Reverse Mathematics and Banach's Theorem

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Cross-Alps Logic Seminar

Banach's Theorem

In his note *Un théorème sur les transformations biunivoques* [1], Stefan Banach proved the following theorem.

Banach's Theorem: If $f_0 : A \to B$ and $f_1 : B \to A$ are injections then there is a bijection $h : A \to B$ such that for all x, either $h(x) = f_0(x)$ or $f_1(h(x)) = x$ (that is, $h(x) = f^{-1}(x)$).

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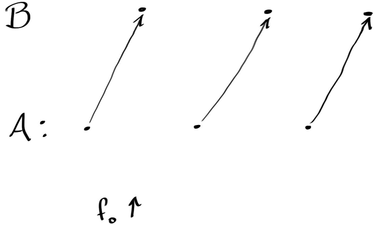

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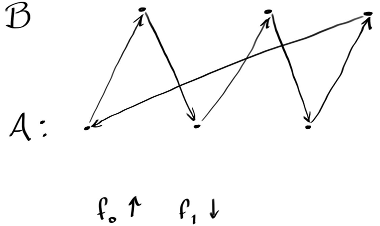

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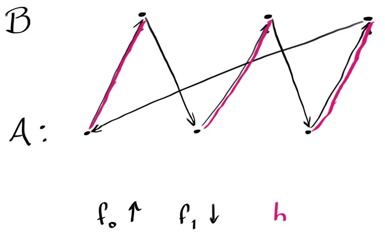

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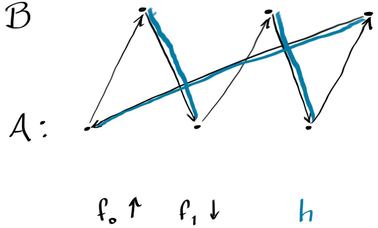

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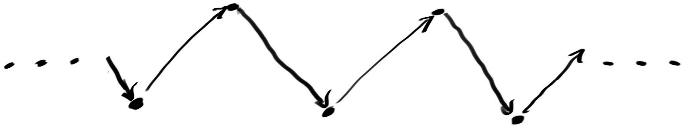

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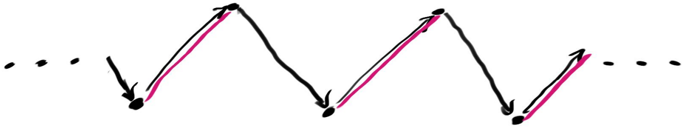

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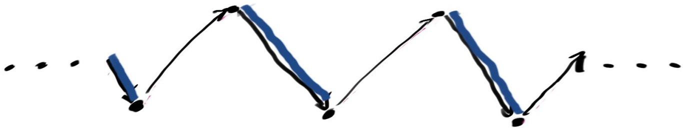

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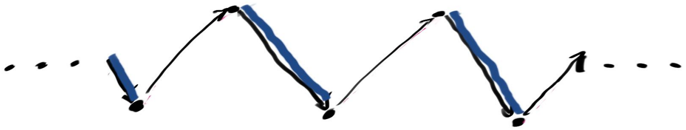

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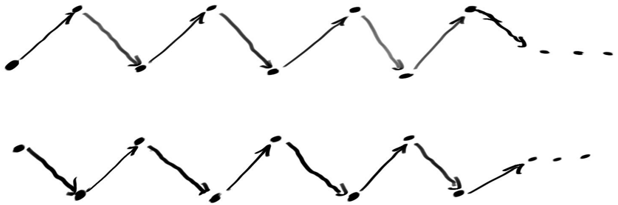

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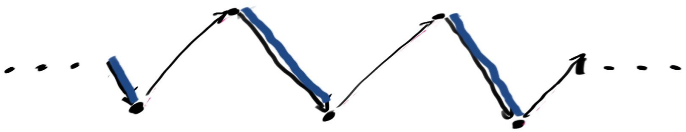

