Hindman's Theorem and Ultrafilters

Jeff Hirst
Appalachian State University
Boone, NC USA

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Hindman's Theorem

Theorem: (Hindman [4]) For any coloring $f : \mathbb{N} \to k$, there is an infinite set H and a color c such that for every finite set $F \subset H$, we have $f(\Sigma F) = c$.

An example:

n	1	2	3	4	5	6	7	8	9	10	11	12
f(n)	▲	■	■	▲	■	▲	■	▲	■	■	▲	■

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How hard is it to find H? (Short answer: we don't know.)

Reverse mathematics

Reverse mathematics uses a hierarchy of axioms of second order arithmetic to measure the strength of theorems.

The language has variables for natural numbers and sets of naturals numbers.

The base system, RCA_0, includes

- arithmetic facts (e.g. $n + 0 = n$),
- an induction scheme (restricted to Σ^0_1 formulas), and
- recursive comprehension
(computable sets exist, i.e. sets with programmable characteristic functions exist).

Adding stronger comprehension axioms creates stronger axiom systems.

ACA_0

The system ACA_0 adds arithmetical comprehension to RCA_0 (sets with arithmetically definable characteristic functions exist).

A theorem of reverse mathematics:

Theorem: Over RCA_0, the following are provably equivalent:

1. ACA_0.
2. Ramsey's theorem for triples and two colors. (Simpson)
3. Every countable sequence of reals in $[0, 1]$ has a convergent subsequence. (Friedman)

Iterating...

Iterated Hindman's Theorem (IHT) If $f_0, f_1, f_2, \ldots$ is a sequence of 2-colorings of $\mathbb{N}$, then there is an infinite set $H = \{h_0, h_1, h_2, \ldots\}$ such that

$H = \{h_0, h_1, \ldots\}$ is sum monochromatic for f_0,
$\{h_1, h_2, \ldots\}$ is sum monochromatic for f_1,
$\{h_2, h_3, \ldots\}$ is sum monochromatic for f_2, and so on.

Iterated Arithmetical Comprehension (ACA_0^+) Suppose $\theta(X, m)$ is an arithmetical formula. Fix X_0 and let $X_{n+1} = \{m \mid \theta(X_n, m)\}$. Then (a code for) the sequence $X_0, X_1, X_2, \ldots$ exists.

Comparative strengths

RCA_0 proves:

$$\mathrm{ACA}_0^+ \rightarrow \mathrm{IHT} \rightarrow \mathrm{HT} \rightarrow \mathrm{ACA}_0$$

(Blass, Hirst, and Simpson [1])

Computability theory:

> There is a computable coloring with no computable sum homogeneous set.
>
> Does every computable coloring have an arithmetically definable sum homogeneous set?

Ultrafilters on $\mathcal{P}(\mathbb{N})$

A filter is a subcollection of $\mathcal{P}(\mathbb{N})$ which is

- does not contain $\emptyset$,
- is closed under superset, and
- is closed under finite intersection.

An ultrafilter contains exactly one of X and X^c for each X

We can think of filters (or ultrafilters) as defining notions of large sets.

An example:
Let $u = \{X \subset \mathbb{N} \mid 2 \in X\}$. $u = \langle 2 \rangle$ is a principal ultrafilter.

A non-example:
Let $v = \{X \subset \mathbb{N} \mid X^c \text{ is finite}\}$. v is a filter, but not an ultrafilter (on $\mathcal{P}(\mathbb{N})$).

Ultrafilters and Hindman's Theorem

Theorem: (Hindman 1972 [3]) Hindman's theorem holds if and only if there is an ultrafilter p on $\mathcal{P}(\mathbb{N})$ such that $\{x \mid A - x \in p\} \in p$ whenever $A \in p$.

Notation: If $A = \{1, 4, 7, 9, 12, \ldots\}$ then $A - 2 = \{2, 5, 7, 10, \ldots\}$. We can think of $A - 2$ as a left shift.

