Reverse Mathematics of Separably Closed Sets

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Abstract

This paper contains a corrected proof that the statement "every non-empty closed subset of a compact complete separable metric space is separably closed" implies the arithmetical comprehension axiom of reverse mathematics.

The result described in the abstract appears as part of Theorem 3.3 of [2]. It is also included in [1] as part of the proof of Theorem 1.15 and in the statement of Theorem 3.12. Professor Brown generously assisted in checking the details of this new corrected proof.

For a comprehensive development of separable metric space theory in reverse mathematics, [4] is an excellent resource. The subsystems referred to below are RCA_0, which includes recursive comprehension, and ACA_0, which includes arithmetical comprehension. In RCA_0 it is useful to distinguish between closed sets and separably closed sets. Closed sets are complements of unions of open balls, and are encoded by the codes for their complements. Separably closed sets are closures of countable sets of points, and are encoded by these countable point sets. In the following theorem, compactness is as in Definition III.2.3 of [4], not Heine-Borel compactness.

Theorem 1. (RCA_0) *The following are equivalent:*

(1) ACA_0.

(2) *Every non-empty closed subset of a compact complete separable metric space is separably closed.*

(3) *Every non-empty closed subset of* $[0, 1]$ *is separably closed.*

Proof. The proof that (1) implies (2) is Theorem 3.2 of [2]. Since RCA_0 proves that $[0, 1]$ is a compact complete separable metric space, (3) follows directly from (2). To complete the argument, we prove that (3) implies (1), correcting the proofs of [1] and [2]. We will work in RCA_0.

To show that (3) implies (1), by Theorem III.2.2 of [4] it suffices to use (3) to deduce the monotone convergence theorem for sequences of rationals in $(0, 1)$. Let $\langle a_i : i \in \mathbb{N}\rangle$ be an increasing sequence of rationals in $(0, 1)$. Define

$B_i = [0, a_i)$ for $i \geq 0$. Each B_i is open relative to the usual topology on $[0,1]$. Let C be the closed subset of $[0,1]$ coded by $\{B_i \ : \ i \geq 0\}$. Intuitively, C is a closed interval with $\lim_n a_n$ as its lower endpoint. We will use statement (3) to prove the existence of this endpoint.

Since $1 \in C$, we know that C is nonempty. Applying (3), we can find $S = \langle x_k \ : \ k \in \mathbb{N}\rangle$ such that $\bar{S} = C$. By Theorem 1 of [3], RCA_0 can prove the existence of a sequence $\langle b_k \ : \ k \in \mathbb{N}\rangle$ such that for each $k \in \mathbb{N}$, $b_k = \min\{x_j \ : \ j \leq k\}$. Thus for all n, $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$. We claim that $\lim_n |a_n - b_n| = 0$. To see this, let $\epsilon > 0$ and choose j so large that $2^{-j} < \min\{\epsilon/2, a_0\}$. By bounded Σ^0_1 comprehension (which is provable in RCA_0, see Theorem II.3.9 in [4]), the set

$$G = \{F \subseteq \{1,2,\ldots,j\} \ : \ \exists n(\textstyle\sum_{i\in F} 2^{-i} \leq a_n)\}$$

exists. Since G is a finite collection, the set $\{\sum_{i\in F} 2^{-i} \ : \ F \in G\}$ has a maximum element. By the definition of G, we can find a k such that a_k is greater than or equal to this maximum. By maximality, we must have $a_k + 2^{-j} \in C$. Since $C = \bar{S}$, there is a b_m such that $|a_k + 2^{-j} - b_m| < 2^{-j}$. If $n \geq \max\{k, m\}$ then

$$|a_n - b_n| \leq |a_k - b_m| \leq |a_k - (a_k + 2^{-j})| + |a_k + 2^{-j} - b_m| < 2^{-j} + 2^{-j} < \epsilon,$$

proving that $\lim_n |a_n - b_n| = 0$. We have verified all the hypotheses of the nested interval completeness theorem (Theorem II.4.8 of [4]). Consequently, $\lim_n a_n$ exists, completing the proof of the monotone convergence theorem and implying ACA_0. $\square$

References

[1] Douglas K. Brown. *Functional analysis in weak subsystems of second order arithmetic*. PhD thesis, The Pennsylvania State University, 1987.

[2] Douglas K. Brown. Notions of closed subsets of a complete separable metric space in weak subsystems of second order arithmetic. In Wilfried Sieg, editor, *Logic and computation (Pittsburgh, PA, 1987)*, volume 106 of *Contemporary Mathematics*, pages 39–50. Amer. Math. Soc., Providence, RI, 1990.

[3] Jeffry L. Hirst. Minima of initial segments of infinite sequences of reals. *Mathematical Logic Quarterly*, 50:47–50, 2004.

[4] Stephen G. Simpson. *Subsystems of second order arithmetic*. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1999.