How hard is it to prove that $\sqrt{2}$ is irrational?

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A first order proof by contradiction

Suppose $(\frac{m}{n})^2 = 2$ where $m, n \in \mathbb{N}$.

$$m^2 = 2n^2$$

If $m = 2^k m_0$ and $n = 2^j n_0$ with m_0 and n_0 odd,

$$(2^k m_0)^2 = 2(2^j n_0)^2$$

$$2^{2k} m_0^2 = 2^{2j+1} n_0^2$$

So $2k = 2j + 1$, a contradiction!

It is easy to prove $\sqrt{2} \notin \mathbb{Q}$ in a first order setting, using some number theory.

First order vs. second order

First order arithmetic formulas use (only) quantifiers over natural numbers.

$$\forall m \forall n \left(\left(\frac{m}{n} \right)^2 \neq 2 \right)$$

Translation: if m/n is a rational, its square is not 2.

Second order arithmetic formulas use quantifiers over sets of natural numbers (and objects coded by sets).

$$\forall \alpha \in \mathbb{R} \, (\alpha^2 = 2 \to \forall m \forall n (\frac{m}{n} \neq \alpha))$$

Translation: If α is a real and $\alpha = \sqrt{2}$, then α is not rational.

Coding of reals

In second order arithmetic, reals are coded by rapidly converging sequences of rationals.

If $\alpha = \langle q_0, q_1, q_2, \ldots \rangle$ codes a real, then $\forall n |q_n - q_{n+j}| \leqslant 2^{-n}$.

Many different sequences could be used to code $\sqrt{2}$. For example:

$$\sqrt{2} = \alpha = \langle 1, 1.4, 1.41, 1.414, \ldots \rangle$$

$$\begin{aligned}\sqrt{2} = \beta &= \langle 1, \frac{3}{2}, \frac{17}{12}, \frac{577}{408}, \frac{665857}{470832}, \ldots \rangle \\ &= \langle 1, 1.5, 1.41\bar{6}, 1.414215, 1.414233562, \ldots \rangle\end{aligned}$$

A conjecture

RCA_0 is an axiom system for second order arithmetic including:

- basic arithmetic axioms,
- induction for some simple formulas,
- an existence axiom for computable sets of natural numbers.

Conjecture: RCA_0 can prove the following:

- The sequence $\beta = \langle 1, \frac{3}{2}, \frac{17}{12}, \frac{577}{408}, \frac{665857}{470832}, \ldots \rangle$ exists and codes a real number.
- $\beta^2 = 2$.
- $\forall m \forall n \, (\beta \neq \frac{m}{n})$

Summarizing, RCA_0 can prove that $\sqrt{2}$ is irrational.

The larger project

Hynek Mlcousek asked on the FOM listserve:
What axioms are needed to prove that π and e are irrational?

Joey Seevers is working on e.
Nicholas Beitzell is working on π.

- What about other irrational algebraic numbers?
- What about other irrational transcendental numbers (e.g. π^e or e^π)?
- How much induction is used in these proofs?
- How does the choice of the representing sequences affect the difficulty of the proofs?
- How hard is it to prove that π (or e, etc.) is transcendental?

References

More about reverse mathematics:

[1] Stephen G. Simpson, *Subsystems of second order arithmetic*, 2nd ed., Perspectives in Logic, Cambridge University Press and ASL, 2009. DOI 10.1017/CBO9780511581007. MR2517689

More about representations of reals:

[2] Jeffry L. Hirst, *Representations of reals in reverse mathematics*, Bull. Pol. Acad. Sci. Math. **55** (2007), no. 4, 303–316. DOI 10.4064/ba55-4-2. MR2369116

More about proofs with weak induction axioms:

[3] Caleb Davis, Denis R. Hirschfeldt, Jeffry Hirst, Jake Pardo, Arno Pauly, and Keita Yokoyama, *Combinatorial principles equivalent to weak induction*, Computability **9** (2020), no. 3-4, 219–229. DOI 10.3233/com-180244. MR4133714