# Graphs, computability, and reverse mathematics

Jeff Hirst
Appalachian State University
Boone, NC

May 21, 2014

Association for Symbolic Logic
2014 North American Annual Meeting
University of Colorado, Boulder

# Reverse Mathematics

**Goal:** Determine exactly which set existence axioms are needed to prove familiar theorems.

**Method:** Prove results of the form $\mathrm{RCA}_0 \vdash \mathbf{AX} \leftrightarrow \mathbf{THM}$

**The base system** $\mathrm{RCA}_0$**:**

Second order arithmetic: integers $n$ and sets of integers $X$

Induction scheme: restricted to $\Sigma^0_1$ formulas

$(\psi(0) \wedge \forall n(\psi(n) \rightarrow \psi(n+1))) \rightarrow \forall n\, \psi(n)$

where $\psi(n)$ has (at most) one number quantifier.

Recursive set comprehension:

If $\theta \in \Sigma^0_1$ and $\psi \in \Pi^0_1$, and $\forall n(\theta(n) \leftrightarrow \psi(n))$,

then there is a set $X$ such that $\forall n(n \in X \leftrightarrow \theta(n))$

# More set comprehension axioms

**Weak König's Lemma:** ($WKL_0$) If $T$ is an infinite tree in which each node is labeled 0 or 1, then $T$ contains an infinite path.

**Arithmetical comprehension:** ($ACA_0$) If $\theta(n)$ doesn't have any set quantifiers, then there is an $X$ such that $\forall n(n \in X \leftrightarrow \theta(n))$

Theorem: $RCA_0$ proves that the following are equivalent:

1. $ACA_0$
2. If $f : \mathbb{N} \to \mathbb{N}$ is an injection then there is a set $Y$ such that $\forall y(y \in Y \leftrightarrow \exists x(f(x) = y))$.

# Why reverse math? Why graph theory?

Work in reverse mathematics can:

- precisely categorize the logical strength of theorems.
- differentiate between different proofs of theorems.
- provide insight into the foundations of mathematics.
- utilize and contribute to work in many subdisciplines of mathematical logic – including proof theory, computability theory, models of arithmetic, etc.

Graph theory is in this talk because:

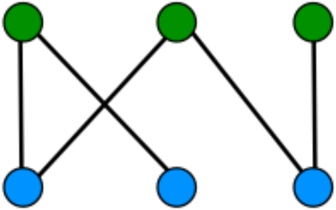
- Friedman's [3, 4] founding work on reverse mathematics includes graph theory.
- The proofs can be described with pictures.

# Matchings

with N. Hughes

**Question:**

When do bipartite graphs contain unique matchings?

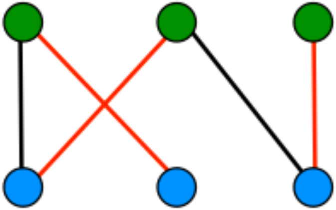


# Matchings

with N. Hughes

**Question:**

When do bipartite graphs contain unique matchings?

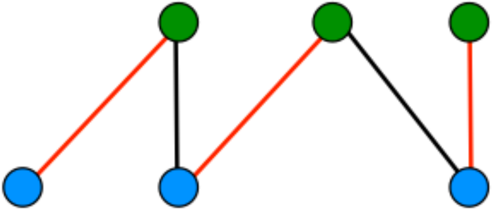


# Matchings

with N. Hughes

**Question:**

When do bipartite graphs contain unique matchings?

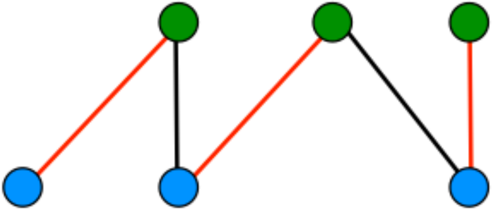


# Matchings

with N. Hughes

**Question:**

When do bipartite graphs contain unique matchings?



A finite bipartite graph $(B, G)$ has a unique matching if and only if there is an enumeration of $B$ as $b_1, b_2, \ldots$ so that for all $n$

$$|G(b_1, b_2, \ldots, b_n)| = n.$$

Alternative terminology: marriage problems, transversals, SDRs (distinct representatives)

# Matchings

with N. Hughes

An extension to infinite graphs:

Theorem: $RCA_0$ proves that the following are equivalent:

1. $WKL_0$
2. Suppose $(B, G)$ is a bipartite graph and $h : B \to G$ is a function such that $h(b)$ is an upper bound on all the vertices in $G$ connected to $b$. If $(B, G)$ contains a unique matching, then there is an enumeration of $B$ as $b_1, b_2, \ldots$ so that for all $n$, $|G(b_1, b_2, \ldots, b_n)| = n$.

