Reverse mathematics, graphs, and matchings

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March 11, 2014

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Reverse Mathematics

Goal: Determine exactly which set existence axioms are needed to prove familiar theorems.

Method: Prove results of the form

$$\mathrm{RCA}_0 \vdash \mathbf{AX} \leftrightarrow \mathbf{THM}$$

where:

- RCA_0 is a weak axiom system,
- **AX** is a set existence axiom selected from a small hierarchy of axioms, and
- **THM** is a familiar theorem.

Why bother?

Work in reverse mathematics can:

- precisely categorize the logical strength of theorems.
- differentiate between different proofs of theorems.
- provide insight into the foundations of mathematics.
- utilize and contribute to work in many subdisciplines of mathematical logic – including proof theory, computability theory, models of arithmetic, etc.

RCA_0

Language:

Integer variables: x, y, z Set variables: X, Y, Z

Axioms:

basic arithmetic axioms

(0, 1, +, $\times$, =, and $<$ behave as usual.)

Restricted induction

$(\psi(0) \wedge \forall n(\psi(n) \to \psi(n+1))) \to \forall n \psi(n)$

where $\psi(n)$ has (at most) one number quantifier.

Recursive set comprehension

If $\theta \in \Sigma^0_1$ and $\psi \in \Pi^0_1$, and $\forall n(\theta(n) \leftrightarrow \psi(n))$,

then there is a set X such that $\forall n(n \in X \leftrightarrow \theta(n))$

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- Elements of countable collections of objects can be identified with natural numbers.

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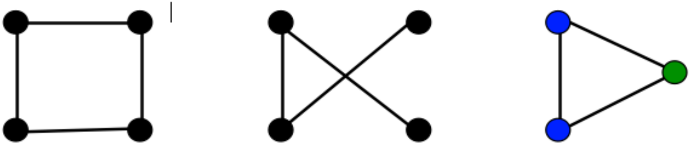
- RCA_0 can prove the arithmetic associated with pairing functions.
- Sets of pairs correspond to functions and/or countable sequences.
- Many mathematical concepts can be encoded in terms of such sequences. Second order arithmetic is remarkably expressive.
- Some coding can be averted: See Friedman's work on Strict Reverse Mathematics or Kohlenbach's *Higher Order Reverse Mathematics* in *Reverse Mathematics 2001*.

An example

Theorem

(RCA_0) *Every finite graph with maximum degree 2 and no cycles of odd length is bipartite (i.e. can be 2-colored).*

The idea behind the proof:



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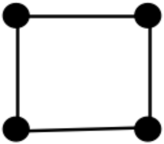

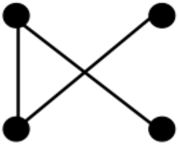

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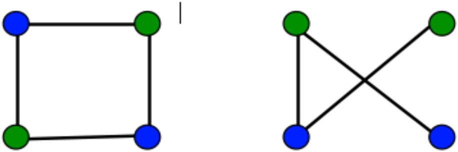

WKL_0

Weak König's Lemma

Statement: Big very skinny trees are tall.

More formally: If T is an infinite tree in which each node is labeled 0 or 1, then T contains an infinite path.

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More formally: If T is an infinite tree in which each node is labeled 0 or 1, then T contains an infinite path.

Or the contrapositive: If a 0-1 tree T has no infinite paths, then it must be finite.

The subsystem WKL_0 is RCA_0 plus Weak König's Lemma.

Note: RCA_0 cannot prove WKL_0

Finally! Some reverse mathematics!

Theorem

(RCA_0) *The following are equivalent:*

1. WKL_0.
2. *Every 2-regular graph with no cycles of odd length is bipartite.*

Note: RCA_0 proves that a graph is bipartite if and only if there is a 2-coloring of its nodes.

WKL_0 implies bipartite graph theorem

Suppose G is a graph with vertices $v_0, v_1, v_2, \ldots$ and no odd cycles.

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We need to use a $0 - 1$ tree to cook up a 2-coloring of G.

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Suppose G is a graph with vertices $v_0, v_1, v_2, \ldots$ and no odd cycles.

We need to use a $0-1$ tree to cook up a 2-coloring of G.

Let T be the tree consisting of sequences of the form $\langle i_0, i_1, \ldots, i_n \rangle$ where the sequence is a correct 2-coloring of the subgraph of G on the vertices $v_0, v_1, \ldots, v_n$.

Since G has no odd cycles, RCA_0 proves T contains infinitely many nodes.

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Since G has no odd cycles, RCA_0 proves T contains infinitely many nodes.

Any path through T is the desired 2-coloring.

