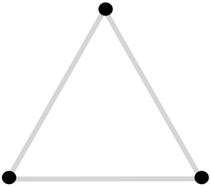
Favorite theorems of the dearly departed: Ramsey’s Theorem for pairs!

Jeff Hirst
Professor Emeritus
Appalachian State University
Boone, NC

March 27, 2026

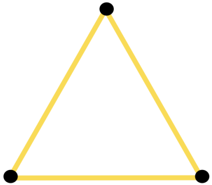
ASU Mathematical Sciences Colloquium

Two coloring edges of complete graphs



K3: A complete graph with three vertices

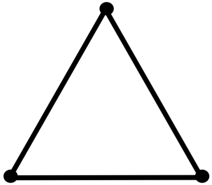
Two coloring edges of complete graphs



K3: A complete graph with three vertices

Edges colored the same. A monochromatic graph

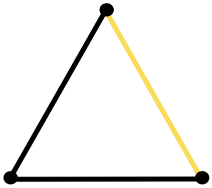
Two coloring edges of complete graphs



K3: A complete graph with three vertices

Edges colored the same. Another monochromatic graph.

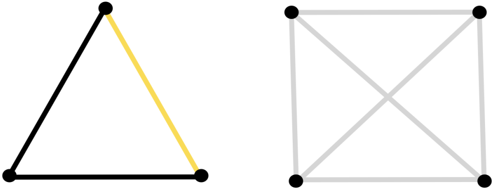
Two coloring edges of complete graphs



K3: A complete graph with three vertices

A non-monochromatic two coloring.

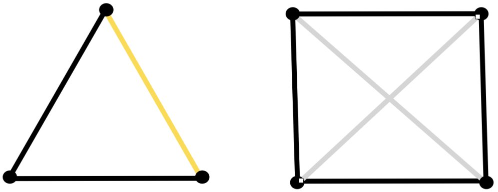
Two coloring edges of complete graphs



K4: A complete graph with four vertices.

Can we two color K4 so that it contains no monochromatic K3s?

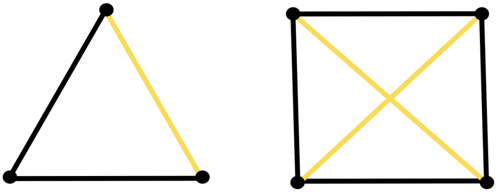
Two coloring edges of complete graphs



K4: A complete graph with four vertices.

Can we two color K4 so that it contains no monochromatic K3s?

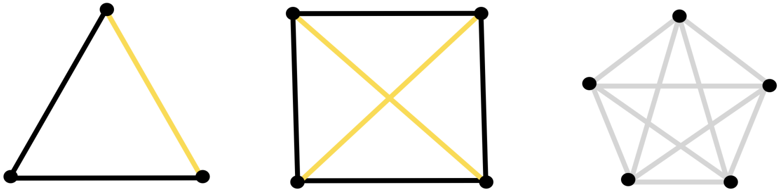
Two coloring edges of complete graphs



K4: A complete graph with four vertices.

Can we two color K4 so that it contains no monochromatic K3s? Yes!

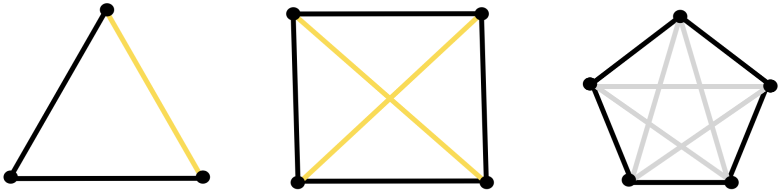
Two coloring edges of complete graphs



How about K5?

Is there a two coloring with no mono. K3s?

Two coloring edges of complete graphs

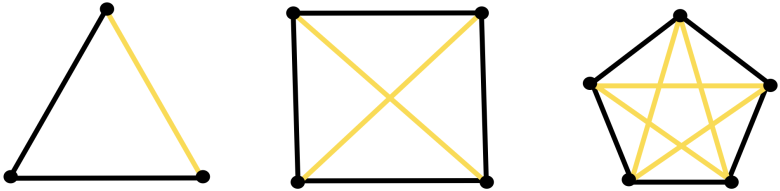


How about K5?

Is there a two coloring with no mono. K3s?

Try the old strategy.

