Ramsey's Theorem on Trees

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Preliminary report on joint work with:
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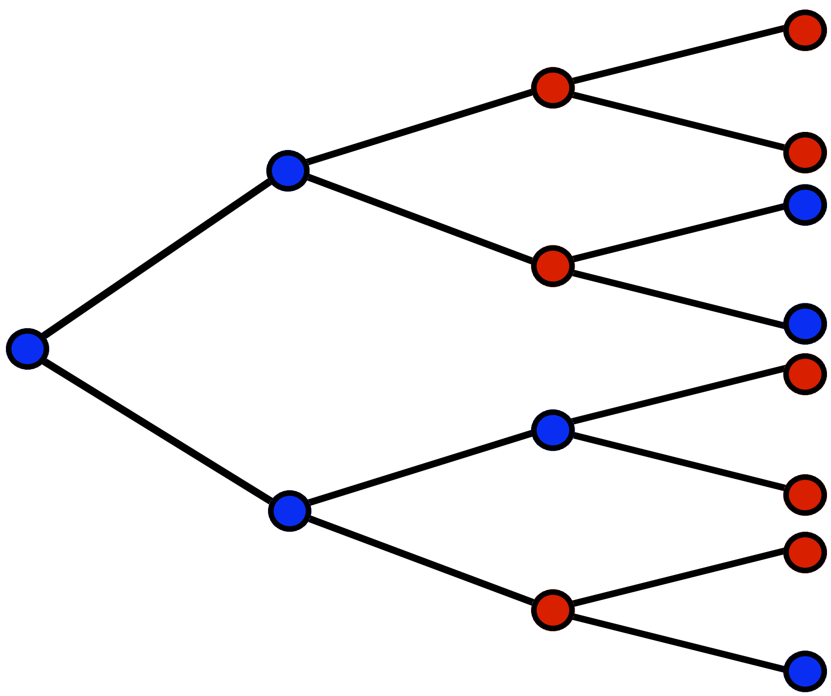
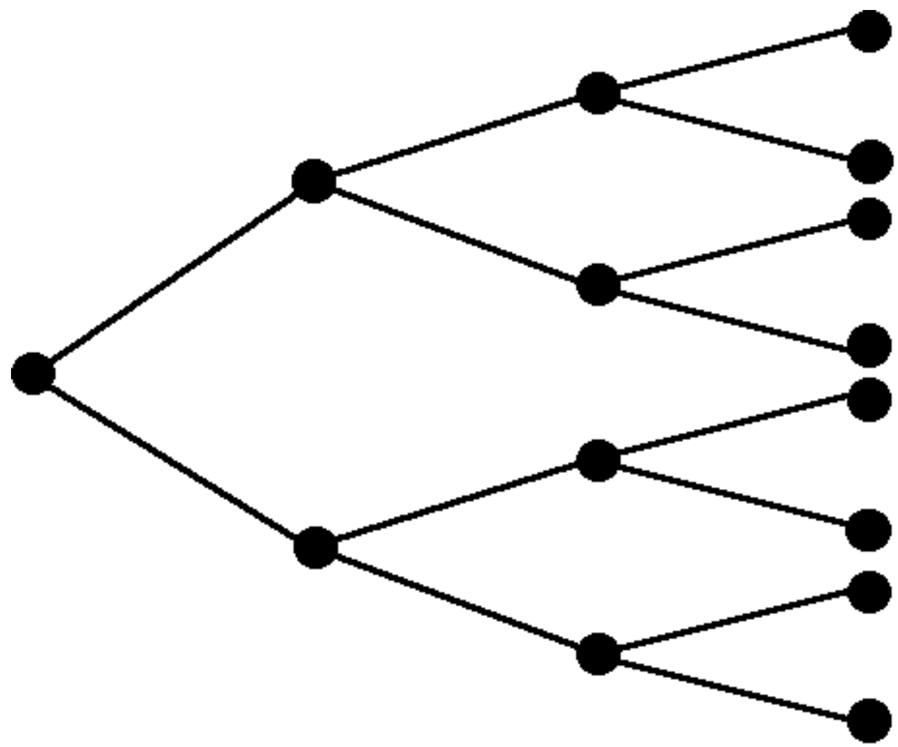
These slides are available at:
`www.mathsci.appstate.edu/~jlh`

Some combinatorics

Theorem 1. $\mathsf{TT}^1_{<\omega}$: *For any finite coloring of* $2^{<\mathbb{N}}$, *there is a monochromatic subtree order-isomorphic to* $2^{<\mathbb{N}}$.