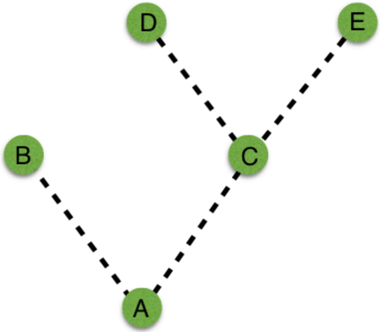
Comment: To show (1) implies (2), use $h$ to construct a bounded tree of initial segments of enumerations of $B$. Any path is an enumeration of $B$.

# Matchings: sketch of the reversal

with N. Hughes

We need to use the existence of the enumeration to show that a tree with no infinite paths is finite.

Here's a tree with no paths. Nodes are green.

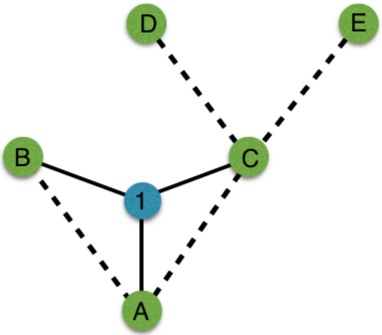


# Matchings: sketch of the reversal

with N. Hughes

We need to use the existence of the enumeration to show that a tree with no infinite paths is finite.

Here's a tree with no paths. Add a blue vertex.

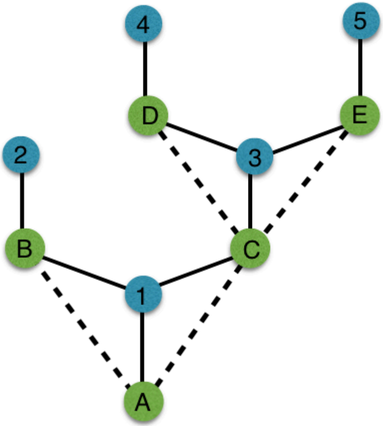


# Matchings: sketch of the reversal

with N. Hughes

We need to use the existence of the enumeration to show that a tree with no infinite paths is finite.

Here's a tree with no paths. Complete the graph.

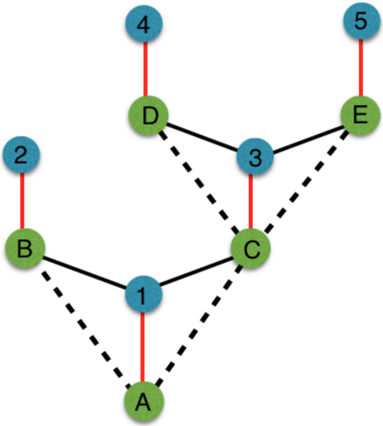


# Matchings: sketch of the reversal

with N. Hughes

We need to use the existence of the enumeration to show that a tree with no infinite paths is finite.

Here's a tree with no paths. Note the unique matching.

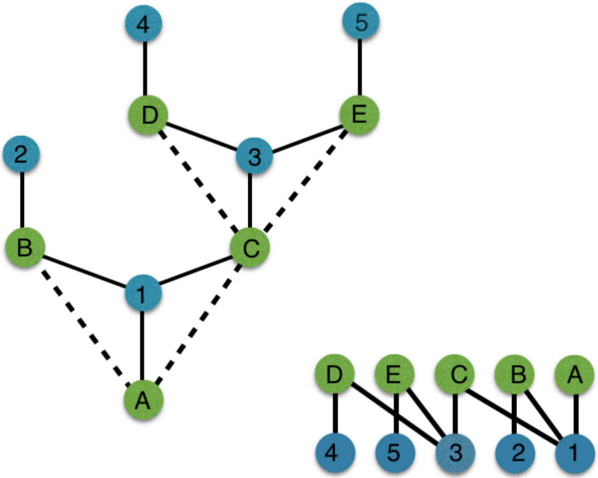


# Matchings: sketch of the reversal

with N. Hughes

We need to use the existence of the enumeration to show that a tree with no infinite paths is finite.

