A weak coloring principle

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The principle ERT

ERT (Eventually repeating tails): Suppose $f : \mathbb{N} \to k$ for some $k \in \mathbb{N}$. Then there is a $b \in \mathbb{N}$ such that for all $x \geqslant b$ there is a $y \geqslant b$ such that $x \neq y$ and $f(x) = f(y)$.

n	0	1	2	3	4	5	6	7	8	9...
$f(n)$	▲	■	■	★	▲	■	■	▲	■	■

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ECT (Eventually constant tail spectra): Suppose $f : \mathbb{N} \to k$ for some $k \in \mathbb{N}$. Then there is a $b \in \mathbb{N}$ such that for all $x \geqslant b$ there is a $y > x$ such that $f(x) = f(y)$.

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spec	▲■★	▲■★	▲■★	▲■★	▲■	▲■	▲■	▲■	▲■	▲■
					b					

Clearly, $\mathrm{RCA}_0 \vdash \mathrm{ECT} \to \mathrm{ERT}$

ECT and $I\Sigma^0_2$

$RCA_0 \vdash I\Sigma^0_2 \to ECT$

Ideas from a proof [5]:

Use bounded Σ^0_2 comprehension to isolate the colors that appear only finitely many times.

$$F = \{c \mid \exists b \forall x (x > b \to f(x) \neq c)\}$$

Use $B\Sigma^0_2$ to find a strict upper bound on all occurrences of colors in F. This is the desired b.

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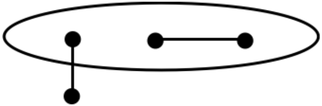
The implication reverses: $RCA_0 \vdash I\Sigma^0_2 \leftrightarrow ECT$

Consequence: $RCA_0 \vdash I\Sigma^0_2 \to ERT$
(We will see that this doesn't reverse.)

What Theorem Follows?

ERT is related to vertex colorings of hypergraphs.

A hypergraph consists of vertices and sets of vertices (edges).



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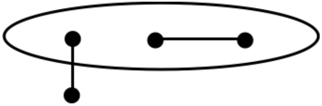
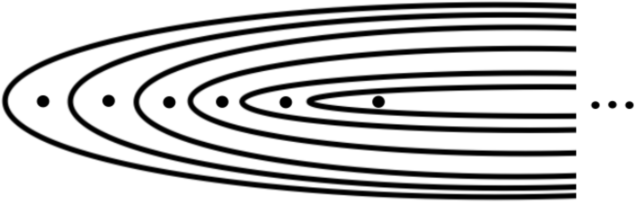




The M-graph

What Theorem Follows?

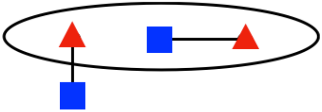
An aside: Here is a version of RT^3_2.

If the edges of the hypergraph $[\mathbb{N}]^3$ are colored with two colors, then there is an infinite set H such that the subhypergraph $[H]^3$ is monochromatic.

In general, Ramsey's theorem can be viewed as addressing edge colorings of hypergraphs.

What Theorem Follows?

A vertex coloring of a hypergraph is *conflict free* if every edge contains a color that appears only once in that edge.



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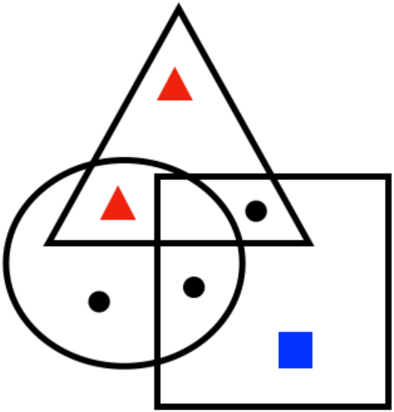

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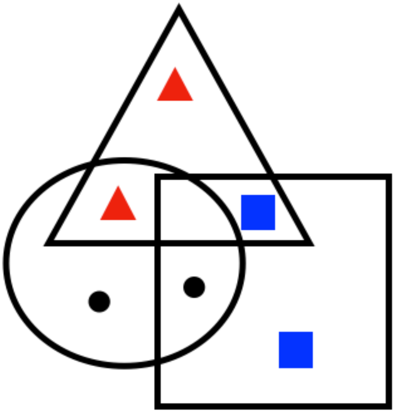

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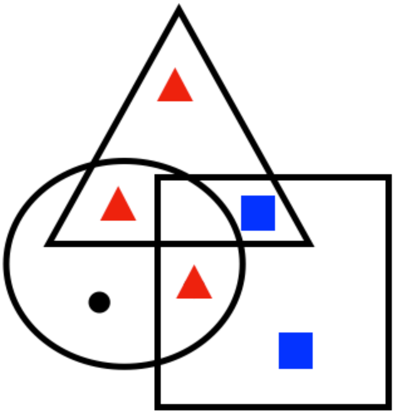