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A formalized version [6]
Theorem: (RCA_0) The following are equivalent:

1. IHT.
2. If $\mathcal{B}$ is a countable boolean algebra closed under left shifts, then there is an ultrafilter p on $\mathcal{B}$ such that there is an $a \in A$ such that $A - a \in p$ whenever $A \in p$.

Galvin-Glazer addition

If u and v are ultrafilters on $\mathcal{P}(\mathbb{N})$, define

$$A \in u + v \leftrightarrow \{x \mid A - x \in u\} \in v$$

An example:

$$\begin{aligned} A \in \langle 2 \rangle + \langle 3 \rangle &\leftrightarrow \{x \mid A - x \in \langle 2 \rangle\} \in \langle 3 \rangle \\ &\leftrightarrow \{x \mid 2 \in A - x\} \in \langle 3 \rangle \\ &\leftrightarrow \{x \mid x + 2 \in A\} \in \langle 3 \rangle \\ &\leftrightarrow \{x \mid x \in A - 2\} \in \langle 3 \rangle \\ &\leftrightarrow A - 2 \in \langle 3 \rangle \\ &\leftrightarrow 3 \in A - 2 \\ &\leftrightarrow 5 \in A \\ &\leftrightarrow A \in \langle 5 \rangle \qquad \text{so } \langle 2 \rangle + \langle 3 \rangle = \langle 5 \rangle \end{aligned}$$

A short proof of Hindman's theorem

Here's the sketch. Comfort [2] fills in details.

For any ultrafilters u and v, $u + v$ is an ultrafilter.

Under the Stone-Čech topology on the ultrafilter space, $u + v$ is right continuous and associative.

A right continuous associative map on a compact space has an idempotent element.

Suppose $p = p + p$. Then

$$A \in p \leftrightarrow \{x \mid A - x \in p\} \in p$$

So p is the ultrafilter appearing in Hindman's 1972 theorem.

Countable Boolean algebras

Motivating question:

Can we port the Galvin-Glazer proof to reverse math?

We want to substitute a countable Boolean algebra for $\mathcal{P}(\mathbb{N})$.

How does this affect the ultrafilter space?

How does this affect ultrafilter addition?

An example: Finite and cofinite sets

The finite and cofinite sets form a countable Boolean algebra closed under left shift. Lets call them $\mathcal{C}$.

In RCA_0, we can construct many representations of $\mathcal{C}$ via sequences of characteristic functions and associated operations.

RCA_0 can prove that every principal ultrafilter of $\mathcal{C}$ exists, and that their sums exist.

What about the rest of the ultrafilters on $\mathcal{C}$?

An example: Finite and cofinite sets

If u is an ultrafilter on $\mathcal{C}$ and u contains a finite set, then u is principal.

If u is an ultrafilter on $\mathcal{C}$ and u contains no finite sets, then u contains every cofinite set.

The cofinite sets form a (unique) nonprincipal ultrafilter on $\mathcal{C}$.

An example: Finite and cofinite sets

Let u be the ultrafilter of cofinite sets on $\mathcal{C}$.

How does addition with u behave?

If X is cofinite, then each of its left shifts is cofinite, so

$$\{x \mid X - x \in u\} = \mathbb{N} \in u.$$

If X is finite, then each of its left shifts is finite, so

$$\{x \mid X - x \in u\} = \emptyset \notin u.$$

Summarizing $u + u = u$.

Using the fact that left shifts of cofinite sets are cofinite, we can also show

$$u + \langle 3 \rangle = \langle 3 \rangle + u = u.$$

Summarizing: Finite and cofinite sets

ACA_0 can prove that

- the Boolean algebra $\mathcal{C}$ exists,
- the ultrafilters on $\mathcal{C}$ consist of the principal ultrafilters and the unique nonprincipal ultrafilter,
- addition is defined for all of the ultrafilters, and
- the addition is commutative.