Here's a tree with no paths. In any enumeration, the root blue vertex is last. The tree is finite.



# Matchings

with N. Hughes

Omitting the bounding function $h$ bumps up the strength of the preceding theorem.

Theorem: $RCA_0$ proves that the following are equivalent:

1. $ACA_0$
2. Suppose $(B, G)$ is a bipartite graph and each vertex in $B$ is connected to only finitely many vertices in $G$. If $(B, G)$ contains a unique matching, then there is an enumeration of $B$ as $b_1, b_2, \ldots$ so that for all $n$, $|G(b_1, b_2, \ldots, b_n)| = n$.

Comment: To show that (1) implies (2), use $ACA_0$ to find $h$ and apply the preceding theorem.

# Matchings: sketch of another reversal

with N. Hughes

We need to use the existence of the enumeration to find the range of an injection. If the injection is:

| $n$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $f(n)$ | 4 | 3 | 0 | 2 |

build the graph like this:

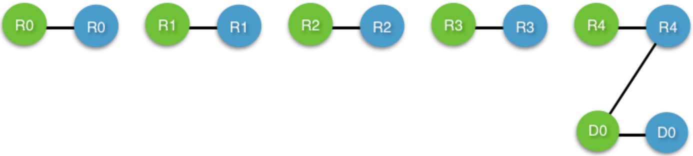
# Matchings: sketch of another reversal

with N. Hughes

We need to use the existence of the enumeration to find the range of an injection. If the injection is:

| $n$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $f(n)$ | 4 | 3 | 0 | 2 |

build the graph like this:




Add vertices and edges for each domain value.

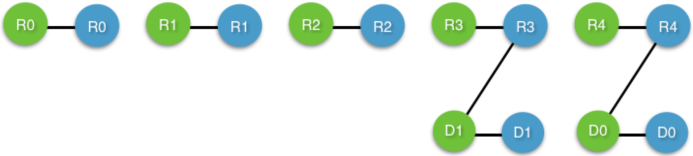
# Matchings: sketch of another reversal

with N. Hughes

We need to use the existence of the enumeration to find the range of an injection. If the injection is:

| $n$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $f(n)$ | 4 | 3 | 0 | 2 |

build the graph like this:




The bipartite graph will contain a unique matching.

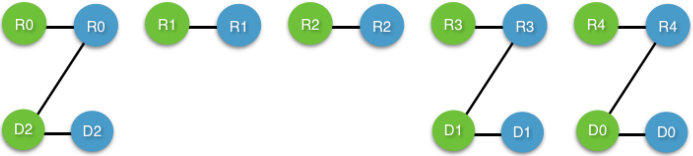
# Matchings: sketch of another reversal

with N. Hughes

We need to use the existence of the enumeration to find the range of an injection. If the injection is:

| $n$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $f(n)$ | 4 | 3 | 0 | 2 |

build the graph like this:




In any enumeration, D2 appears before R0.

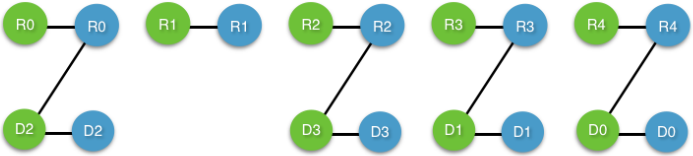
# Matchings: sketch of another reversal

with N. Hughes

We need to use the existence of the enumeration to find the range of an injection. If the injection is:

| $n$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $f(n)$ | 4 | 3 | 0 | 2 |

build the graph like this:




$n \in \text{Range}(f)$ iff Dm appears before Rn and $f(m) = n$.

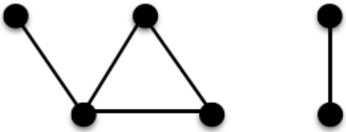
# Matchings: an open question

with N. Hughes

How strong is the following statement?

Lemma: Suppose $(B, G)$ is an infinite bipartite graph such that $G(b)$ is finite for each $b$ and $(B, G)$ contains a unique matching. Then some $b$ in $B$ is connected to exactly one $g$ in $G$.

Our initial attempts used arguments involving connected components of graphs. . .



# Connected components

with K. Gura and C. Mummert, preliminary

Theorem: $\mathsf{RCA}_0$ proves that the following are equivalent:

1. $\mathsf{ACA}_0$
2. Every countable graph has a connected component.

Comments on the proof: The connected component containing a given vertex is arithmetically definable in the vertex and the graph, so (1) implies (2).