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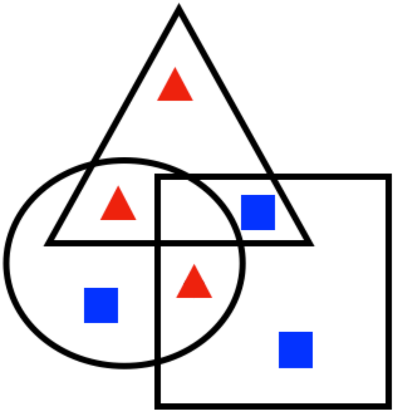

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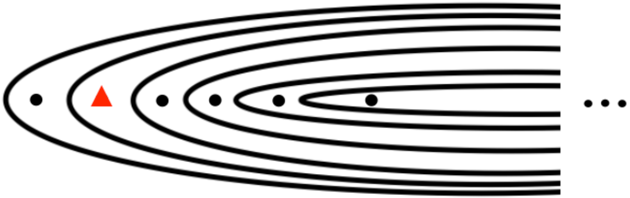

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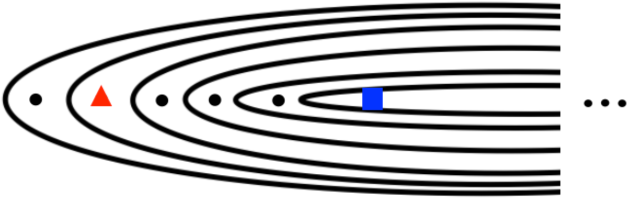

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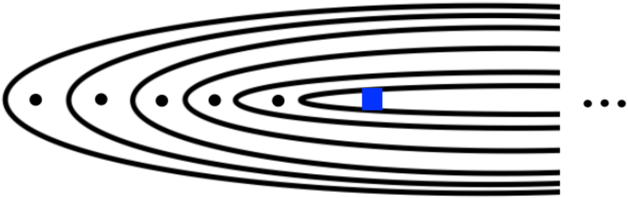

Every finite partial subhypergraph of the M-graph has a conflict free 2-coloring.

What Theorem Follows? Finally answered.

Theorem: (RCA_0) The following are equivalent:

1. ERT
2. The M-graph has no finite conflict free coloring.

Sketch:
$\rightarrow$ Finitely color the M-graph. Apply ERT to the coloring; get b. The edge starting at vertex b has no singleton color. The coloring is not conflict free.
$\leftarrow$ Finitely color $\mathbb{N}$. Copy to the M-graph. Some E_b has no singleton color. b witnesses ERT.

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Finite partial subhypergraphs of the M-graph have conflict free 2-colorings, but the M-graph has no conflict free 2-coloring (or finite coloring). In this setting, compactness does not hold.

How strong is ERT?

Is it provable in RCA_0? Maybe, but not in any obvious fashion.

Is it equivalent to $\mathsf{I}\Sigma^0_2$ over RCA_0? No.

Chong, Slaman, and Yang [1] proved that SRT^2_2 does not imply $\mathsf{I}\Sigma^0_2$. If we prove ERT from SRT^2_2, then we will know that ERT is strictly weaker than $\mathsf{I}\Sigma^0_2$.

Thm: $\mathsf{RCA}_0 \vdash \mathsf{SRT}^2_2 \to \mathsf{ERT}$

Goal: convert a finite coloring of $\mathbb{N}$ into a 2-coloring of pairs.

Method:
Color an interval 1 if and only if it contains a singleton color.

Example:

n	0	1	2	3	4	5	6	7	8	9...
$f(n)$	▲	■	■	★	▲	■	★	★	▲	★
			[	1	)					

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Note that the coloring is stable. Double swaps use a color.

Thm: $\mathsf{RCA}_0 \vdash \mathsf{SRT}^2_2 \to \mathsf{ERT}$

Apply SRT^2_2. Consider a big (e.g. $3 \cdot 2^{k-1}$) homogeneous set.
Suppose every interval contains a singleton color.

h_0 h_1 • • • • • • • • •

[s_0)

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[& & & & & & & & s_0 & &) \\
[& s_1 & & & & & & &) & & \leftarrow \text{no } s_0
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[s_1) $\leftarrow$ no s_0

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In k steps we find an interval with no possible color. $\rightarrow\leftarrow$
So every interval contains no singleton colors.
h_0 is the b for ERT.

Partial functions and $P\Sigma_0^0$

$P\Sigma_0^0$ asserts the existence of certain sequences for partial functions.

First order version: See Hájek and Pudlák [4]

Second order formulation: See Kreuzer and Yokoyama [6]

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Example:

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$f(n)$	2	?	4	6	?	9	3	?

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$f(n)$	2	?	4	6	?	9	3	?
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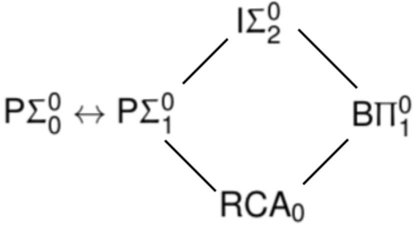
$P\Sigma^0_0$: for any partial f and any k, we can find $s_0, s_1 \dots s_k$.