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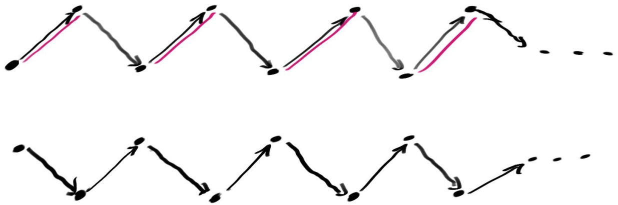

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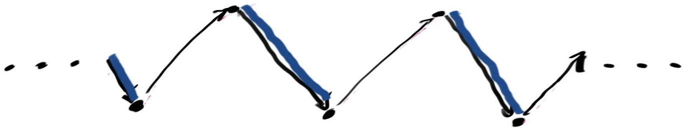

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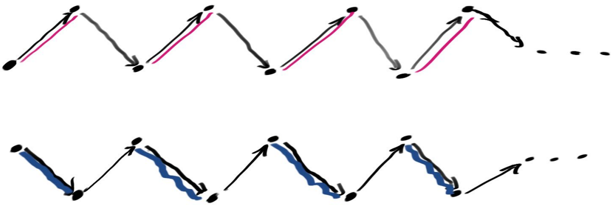


A sketch of a proof of Banach's theorem:

Suppose $f_0 : A \to B$ and $f_1 : B \to A$ are injections.

Every element of A (and B) lies in exactly one component.

For $a \in A$, define:

$$h(a) = \begin{cases} f_1^{-1}(a) & \text{if the component for } a \text{ has a high endpoint} \\ f_0(a) & \text{otherwise} \end{cases}$$

Old reverse mathematics results

We will look at two restrictions of Banach's theorem in subsystems of second order arithmetic, first considered long ago in my dissertation [2] and a related article [3].

Theorem (RCA_0) The following are equivalent:

(1) ACA_0 (Arithmetical comprehension axiom)

(2) If $f_0 : \mathbb{N} \to \mathbb{N}$ and $f_1 : \mathbb{N} \to \mathbb{N}$ are injections then there is a bijection $h : \mathbb{N} \to \mathbb{N}$ such that for all n, $h(n) = f_0(n)$ or $f_1(h(n)) = n$.

Overview of reverse mathematics

Reverse mathematics is a program for the study of the strength of mathematical statements, based on a hierarchy of axiom systems for second order arithmetic. Simpson's book [5] is an excellent reference.

The language includes variables for natural numbers n and sets of natural numbers X, and symbols for arithmetic operations and relations like $m \times (a + b) = c$ and $a \in X$.

The base theory, RCA_0, includes axioms for restricted induction and the recursive comprehension axiom, which (informally) asserts that computable sets (using parameters) exist.

The system ACA_0 consists of RCA_0 plus a comprehension scheme for sets defined by formulas with quantification limited to numbers.

Proving Banach's Theorem in ACA_0

Suppose $f_0 : \mathbb{N} \to \mathbb{N}$ and $f_1 : \mathbb{N} \to \mathbb{N}$ are injections.

For $n \in \mathbb{N}$, the component containing n has a high endpoint if and only if there is a finite sequence $m_0, m_1, m_2, \ldots, m_k$ such that

$$f_1(m_0) = n \quad f_0(m_1) = m_0 \quad \ldots \quad f_1(m_k) = m_{k-1}$$

and

$$\forall j \; f_0(j) \neq m_k$$

which can be written as an arithmetical formula. Thus the function

$$h(n) = \begin{cases} \mu m(f_1(m) = n) & \text{if the component for } n \text{ has a high endpoint} \\ f_0(n) & \text{otherwise} \end{cases}$$

is also defined by an arithmetical formula.

Preparation for a reversal

Now we want to use Banach's theorem to deduce ACA_0.

The main tool for reversals to ACA_0 is Lemma III.1.3 of Simpson [5]:

Lemma (RCA_0) The following are equivalent:

(1) ACA_0

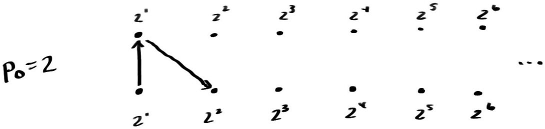
(2) If $g : \mathbb{N} \to \mathbb{N}$ is an injection, then $\{m \mid \exists n\, g(n) = m\}$ exists.