For the reversal, we need to use any connected component to find the range of an injection.

If the injection is:

| $n$ | 0 | 1 | 2 |
|---|---|---|---|
| $f(n)$ | 4 | 3 | 0 |

build the graph like this. . .

# Connected components

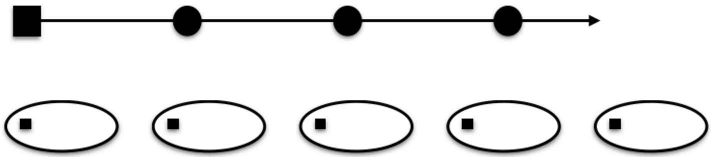
with K. Gura and C. Mummert, preliminary

For the reversal, we need to use any connected component to find the range of an injection.

If the injection is:

| $n$ | 0 | 1 | 2 |
|---|---|---|---|
| $f(n)$ | 4 | 3 | 0 |

build the graph like this. . .



# Connected components

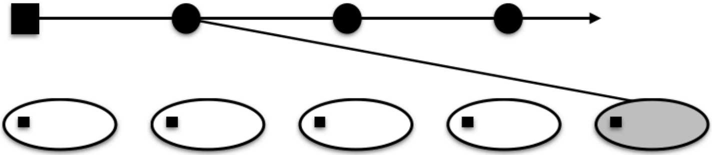
with K. Gura and C. Mummert, preliminary

For the reversal, we need to use any connected component to find the range of an injection.

If the injection is:

| $n$ | 0 | 1 | 2 |
|---|---|---|---|
| $f(n)$ | 4 | 3 | 0 |

build the graph like this...



# Connected components

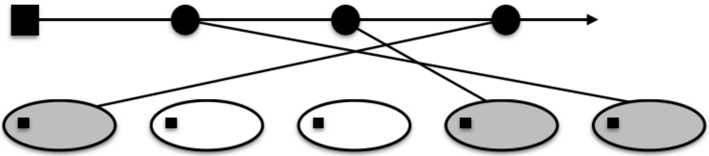
with K. Gura and C. Mummert, preliminary

For the reversal, we need to use any connected component to find the range of an injection.

If the injection is:

| $n$ | 0 | 1 | 2 |
|---|---|---|---|
| $f(n)$ | 4 | 3 | 0 |

build the graph like this...



# Connected components

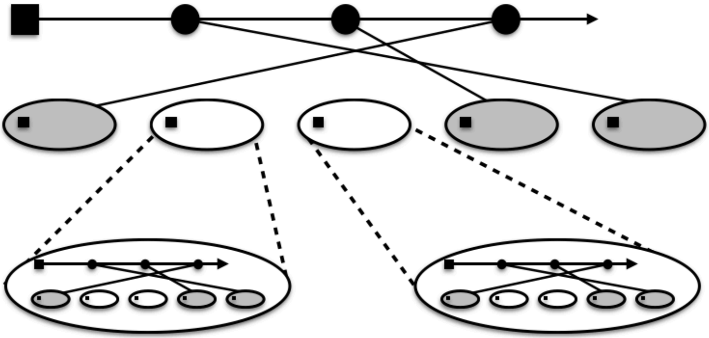
with K. Gura and C. Mummert, preliminary

For the reversal, we need to use any connected component to find the range of an injection.

If the injection is:

| $n$ | 0 | 1 | 2 |
|---|---|---|---|
| $f(n)$ | 4 | 3 | 0 |

build the graph like this. . .



# Connected components

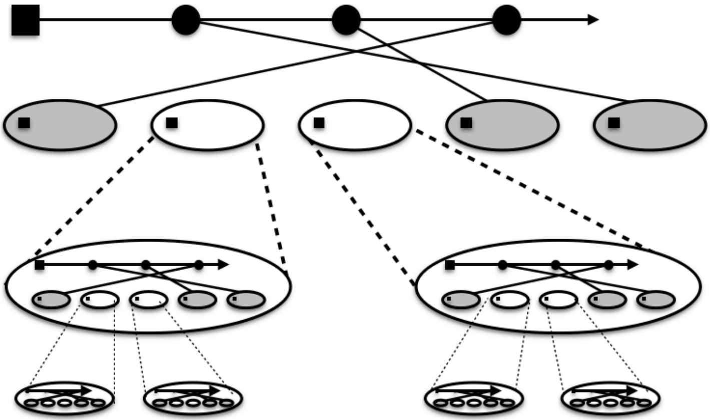
with K. Gura and C. Mummert, preliminary

For the reversal, we need to use any connected component to find the range of an injection.