Summarizing: Finite and cofinite sets

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- the Boolean algebra $\mathcal{C}$ exists,
- the ultrafilters on $\mathcal{C}$ consist of the principal ultrafilters and the unique nonprincipal ultrafilter,
- addition is defined for all of the ultrafilters, and
- the addition is commutative.

Ultrafilter addition is commutative on some Boolean algebras, but not on others. For example, ultrafilter addition on $\mathcal{P}(\mathbb{N})$ is not commutative; see [5, Thm 4.27].

Summarizing: Finite and cofinite sets

Where did we use ACA_0?

Theorem:(RCA_0) The following are equivalent:

1. ACA_0.
2. Every infinite Boolean algebra has a nonprincipal ultrafilter.
3. $\mathcal{C}$ has a nonprincipal ultrafilter.
4. $\mathcal{C}$ has an idempotent for ultrafilter addition.

Ideas from the proof:
1→2: The algebra is countable, so we can list the sets. Make choices so that the intersection of the chosen sets is always infinite.
3→1: Sets can be repeated in the presentation of $\mathcal{C}$. We can insert sets A_0 and A_1 so that $A_0^c = A_1$ and which one is finite is determined at a stage in the construction.

More differences

The ultrafilters on $\mathcal{P}(\mathbb{N})$ have a different topology from the ultrafilters on a countable algebra.

The topology for $\mathcal{P}(\mathbb{N})$ is $\beta\mathbb{N}$.

In a countable Boolean algebra, we can list all the sets, and mark them 1 or 0 as we put them into an ultrafilter. So an ultrafilter is an infinite string of 0s and 1s.

The ultrafilters on a countable Boolean algebra can be viewed as a closed subset of Cantor space. They form a closed compact subset of a complete separable metric space. The principal filters are dense in the space.

Conjectures

Simpson: ACA_0 proves Hindman's Theorem.

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Hirst: It is seldom profitable to bet against Simpson.

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Conjecture: (RCA_0) The following are equivalent:

1. IHT.
2. If $\mathcal{B}$ is a countable shift algebra including all finite sets, then there is an extension $\mathcal{B}^*$ of $\mathcal{B}$ and an ultrafilter u on $\mathcal{B}^*$ such that $u + u = u$.

References

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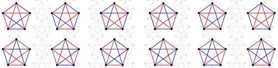
How many 2-colorings of K5 have no 1-colored K3 ?

Ramsey Interest Group: Anthony Hengst, Sergei Miles, Isaac Medina Silva, Allison Staley **Faculty Mentor: Jeff Hirst**

Appalachian State University, Department of Mathematical Sciences, Boone, North Carolina 28608

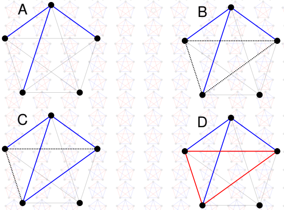
Introduction

Of the 1024 possible 2-colorings of K5, only 12 have no 1-colored triangles.



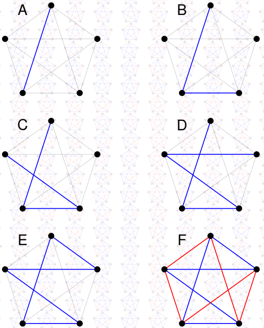
Claim 1

If any 3 edges match, then there is a 1-colored triangle.



Claim 2

If G has no 1-colored triangles, then G has a 1-colored 5-cycle.

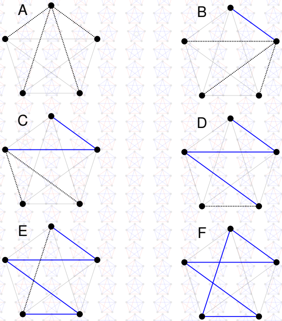


E: 1-colored 5-cycle

F: Remaining edges form a 5-cycle

Claim 3

There are 12 ways to construct a 1-colored 5-cycle.



$$\frac{4 \cdot 3 \cdot 2 \cdot 1 \cdot 1}{2} = 12$$