To prove the reversal, given an injection $g : \mathbb{N} \to \mathbb{N}$, we want to compute injections f_0 and f_1 such that if h satisfies Banach's theorem, we can compute the range of g using h.

A sample construction of f_0 and f_1 from g

Suppose the injection g has these values:

n	0	1	2	3
$g(n)$	3	2	4	0



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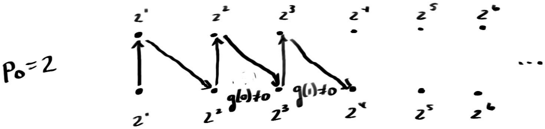

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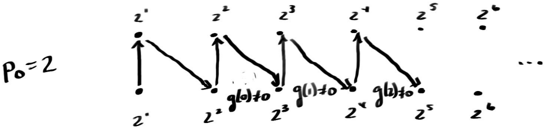

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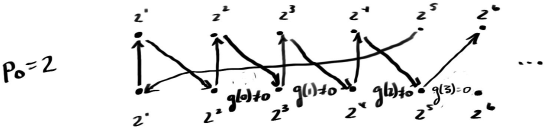

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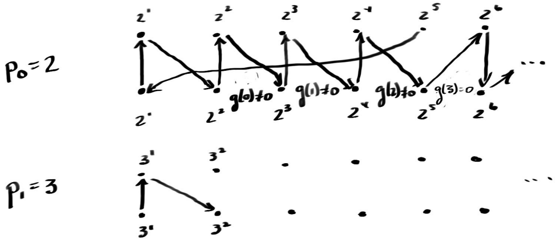

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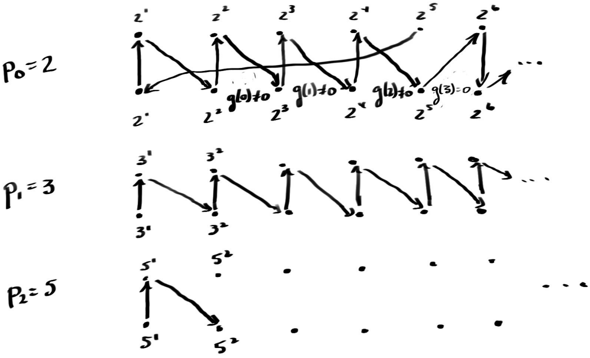

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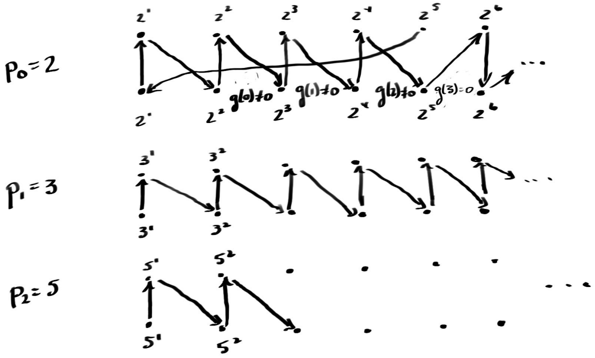

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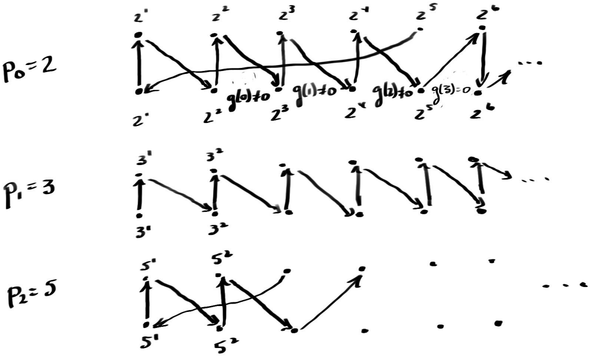