If the injection is:

| $n$ | 0 | 1 | 2 |
|---|---|---|---|
| $f(n)$ | 4 | 3 | 0 |

build the graph like this...



# Connected components

with K. Gura and C. Mummert, preliminary

Another result on connected components:

Theorem: $RCA_0$ proves that the following are equivalent:

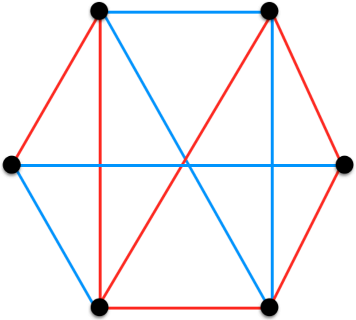
1. $ACA_0$
2. If $G$ is a graph then there is an infinite set of vertices all of which lie in the same connected component or no two of which lie in the same connected component.

This is reminiscent of, but not equivalent to, Ramsey's theorem for pairs (by Seetapun and Slaman).

# Ramsey's Theorem

with Dorais, Dzhafarov, Mileti, and Shafer

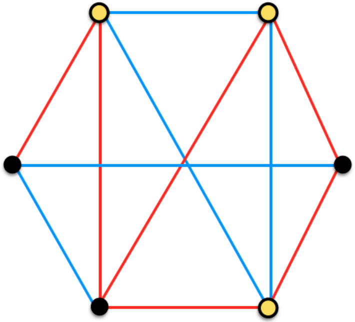
Theorem: ($RT^2_2$) If $G$ is the complete graph with vertices $V = \{v_0, v_1, \ldots\}$, and $f : [V]^2 \to \{\text{red}, \text{blue}\}$ colors the edges of $G$, then there is an infinite $S \subset V$ such that the subgraph with vertices from $S$ is monochromatic.



# Ramsey's Theorem

with Dorais, Dzhafarov, Mileti, and Shafer

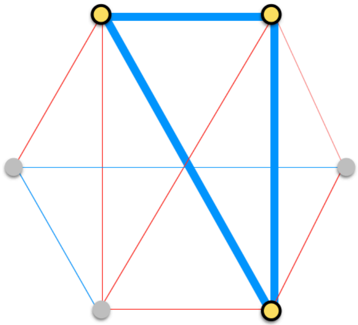
Theorem: ($RT^2_2$) If $G$ is the complete graph with vertices $V = \{v_0, v_1, \ldots\}$, and $f : [V]^2 \to \{\text{red}, \text{blue}\}$ colors the edges of $G$, then there is an infinite $S \subset V$ such that the subgraph with vertices from $S$ is monochromatic.



# Ramsey's Theorem

with Dorais, Dzhafarov, Mileti, and Shafer

Theorem: ($RT^2_2$) If $G$ is the complete graph with vertices $V = \{v_0, v_1, \ldots\}$, and $f : [V]^2 \to \{\text{red, blue}\}$ colors the edges of $G$, then there is an infinite $S \subset V$ such that the subgraph with vertices from $S$ is monochromatic.



# Ramsey's Theorem

with Dorais, Dzhafarov, Mileti, and Shafer

In a proof in $\mathsf{RCA}_0$, we can replace two applications of $\mathsf{RT}^2_2$ with one application of $\mathsf{RT}^2_4$.

For example, given $f : [\mathbb{N}]^2 \to 2$ and $g : [\mathbb{N}]^2 \to 2$, define

$$h(\text{edge}) = 2 \cdot f(\text{edge}) + g(\text{edge})$$

Any subgraph monochromatic for $h$ is also monochromatic for both $f$ and $g$.

Question: Can we replace two uses of $\mathsf{RT}^2_2$ with one use of $\mathsf{RT}^2_2$?

When we replace two uses of $\mathsf{RT}^2_2$ with one use of $\mathsf{RT}^2_4$ ...

We have Turing reductions $\Phi$ and $\Psi$ such that given colorings $f$ and $g$, $\Phi(f,g)$ computes the new coloring $h$, and given any monochromatic subgraph $X$ for $h$, $\Psi(X)$ computes monochromatic subgraphs for $f$ and $g$.

