Pi^0_j - \mathsf{TI}_0$.
6. $\Sigma^0_j - \mathsf{TLE}_0$.

PROOF. Simpson proves that ACA_0 implies ATI_0 in Lemma V.2.1 of [13]. The least element schemes can be deduced from ATI_0 by the usual arguments. Corollary 3 and Corollary 4 of [8] provide the reversals. ⊣

This survey only addresses well orderings and the formalized operations of ordinal arithmetic. It does not include references to a substantial body of literature related to this topic. For example, the formalization of ordinal notations in reverse mathematics, proof theoretic ordinals for the subsystems, and various applications of ordinal arithmetic to algebra, analysis, topology, graph theory and quasi-ordering theory have all been omitted. As always, [13] is strongly recommended as a source for further reading and references.

The interested reader will be able to discover numerous open questions on the reverse mathematics of ordinal arithmetic. Most of the results in this survey were motivated by Cantor's introductory articles [1]. Many topics in Chapter XIV of Sierpiński's book [12] could be formalized in second order arithmetic and analyzed. Sometimes even a single exercise can motivate a substantial number of new results.

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