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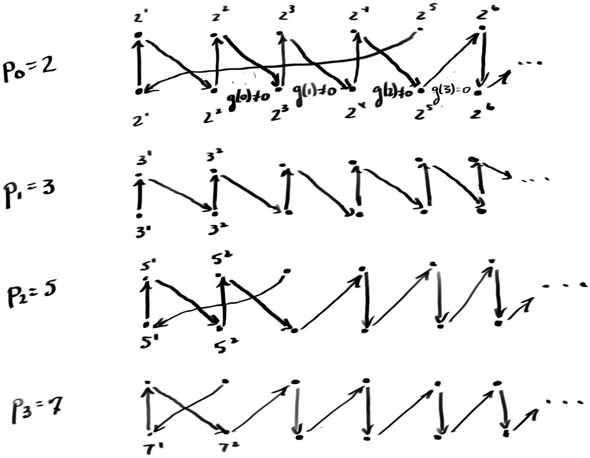

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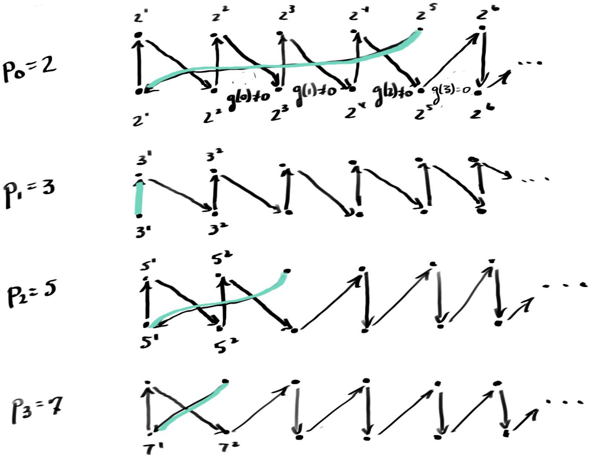

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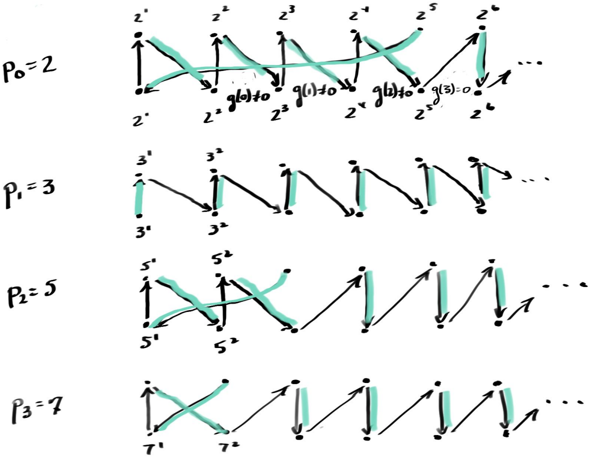

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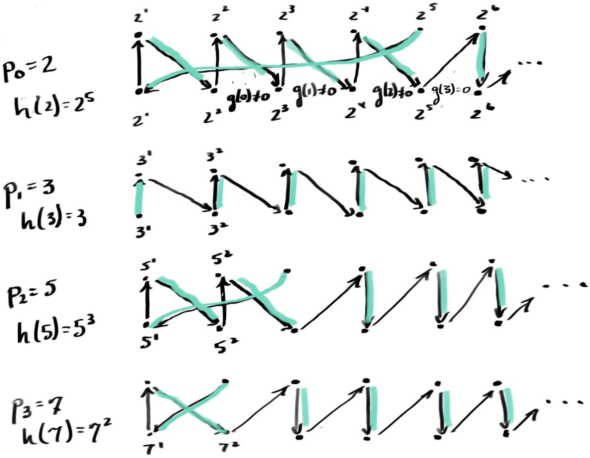

A bounded restriction

Definition: The function $b : \mathbb{N} \to \mathbb{N}$ is a bounding function for $f : \mathbb{N} \to \mathbb{N}$ if for all n, if $\exists t\ (f(t) = n)$ then $(\exists t \leqslant b(n))\ (f(t) = n)$.

Theorem (RCA_0) The following are equivalent:

(1) WKL_0. Weak König's Lemma: Every infinite 0-1 tree has an infinite path.