Given these Turing reductions, we write $\langle \mathsf{RT}^2_2, \mathsf{RT}^2_2 \rangle \leqslant_{sW} \mathsf{RT}^2_4$ and say "two uses of $\mathsf{RT}^2_2$ are strongly Weihrauch reducible to one use of $\mathsf{RT}^2_4$."

Revised question: Is $\langle \mathsf{RT}^2_2, \mathsf{RT}^2_2 \rangle \leqslant_{sW} \mathsf{RT}^2_2$?

# Ramsey's Theorem

with Dorais, Dzhafarov, Mileti, and Shafer

When we replace two uses of $\mathsf{RT}^2_2$ with one use of $\mathsf{RT}^2_4 \ldots$

We have Turing reductions $\Phi$ and $\Psi$ such that given colorings $f$ and $g$, $\Phi(f, g)$ computes the new coloring $h$, and given any monochromatic subgraph $X$ for $h$, $\Psi(X)$ computes monochromatic subgraphs for $f$ and $g$.

Given these Turing reductions, we write $\langle \mathsf{RT}^2_2, \mathsf{RT}^2_2 \rangle \leqslant_{sW} \mathsf{RT}^2_4$ and say "two uses of $\mathsf{RT}^2_2$ are strongly Weihrauch reducible to one use of $\mathsf{RT}^2_4$."

Revised question: Is $\langle \mathsf{RT}^2_2, \mathsf{RT}^2_2 \rangle \leqslant_{sW} \mathsf{RT}^2_2$? (Hint: No.)

# Ramsey's Theorem

with Dorais, Dzhafarov, Mileti, and Shafer

Definition: A $\Pi^1_2$ statement $P$

- is *total* if every element of $2^\omega$ codes an instance of $P$, and
- has *finite tolerance* if there is a Turing functional $\Theta$ such that if $B_1$ and $B_2$ agree after $m$ and $S$ is a solution of $B_1$ then $\Theta(S, m)$ is a solution of $B_2$.

Squashing Theorem:
Let $P$ be a total $\Pi^1_2$ statement with finite tolerance. Then:

$$\langle P, P\rangle \leqslant_{sW} P \textit{ implies } \mathrm{Seq}P \leqslant_{sW} P$$

Informally, if two uses of $P$ can be reduced to one use, then infinitely many uses of $P$ can be reduced to one use.

# Ramsey's Theorem

with Dorais, Dzhafarov, Mileti, and Shafer

Squashing Theorem:

Let $P$ be a total $\Pi^1_2$ statement with finite tolerance. Then:

$$\langle P, P\rangle \leqslant_{sW} P \textit{ implies } \mathrm{Seq}P \leqslant_{sW} P$$

Application to the $\mathrm{RT}^2_2$ problem:

$\mathrm{RT}^2_2$ is total and has finite tolerance.

There is a computable instance of $\mathrm{SeqRT}^2_2$ such that $0''$ is computable from every solution.

There is no computable instance of $\mathrm{RT}^2_2$ such that $0''$ is computable from every solution. (Jockusch)

$\mathrm{SeqRT}^2_2 \not\leqslant_{sW} \mathrm{RT}^2_2$, and so $\langle \mathrm{RT}^2_2, \mathrm{RT}^2_2\rangle \not\leqslant_{sW} \mathrm{RT}^2_2$.

# Ramsey's Theorem

with Dorais, Dzhafarov, Mileti, and Shafer

A trick from the proof of the:

**Squashing Theorem:** $\langle P, P\rangle \leqslant_{sW} P$ *implies* $\mathrm{Seq}P \leqslant_{sW} P$

Compress the sequence $f0, f1, \ldots$ into a single instance $h0$.

$$h0 \left\{ \begin{array}{lcccccc} f0 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ f1 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \end{array} \right.$$

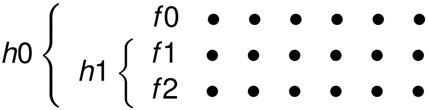
# Ramsey's Theorem

with Dorais, Dzhafarov, Mileti, and Shafer

A trick from the proof of the:

Squashing Theorem: $\langle P, P \rangle \leqslant_{sW} P$ *implies* $\mathrm{Seq}P \leqslant_{sW} P$

Compress the sequence $f0, f1, \ldots$ into a single instance $h0$.