Partial functions and $P\Sigma^0_0$

$P\Sigma^0_0$: for any partial f and any k, we can find $s_0, s_1 \ldots s_k$.

$P\Sigma^0_n$: sequences for Σ^0_n definable partial functions.



Thm: $\mathsf{RCA}_0 \vdash \mathsf{P}\Sigma^0_0 \to \mathsf{ERT}$

Goal: Convert a k coloring of $\mathbb{N}$ into a partial function.

Method: Values point to next matching location.

Example:

	▲	■	■	★	▲	★	★	★	▲	★	★
n	0	1	2	3	4	5	6	7	8	9	10
f(n)	4										

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Example:

	▲	■	■	★	▲	★	★	★	▲	★	★
n	0	1	2	3	4	5	6	7	8	9	10
f(n)	4	2	?	5	8	6	7	9	?	10	11

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	▲	■	■	★	▲	★	★	★	▲	★	★
n	0	1	2	3	4	5	6	7	8	9	10
f(n)	4	2	?	5	8	6	7	9	?	10	11
					s_0				s_1	s_2	s_3

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	▲	■	■	★	▲	★	★	★	▲	★	★
n	0	1	2	3	4	5	6	7	8	9	10
f(n)	4	2	?	5	8	6	7	9	?	10	11
					s_0				s_1	s_2	s_3
spec	$<$ ■	▲	★		$>$	$<$	▲	★	$>$	$<$ ★ $>$	$<$ ★ $>$

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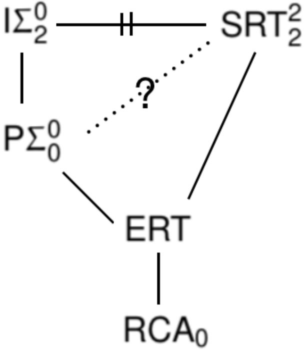
n	0	1	2	3	4	5	6	7	8	9	10
f(n)	4	2	?	5	8	6	7	9	?	10	11
					s_0				s_1	s_2	s_3
spec	$<$ ■	▲	★		$>$	$<$	▲	★	$>$	$<$ ★ $>$	$<$ ★ $>$

The spectra are descending subsets of the colors.
When they match, the leading edge is b for ERT.

Summary for ERT



Hypergraphs and compactness

A vertex coloring of a hypergraph is *strong* if it is injective on every edge.

Theorem: (RCA_0) The following are equivalent:

1. WKL_0
2. Let H be a hypergraph with a set of finite sets for edges. If every finite partial hypergraph of H has a strong 3-coloring, then H has a strong k-coloring for some k.
3. Let H be a hypergraph with a sequence of finite sets for edges. If every finite partial hypergraph of H has a strong 2-coloring, then H has a strong k-coloring for some k.

$RCA_0 + I\Sigma^0_2 \vdash WKL_0 \leftrightarrow$ every locally 2-colorable graph is finitely colorable. See Schmerl [7] and section 5 of [3].

References

[1] C. T. Chong, Theodore A. Slaman, and Yue Yang, *The metamathematics of stable Ramsey's theorem for pairs*, J. Amer. Math. Soc. **27** (2014), no. 3, 863–892. DOI 10.1090/S0894-0347-2014-00789-X MR3194495.

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How many 2-colorings of K5 have no 1-colored K3 ?

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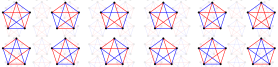
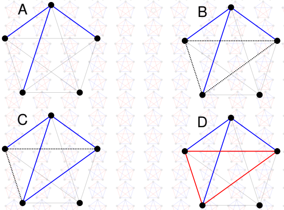
Introduction

Of the 1024 possible 2-colorings of K5, only 12 have no 1-colored triangles.



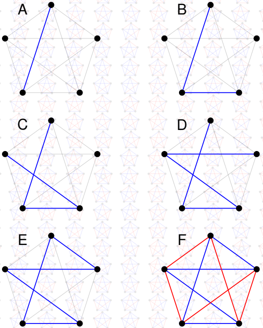
Claim 1

If any 3 edges match, then there is a 1-colored triangle.



Claim 2

If G has no 1-colored triangles, then G has a 1-colored 5-cycle.

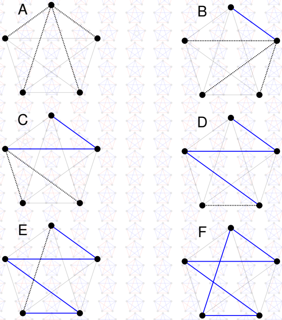


E: 1-colored 5-cycle

F: Remaining edges form a 5-cycle

Claim 3

There are 12 ways to construct a 1-colored 5-cycle.



$$\frac{4 \cdot 3 \cdot 2 \cdot 1 \cdot 1}{2} = 12$$

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