Two coloring edges of complete graphs



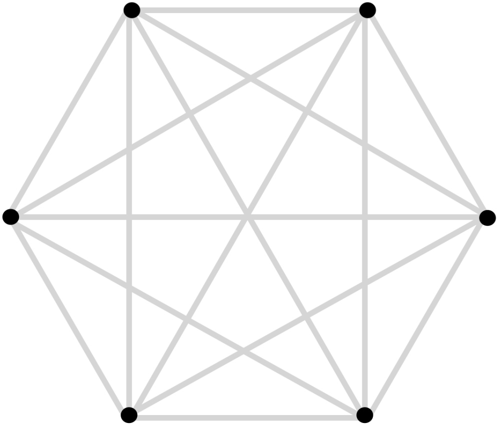
How about K5?

Is there a two coloring with no mono. K3s?

Try the old strategy. It works!

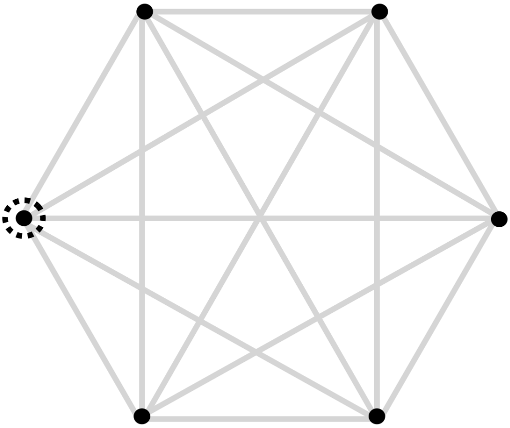
Two coloring K6

Every two coloring of K6 contains a monochromatic K3.



Two coloring K6

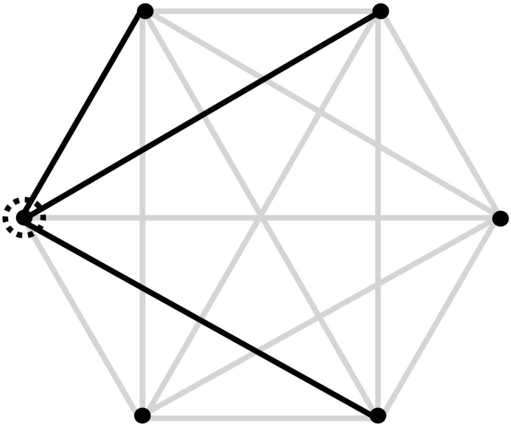
Every two coloring of K6 contains a monochromatic K3.



Concentrate on one vertex.

Two coloring K6

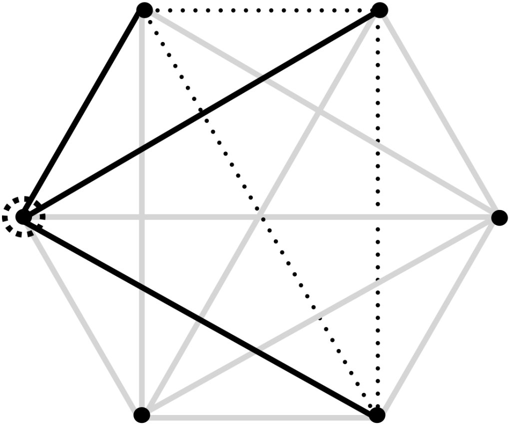
Every two coloring of K6 contains a monochromatic K3.



Three of the five edges must match.

Two coloring K6

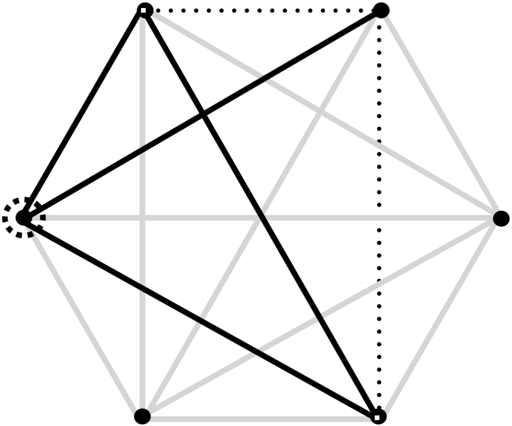
Every two coloring of K6 contains a monochromatic K3.



Consider the edges at the far endpoints.

Two coloring K6

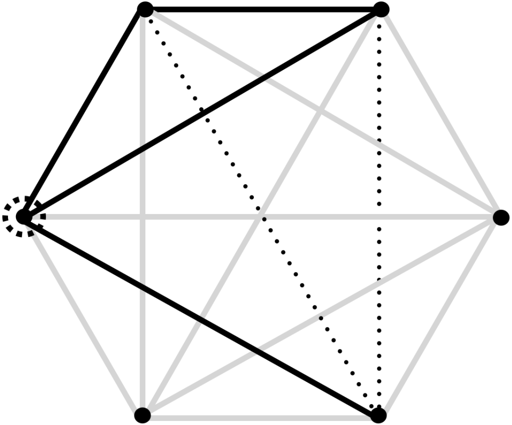
Every two coloring of K6 contains a monochromatic K3.