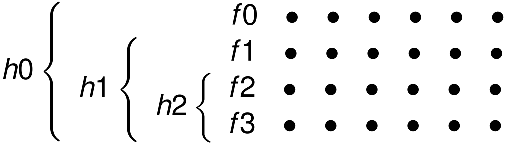
# Ramsey's Theorem

with Dorais, Dzhafarov, Mileti, and Shafer

A trick from the proof of the:

Squashing Theorem: $\langle P, P\rangle \leqslant_{sW} P$ *implies* $\mathrm{Seq}P \leqslant_{sW} P$

Compress the sequence $f0, f1, \ldots$ into a single instance $h0$.



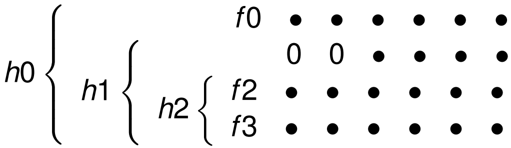
# Ramsey's Theorem

with Dorais, Dzhafarov, Mileti, and Shafer

A trick from the proof of the:

Squashing Theorem: $\langle P, P \rangle \leqslant_{sW} P$ *implies* $\mathrm{Seq}P \leqslant_{sW} P$

Assume the initial outputs of $h1$ are 0.



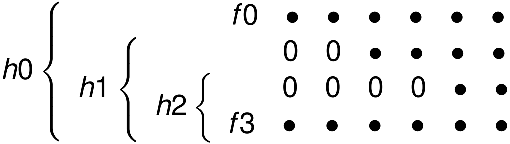
# Ramsey's Theorem

with Dorais, Dzhafarov, Mileti, and Shafer

A trick from the proof of the:

Squashing Theorem: $\langle P, P \rangle \leqslant_{sW} P$ *implies* $\mathrm{Seq}P \leqslant_{sW} P$

Assume the initial outputs of $h2$ are 0.



# Reductions

with Dorais, Dzhafarov, Mileti, and Shafer

Question: If $C$ is the problem of finding a connected component of a graph, then $\langle C, C\rangle$ is strongly Weihrauch reducible to $C$.

- Does $C$ have finite tolerance?
- Is Seq$C$ reducible to $C$?
- Can we usefully strengthen the Squashing Theorem?

Question: In a proof in $\mathrm{RCA}_0$, can we replace two uses of $\mathrm{RT}^2_2$ by a single use of $\mathrm{RT}^2_2$ in a nonuniform fashion?

Question: If we use an axiom twice in a proof, how can we know if the second use is necessary?

# References


[1] Vasco Brattka and Guido Gherardi, *Weihrauch degrees, omniscience principles and weak computability*, J. Symbolic Logic **76** (2011), no. 1, 143–176. DOI 10.2178/jsl/1294170993 MR2791341 (2012c:03186).

[2] François Dorais, Damir Dzhafarov, Jeffry Hirst, Joseph Mileti, and Paul Shafer, Transactions of the AMS (to appear). http://arxiv.org/abs/1212.0157.

[3] Harvey Friedman, *Some systems of second order arithmetic and their use*, Proceedings of the International Congress of Mathematicians (Vancouver, B. C., 1974), Vol. 1, 1975, pp. 235–242. http://www.mathunion.org/ICM/ MR0429508.

[4] Harvey Friedman, *Abstracts: Systems of second order arithmetic with restricted induction, I and II*, J. Symbolic Logic **41** (1976), 557–559. http://www.jstor.org/stable/2272259.

[5] Kirill Gura, Jeffry Hirst, and Carl Mummert, *On the existence of a connected component of a graph*. in preparation.

[6] Jeffry Hirst and Noah Hughes, *Reverse mathematics and marriage problems with unique solutions* (submitted). Draft version.

[7] Jeffry Hirst, *Marriage theorems and reverse mathematics*, Logic and computation (Pittsburgh, PA, 1987), Contemp. Math., vol. 106, Amer. Math. Soc., Providence, RI, 1990, pp. 181–196. DOI 10.1090/conm/106/1057822 MR1057822 (91k:03141).

[8] Jeffry Hirst, *Connected components of graphs and reverse mathematics*, Arch. Math. Logic **31** (1992), no. 3, 183–192. DOI 10.1007/BF01269946 MR1147740 (92j:03052).

[9] Stephen G. Simpson, *Subsystems of second order arithmetic*, 2nd ed., Perspectives in Logic, Cambridge University Press, Cambridge, 2009. DOI 10.1017/CBO9780511581007 MR2517689.

Graphs, computability, and reverse mathematics
Jeff Hirst, Appalachian State University