A tool for reversals

Theorem

(RCA$_0$) *The following are equivalent:*

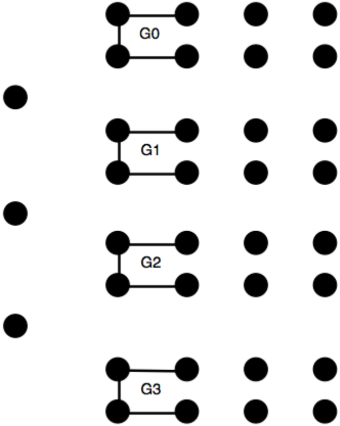
1. WKL$_0$.
2. *If* f *and* g *are injective functions from* $\mathbb{N}$ *into* $\mathbb{N}$ *and* $Ran(f) \cap Ran(g) = \emptyset$*, then there is a set* X *such that* $Ran(f) \subset X$ *and* $X \cap Ran(g) = \emptyset$*.*

Comment: X in (2) is like a separating set for disjoint computably enumerable sets.

The bipartite graph theorem implies WKL_0. A reversal!

Suppose we are given f and g with $Ran(f) \cap Ran(g) = \emptyset$.

If, for example, $f(0) = 1$ and $g(0) = 2$, build G like this:



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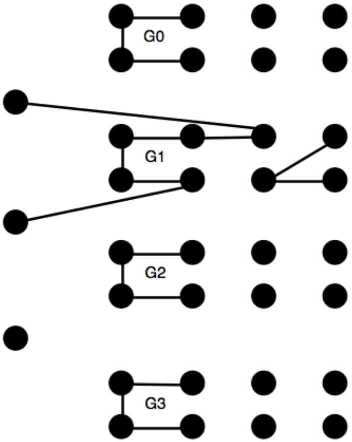
Suppose we are given f and g with $Ran(f) \cap Ran(g) = \emptyset$.

If, for example, $f(0) = 1$ and $g(0) = 2$, add edges for f



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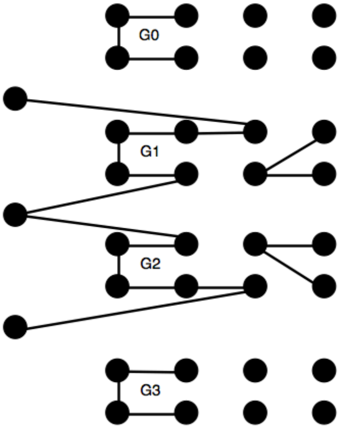

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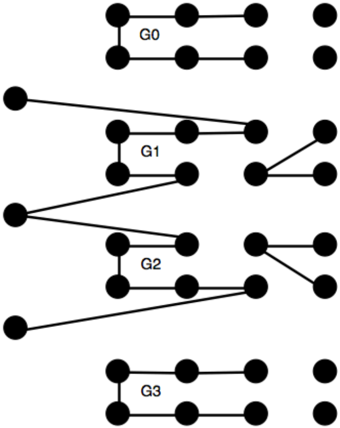
If, for example, $f(0) = 1$ and $g(0) = 2$, extend elsewhere



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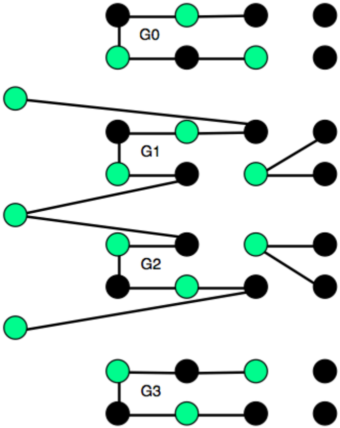
If, for example, $f(0) = 1$ and $g(0) = 2$, color and finish. . .



A few other theorems equivalent to WKL_0

Theorem

(RCA_0) *The following are equivalent:*

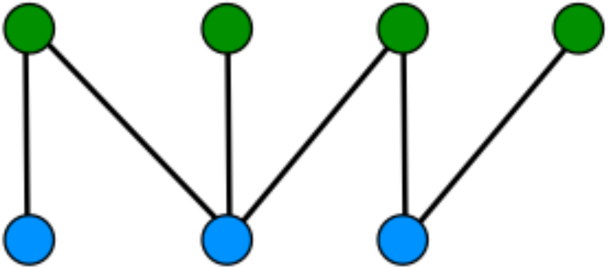
1. WKL_0.
2. *Every ctn. function on* $[0,1]$ *is bounded. (Simpson)*
3. *The closed interval* $[0,1]$ *is compact. (Friedman)*
4. *Every closed subset of* $\mathbb{Q} \cap [0,1]$ *is compact. (Hirst)*
5. *Existence theorem for solutions to ODEs. (Simpson)*
6. *If* $\langle x_n \rangle_{n \in \mathbb{N}}$ *is a sequence of real numbers then there is a sequence of natural numbers* $\langle i_n \rangle_{n \in \mathbb{N}}$ *such that for each* j, $x_{i_j} = \min\{x_n \mid n \leqslant j\}$. *(Hirst)*