If one is colored black, there is a monochromatic K3.

Two coloring K6

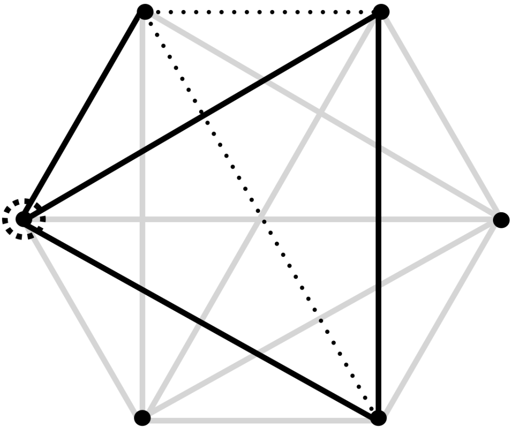
Every two coloring of K6 contains a monochromatic K3.



If one is colored black, there is a monochromatic K3.

Two coloring K6

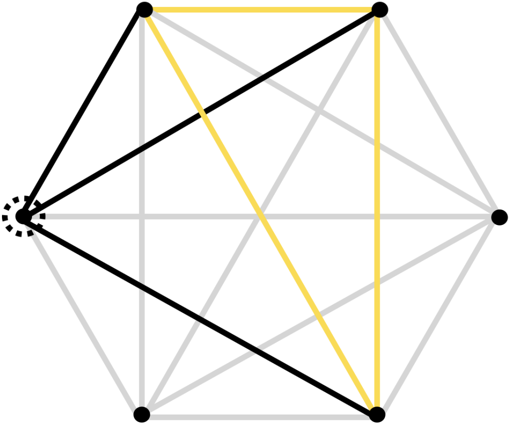
Every two coloring of K6 contains a monochromatic K3.



If any one is colored black, there is a monochromatic K3.

Two coloring K6

Every two coloring of K6 contains a monochromatic K3.



If none are black, there is a gold K3.

Classical Ramsey numbers

We know that:

Every two coloring of K6 contains a monochromatic K3.

Every two coloring of Kn for n$\geqslant$ 6 has this property.

Also, 6 is the least number with this property.

We say the classical Ramsey number $r_2(3)$ equals 6.

Classical Ramsey numbers

We know that:

Every two coloring of K6 contains a monochromatic K3.

Every two coloring of Kn for n$\geqslant$ 6 has this property.

Also, 6 is the least number with this property.

We say the classical Ramsey number $r_2(3)$ equals 6.

What about monochromatic K4s?

$r_2(4) = 18$. Consequently, every two coloring of K18 contains a monochromatic K4, but there is a two coloring of K17 with no monochromatic K4s.

A finite Ramsey theorem

The finite Ramsey theorem (for graphs and two colors) says that for every m there is an n such that $r_2(m) = n$.

A finite Ramsey theorem

The finite Ramsey theorem (for graphs and two colors) says that for every m there is an n such that $r_2(m) = n$.

This can be proved using an induction argument, or by applying the following infinite Ramsey theorem.

RT(2,2): Every two coloring of $K\mathbb{N}$ contains a monochromatic $K\mathbb{N}$.

Ideas from the proof of RT(2,2)

Suppose we have a two coloring of $K\mathbb{N}$.

0 1 2 3 4 5 6 7 8 9 10 11 12 ...

We have a vertex for each natural number.

Ideas from the proof of RT(2,2)

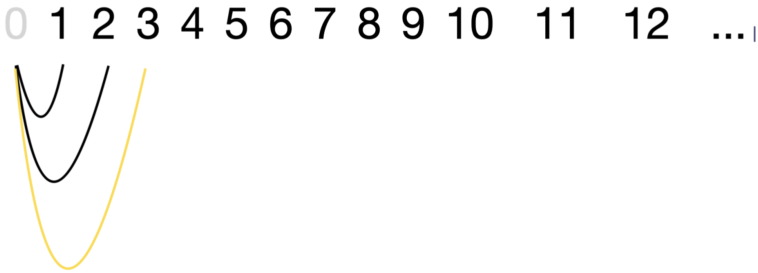
Suppose we have a two coloring of $K\mathbb{N}$.