(2) If $f_0 : \mathbb{N} \to \mathbb{N}$ and $f_1 : \mathbb{N} \to \mathbb{N}$ are injections with bounding function $b : \mathbb{N} \to \mathbb{N}$ then there is a bijection $h : \mathbb{N} \to \mathbb{N}$ such that for all n, $h(n) = f_0(n)$ or $f_1(h(n)) = n$.

Proving Bounded Banach's Theorem in WKL_0

Given injections f_0 and f_1 bounded by b, construct a tree of possible initial segments of h in the manner of this example:

Include $\langle 0, 0, 1, 0 \rangle$ in T if the initial segment of h defined by

$$h(0) = f_0(0) \qquad h(1) = f_0(1) \qquad h(2) = f_1^{-1}(2) \qquad h(3) = f_0(3)$$

is "consistent up to 4," meaning:

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is "consistent up to 4," meaning:

- h is defined: There is a $t \leqslant b(2)$ such that $f(t) = 2$
- h is injective: $f^{-1}(2) \notin \{f_0(0), f_0(1), f_0(3)\}$
- h is onto: Any endpoints of components before 4 are included in the graph of h.

Any infinite path through the tree computes the desired bijection.

Preparation for a reversal

Now we want to to use bounded Banach's Theorem to prove WKL_0.

An important reversal tool for WKL_0 is Lemma IV.4.4 of Simpson [5]:

Lemma (RCA_0) The following are equivalent:

(1) WKL_0

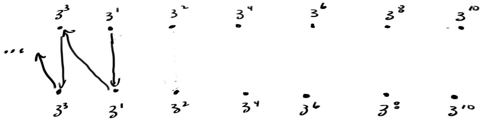
(2) If g_0 and g_1 are injections with disjoint ranges then there is a set X that separates their ranges, that is:

$$\forall n (g_0(n) \in X \wedge g_1(n) \notin X)$$

Sketch: Bounded Banach's Theorem implies WKL_0

Given bijections g_0 and g_1 with disjoint ranges we need bounded injections f_0 and f_1 such that any bijection satisfying Banach's Theorem can be used to calculate a separating set.

Example: Values of f_0 and f_1 on powers of $p_1 = 3$ reflect whether or not 1 is in the separating set.



Suppose $g_0(2) = 1$. Omit the vertical link at 3^{2m+2} for $m = 2$.

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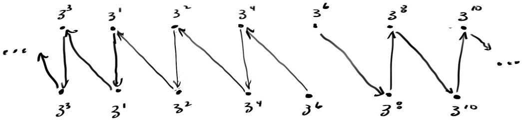

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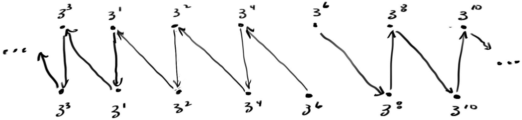

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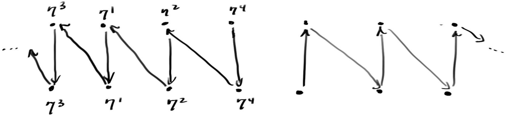

Suppose $g_1(1) = 3$. Omit the slanted link at 7^{2m+2} for $m = 1$.

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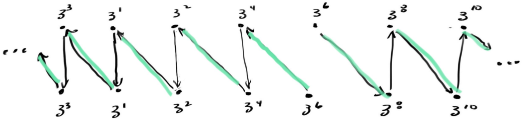

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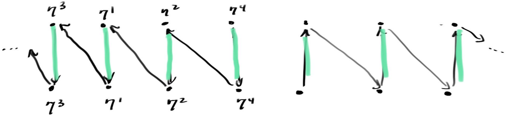

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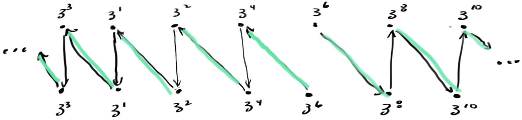

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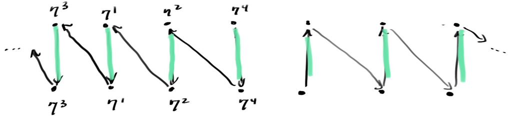

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$\{n \mid h(p_n) \neq p_n\}$ is a separating set.