Theorem 2. $\mathsf{TT}^{\mathsf{n}}_{<\omega}$: *For any finite coloring of the n-tuples of comparable nodes in $2^{<\mathbb{N}}$, there is a color and a subtree order-isomorphic to $2^{<\mathbb{N}}$ in which all n-tuples of comparable nodes have the specified color.*



Related combinatorial results

Deuber, Prömel, and Voigt:
A canonical partition theorem for chains in regular trees

Includes $\mathsf{TT}^1_{<\omega}$. For higher exponents, modifies definition of monochromatic in order to use graph-isomorphic subtrees. (Thanks to Ali Enayat for this reference.)

Milliken
A partition theorem for the infinite subtrees of a tree
and
A Ramsey theorem for trees

Subtrees are colored. Monochromatic defined in terms of strongly embeddable subtrees.

Theorem 3. ($\mathsf{RCA_0}+\Sigma^0_2-\mathsf{IND}$) *For all* k, TT^1_k. *That is, for any finite coloring of* $2^{<\mathbb{N}}$, *there is a monochromatic subtree isomorphic to* $2^{<\mathbb{N}}$.

Proof concept:

$s_1, \ldots, s_{2^k-1}$: nonempty subsets of colors listed in increasing cardinality

$\exists n \exists \tau \forall \sigma (\sigma$ extends $\tau \rightarrow$color of σ is in $s_n)$

Pick least such n, using Σ^0_2 least element principle.

Theorem 4. ($\mathsf{ACA_0}$) *For all k, TT^2_k. That is, for any finite coloring of pairs of comparable nodes of $2^{<\mathbb{N}}$, there is a monochromatic subtree isomorphic to $2^{<\mathbb{N}}$.*

Theorem 5. ($\mathsf{ACA_0}$) *For all $n \geq 1$, $\mathsf{TT}^\mathsf{n}_{<\omega}$ implies $\mathsf{TT}^{\mathsf{n+1}}_{<\omega}$.*

Theorem 6. *For $n \geq 3$ and $k \geq 2$, $\mathsf{RCA_0}$ proves that the following are equivalent:*

(1) $\mathsf{ACA_0}$.

(2) $\mathsf{TT}^\mathsf{n}_{<\omega}$.

(3) $\mathsf{TT}^\mathsf{n}_\mathsf{k}$.

Computability theory

In *Ramsey's theorem and recursion theory*, Carl Jockusch proves:

> Corollary 3.2: There exists a recursive basic partition P such that $H(P)$ contains no Σ^0_2 set.
>
> Theorem 4.2: If P is a recursive partition of all pairs of integers into p classes, then $H(P)$ contains a Π^0_2 set.
>
> Theorem 5.1: If $n \geq 2$, there exists a recursive partition P of $[N]^n$ into two classes such that $H(P)$ contains no set recursive in 0^{n-1} and hence no Σ^0_n set.
>
> Theorem 5.5: If P is a recursive partition of $[N]^n$ into finitely many classes, then $H(P)$ contains a Π^0_n set.

Note: The Σ^0_2 bounds are free in the tree case.

Π^0_n bounds for trees

Theorem 7. *Every computable finite coloring of pairs of comparable nodes of* $2^{<\mathbb{N}}$ *has a* Π^0_2 *monochromatic subtree that is isomorphic to* $2^{<\mathbb{N}}$*.*