0 1 2 3 4 5 6 7 8 9 10 11 12 ...

Consider edges connected to 0.

Ideas from the proof of RT(2,2)

Suppose we have a two coloring of K$\mathbb{N}$.



Consider edges connected to 0.

Ideas from the proof of RT(2,2)

Suppose we have a two coloring of $K\mathbb{N}$.

0 1 2 3 4 5 6 7 8 9 10 11 12 ...

Color the integers to match the edges.

Ideas from the proof of RT(2,2)

Suppose we have a two coloring of $K\mathbb{N}$.

0 1 2 3 4 5 6 7 8 9 10 11 12 ...

If infinitely many are black, color 0 black.

Ideas from the proof of RT(2,2)

Suppose we have a two coloring of K$\mathbb{N}$.

0 1 2 3 4 5 6 7 8 9 10 11 12 ...

1 2 4 6 7 9 10 12 ...

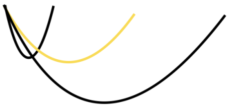
Select the integers corresponding to black edges.

Ideas from the proof of RT(2,2)

Suppose we have a two coloring of $K\mathbb{N}$.

0 1 2 3 4 5 6 7 8 9 10 11 12 ...

1 2 4 6 7 9 10 12 ...



Repeat the process with the integers in the new list.

Ideas from the proof of RT(2,2)

Suppose we have a two coloring of K$\mathbb{N}$.

0 1 2 3 4 5 6 7 8 9 10 11 12 ...

1 2 4 6 7 9 10 12 ...

Color the new list based on edges connected to 1.

Ideas from the proof of RT(2,2)

Suppose we have a two coloring of $K_{\mathbb{N}}$.

0 1 2 3 4 5 6 7 8 9 10 11 12 ...

1 2 4 6 7 9 10 12 ...

4 9 10 12 ...

If infinitely many are gold, color 1 gold and thin.

Ideas from the proof of RT(2,2)

Suppose we have a two coloring of $K\mathbb{N}$.

0 1 2 3 4 5 6 7 8 9 10 11 12 ...

1 2 4 6 7 9 10 12 ...

4 9 10 12 ...

Using edges connected to 4 color. Color 4 appropriately.

Ideas from the proof of RT(2,2)

Suppose we have a two coloring of $K\mathbb{N}$.

0 1 2 3 4 5 6 7 8 9 10 11 12 ...

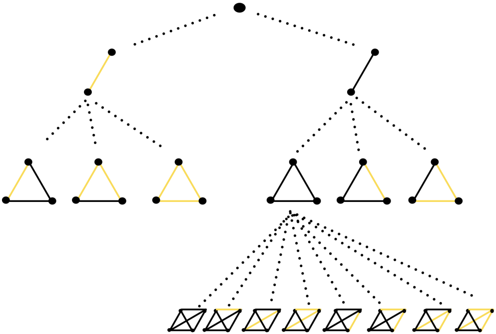
4 9 10 12 ...

9 12 ...

Pick an infinite mono set on the diagonal. It's a mono $K\mathbb{N}$!

Finite Ramsey's theorem from RT(2,2)

Suppose (erroneously) that for every n, some two coloring of Kn has no monochromatic Km. Build a tree of colorings with no monochomatic Km.



An infinite path through the tree describes a two coloring of K$\mathbb{N}$ with no monochromatic Km, and so no monochromatic K$\mathbb{N}$, contradicting RT(2,2).

Known Ramsey numbers

We know that for every m there is a Ramsey number $r_2(m)$.

We have seen that $r_2(3) = 6$ and $r_2(4) = 18$.

It is known that $43 \leqslant r_2(5) \leqslant 46$.

References

[1] F. P. Ramsey, *On a Problem of Formal Logic*, Proc. London Math. Soc. (2) **30** (1929), no. 4, 264–286, DOI 10.1112/plms/s2-30.1.264. MR1576401

[2] Ronald L. Graham, Bruce L. Rothschild, and Joel H. Spencer, *Ramsey theory*, Paperback edition, Wiley Series in Discrete Mathematics and Optimization, John Wiley & Sons, Inc., Hoboken, NJ, 2013. MR3288500

[3] Denis R. Hirschfeldt, *Slicing the truth*, Lecture Notes Series. Institute for Mathematical Sciences. National University of Singapore, vol. 28, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2015. On the computable and reverse mathematics of combinatorial principles; Edited and with a foreword by Chitat Chong, Qi Feng, Theodore A. Slaman, W. Hugh Woodin and Yue Yang. MR3244278