Results about matchings: joint work with Noah Hughes

RCA_0 proves the following theorem of Philip Hall

Theorem

(RCA_0) *If $M = (B, G)$ is a finite society such that $|G(B_0)| \geqslant |B_0|$ for every $B_0 \subset B$, then M is espousable.*



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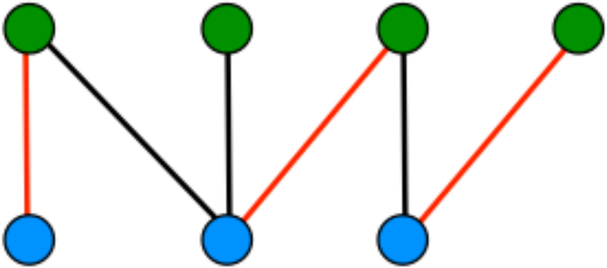
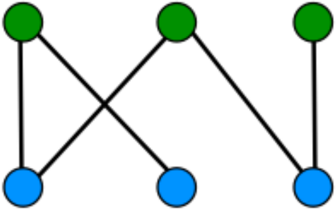


Results about matchings: joint work with Noah Hughes

...and RCA_0 proves the following theorem of Marshall Hall, Jr.

Theorem

(RCA_0) *If $M = (B, G)$ is a finite society with a unique espousal, then there is an enumeration of B such that for every i, $|G(\{b_0, \ldots, b_{i-1}\})| = i$.*



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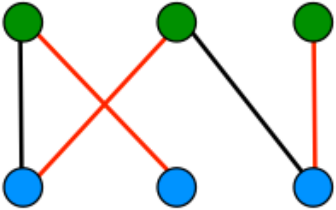

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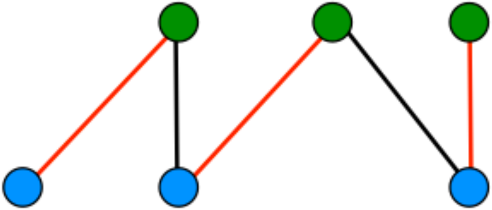


A result about infinite matchings (with Noah)

Theorem

(RCA_0) *The following are equivalent:*

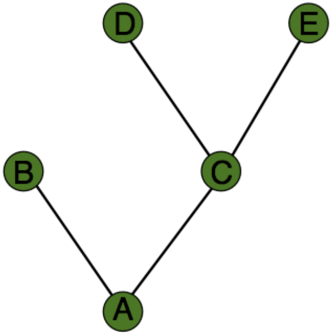
1. WKL_0.
2. *Suppose* $M = (B, G)$ *is a society and* $h(b) = |G(b)|$ *for every* $b \in B$. *If* M *has a unique espousal, then there is an enumeration of* B *such that for every* i, $|G(\{b_0, \ldots, b_{i-1}\})| = i$.

Note: The existence of the enumeration is actually a necessary and sufficient condition for the existence of a unique espousal.

Sketch of the reversal

We need to use the existence of the enumeration to show that a tree with no infinite paths is finite.

Here's a tree with no paths. Nodes are girls



Sketch of the reversal

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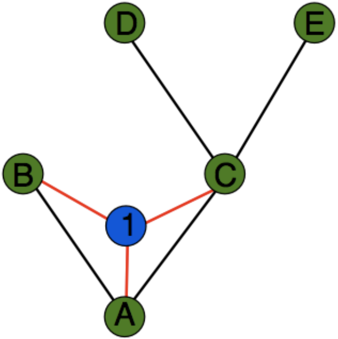
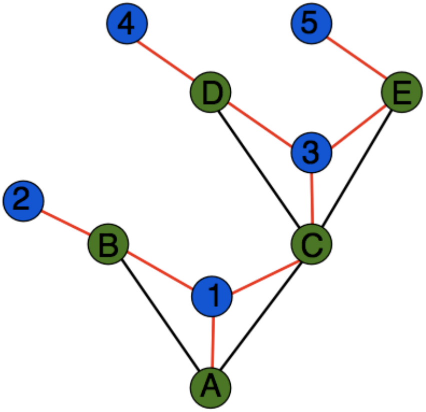
Here's a tree with no paths. Add a boy.