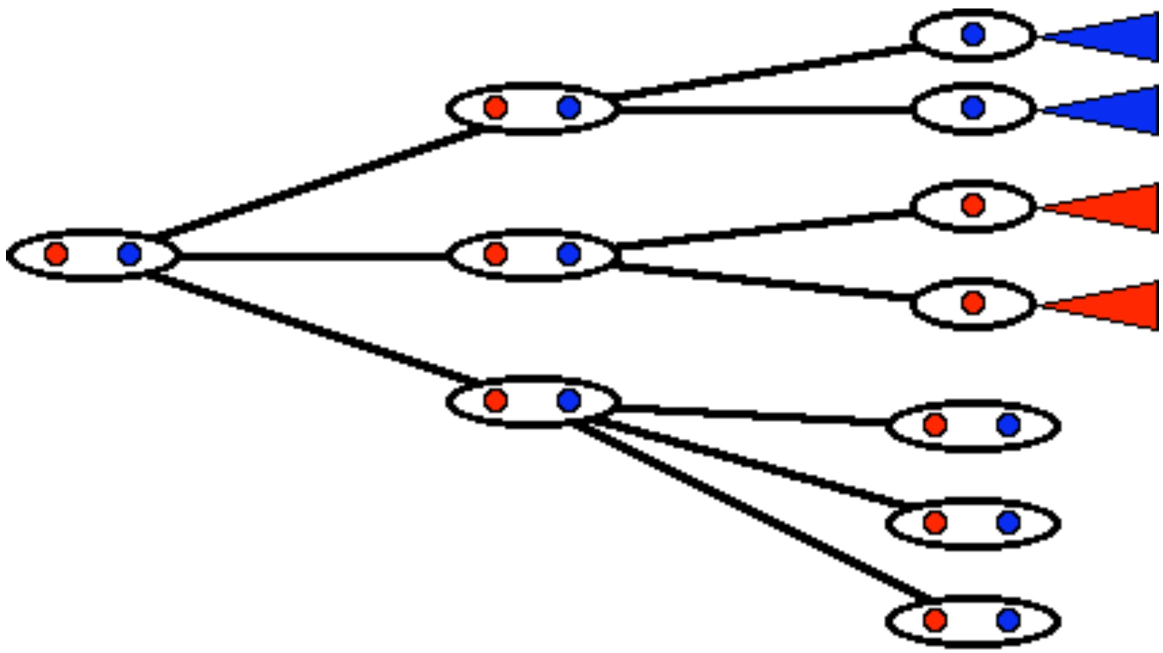
Theorem 8. *Every computable finite coloring of* n*-tuples of comparable nodes of* $2^{<\mathbb{N}}$ *has a* Π^0_n *monochromatic subtree that is isomorphic to* $2^{<\mathbb{N}}$*.*

Proof strategy: Emulate Carl.

Picture from the proof of the Π^0_n bound

color blocks:

$j+1$ chains of length j above which j colors appear



Questions

1. Does $\mathsf{TT}^1_{<\omega}$ imply $\Sigma^0_2 - \mathsf{IND}$?

2. Does TT^2_2 imply $\mathsf{ACA_0}$?
   (Can someone emulate Seetapun?)

3. Does TT^2_2 imply $\mathsf{TT}^2_{<\omega}$?
   (Can someone emulate Cholak, Jockusch, and Slaman?)

4. What's so special about binary trees?
   (Answer: Nothing. ω^ω can be embedded in $2^{<\mathbb{N}}$)

More questions

1. Can the following be formulated for trees? What are the corresponding reverse mathematics and computability theoretic results?
   - Hindman's theorem
   - Erdös-Rado theorem
   - Coh
   - Stable Ramsey theorem
   - Free set theorem, thin set theorem, etc.
2. What's so special about trees? What about other partial orders?

References

[1] W. Deuber, H. J. Prömel, and B. Voigt, *A canonical partition theorem for chains in regular trees*, Combinatorial theory (Schloss Rauischholzhausen, 1982), Lecture Notes in Math., vol. 969, Springer, Berlin, 1982, pp. 115–132. MR **692237 (84k:**05066)

[2] Keith R. Milliken, *A partition theorem for the infinite subtrees of a tree*, Trans. Amer. Math. Soc. **263** (1981), no. 1, 137–148. MR **590416 (82g:**04003)

[3] Carl G. Jockusch Jr., *Ramsey's theorem and recursion theory*, J. Symbolic Logic **37** (1972), 268–280. MR 0376319 (51 #12495)

[4] Jennifer Chubb, Jeff Hirst, and Tim McNichol, *Reverse mathematics and partitions of trees (draft)*, available at `www.mathsci.appstate.edu/~jlh/pdf/rt.pdf`.