Higher order reverse mathematics

A framework for doing reverse mathematics in all finite types was introduced by Kohlenbach [4]. It allows the introduction of functionals from sets to numbers and from sets to sets (and more).

The axiom system RCA_0^ω is based on functions (rather than sets), and incorporates limited versions of induction, primitive recursion, and choice. RCA_0^ω is a conservative extension of RCA_0.

Many theorems of second order arithmetic are of the form $\forall X \exists Y\ \theta(X, Y)$. The language of RCA_0^ω can express *Skolemized versions* of the form

$$\exists \Psi \forall X\ \theta(X, \Psi(X))$$

We can study the strength of these functional existence statements over RCA_0^ω.

Preliminary results with Carl Mummert

Theorem (RCA_0^ω) The following are equivalent:

(1) (WKL): There is a functional $\mathsf{WKL} : \mathbb{N}^\mathbb{N} \to 2^\mathbb{N}$ such that if T is a code for an infinite tree in $2^\mathbb{N}$, then $\mathsf{WKL}(T)$ is an infinite path in T.

(2) (LLPO): There is a functional $\mathsf{LLPO} : \mathbb{N}^\mathbb{N} \to 2$ such that the value of $\mathsf{LLPO}(f)$ is the parity of the location of the first 0 in the range of f, provided such a location exists.

(3) $(\mathsf{bB}_\mathbb{N})$: There is a functional $\mathsf{bB}_\mathbb{N}(f_0, f_1, b) = h$ such that if f_0 and f_1 are injections with bound b then h is a bijection satisfying bounded Banach's Theorem.

The proof of the reversal uses (2) to avoid the prime power arguments. The forward direction relies on b (as opposed to characteristic functions for the ranges of f_0 and f_1) to ensure that $\mathsf{bB}_\mathbb{N}$ is total, even when the inputs are incorrect.

Higher order Banach's Theorem on $\mathbb{N}$

Theorem (RCA_0^ω) The following are equivalent:

(1) $(\exists^2)$: There is functional $\mathrm{LPO} : \mathbb{N}^\mathbb{N} \to 2$ such that $\mathrm{LPO}(f) = 0$ if and only if $\exists n\, f(n) = 0$.

(2) $(\mathrm{B}_\mathbb{N})$: There is a functional $\mathrm{B}_\mathbb{N}(f_0, f_1) = h$ such that if f_0 and f_1 are injections then h is a bijection satisfying Banach's Theorem.

The proof of the reversal uses (1) to avoid prime power arguments.

Conjectured extension

If f_0 and f_1 are injections from a complete separable metric space X into X with modulus of uniform continuity m, we can define a functional B_X such that $\mathrm{B}_X(f_0, f_1, m)$ is a bijection $h : X \to X$ satisfying Banach's Theorem.

Conjecture (RCA_0^ω) The following are equivalent:

(1) $(\exists^2)$

(2) If X is a compact complete sep. metric space then (B_X).

(3) $(\mathrm{B}_{[0,1]})$

(4) $(\mathrm{B}_{2^{\mathbb{N}}})$

We believe relaxing the restrictions on uniform continuity and compactness results in strictly stronger functional existence statements.

Links to traditional reverse mathematics

Known conservation results:

$\mathsf{RCA}_0^\omega + (\exists^2)$ is conservative over ACA_0 for Π^1_2 formulas.

$\mathsf{RCA}_0^\omega + (\mathsf{S})$ (Souslin functional) is conservative over $\Pi^1_1 - CA_0$ for Π^1_2 formulas.

Preliminary conservation results:

$\mathsf{RCA}_0^\omega + (\mathsf{WKL})$ is conservative over WKL_0 for Π^1_2 formulas.

$\mathsf{RCA}_0^\omega + (\mathsf{wS})$ is conservative over ATR_0 for Π^1_2 formulas.

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