Sketch of the reversal

We need to use the existence of the enumeration to show that a tree with no infinite paths is finite.

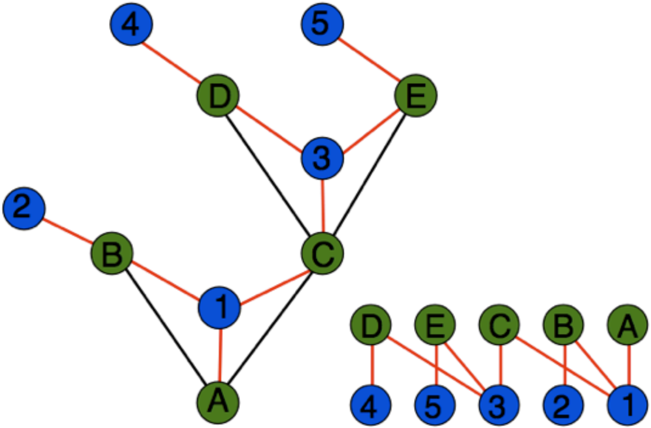
Here's a tree with no paths. Complete the society.



Sketch of the reversal

We need to use the existence of the enumeration to show that a tree with no infinite paths is finite.

Here's a tree with no paths. In any enumeration, the root boy is last and has finitely many predecessors. The tree is finite.



Arithmetical Comprehension

ACA_0 is RCA_0 plus the following comprehension scheme:

> For any formula $\theta(n)$ with only number quantifiers, the set $\{n \in \mathbb{N} \mid \theta(n)\}$ exists.

The minimum ω model of ACA_0 contains all the arithmetically definable sets.

Note: $WKL_0 \nvdash ACA_0$, but $ACA_0 \vdash WKL_0$.

ACA_0 and Graph Theory

Theorem

(RCA_0) *The following are equivalent:*

1. ACA_0
2. *Every graph can be decomposed into its connected components.*

Half of the proof: To prove that 1) implies 2), let G be a graph with vertices $v_0, v_1, \ldots$

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Define f by letting $f(n)$ be the least j such that there is a path from v_n to v_j.

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Define f by letting $f(n)$ be the least j such that there is a path from v_n to v_j.

By ACA_0, f exists. f is the desired decomposition.

A tool for reversals to ACA_0

Theorem

(RCA_0) *The following are equivalent:*

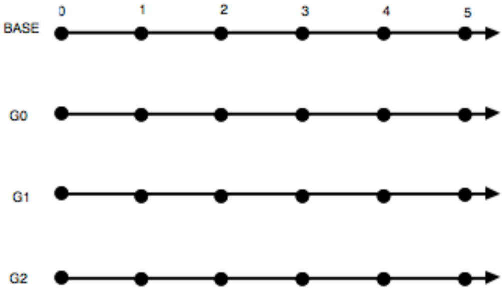
1. ACA_0
2. *If $f : \mathbb{N} \to \mathbb{N}$ is 1-1, then Ran(f) exists.*

To prove that the graph decomposition theorem implies ACA_0, we want to use a graph decomposition to calculate the range of a function.

The graph decomposition theorem implies ACA_0

Suppose we are given an injection f.

If, for example, $f(0) = 2$ and $f(1) = 0$, we will construct the graph G as follows:



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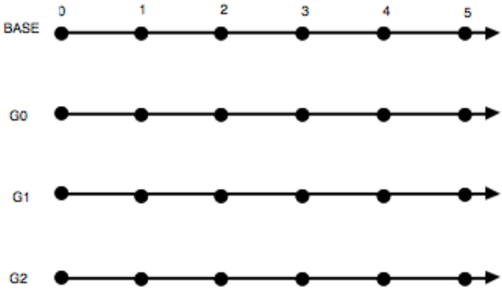

Add links for each value of f.

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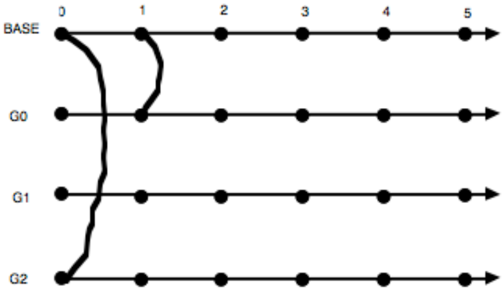

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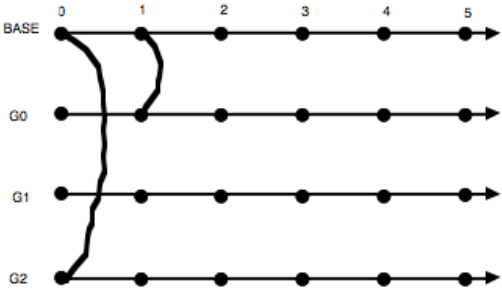

Add links for each value of f.Decompose G.

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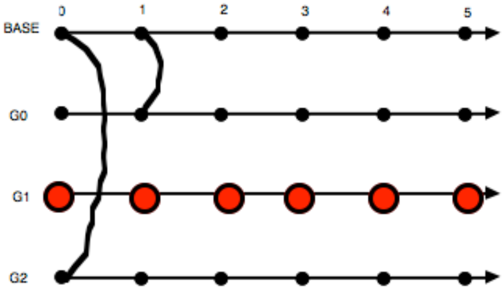

The range of f is computable from the decomposition.

Other theorems equivalent to ACA_0

Theorem

(RCA_0) *The following are equivalent:*

1. ACA_0.
2. *Bolzano-Weierstraß theorem. (Friedman)*
3. *Cauchy sequences converge. (Simpson)*
4. *Ramsey's theorem for triples. (Simpson)*
5. Σ^0_1*-CE (a maximality principle).*

(Dzhafarov and C. Mummert)

Arithmetical Transfinite Recursion

ATR_0 consists of RCA_0 plus axioms that allow iteration of arithmetical comprehension along any well ordering. This allows transfinite constructions.

Theorem

(RCA_0) *The following are equivalent:*

1. ATR_0.
2. *Lusin's Separation Theorem: Any two disjoint analytic sets can be separated by a Borel set. (Simpson)*
3. *Mahlo's Theorem: Given any two countable closed compact subsets of the reals, one can be homeomorphically embedded in the other. (Friedman and Hirst)*
4. *Every countable reduced Abelian p-group has an Ulm resolution. (Friedman, Simpson, and Smith)*
5. *Sherman's Inequality: If α, β, and γ are countable well orderings, then $(\alpha + \beta)\gamma \leqslant \alpha\gamma + \beta\gamma$. (Hirst)*

Π^1_1 comprehension

The system $\Pi^1_1 - CA_0$ is RCA_0 plus the axioms asserting the existence of the set $\{n \in \mathbb{N} \mid \theta(n)\}$ for $\theta \in \Pi^1_1$. (That is, θ has one universal set quantifier and no other set quantifiers.)

Theorem

(RCA_0) *The following are equivalent:*

1. $\Pi^1_1 - CA_0$.
2. *If* $\langle T_i \rangle_{n \in \mathbb{N}}$ *is a sequence of trees then there is a function* $f : \mathbb{N} \to 2$ *such that* $f(n) = 1$ *iff* T_n *is well founded.*
3. *Cantor/Bendixson Theorem: Every closed subset of* $\mathbb{R}$ *is the union of a countable set and a perfect set.*

An abbreviated list of references

[1] Harvey Friedman, *Some systems of second order arithmetic and their use*, Proceedings of the International Congress of Mathematicians (Vancouver, B. C., 1974), Vol. 1, 1975, pp. 235–242. http://www.mathunion.org/ICM/ MR0429508.

[2] Harvey Friedman, *Abstracts: Systems of second order arithmetic with restricted induction, I and II*, J. Symbolic Logic **41** (1976), 557–559. http://www.jstor.org/stable/2272259.

[3] Stephen G. Simpson, *Subsystems of second order arithmetic*, 2nd ed., Perspectives in Logic, Cambridge University Press, Cambridge, 2009. DOI 10.1017/CBO9780511581007 MR2517689.

Things that don't fit

Theorems are interesting when they are equivalent to one of the "big five," and also when they aren't.

- The infinite pigeon-hole principle, $\mathsf{RT}(1)$, is not provable in WKL_0. $\mathsf{RT}(1)$ is equivalent to a Σ^0_2 bounding principle.
- Ramsey's theorem for pairs on trees, $\mathsf{TT}(2)$, implies the usual form of Ramsey's theorem for pairs, $\mathsf{RT}(2)$. Can the proof that $\mathsf{RT}(2)$ does not imply ACA_0 be adapted to show that $\mathsf{TT}(2)$ does not imply ACA_0?
- The statement "every graph with finitely many connected components can be decomposed into its connected components" is equivalent to induction for Σ^0_2 formulas over RCA_0. RCA_0 proves that every graph with finitely many components has a connected component. Does RCA_0 prove that every graph has a connected component?
- Full Ramsey's theorem is equivalent to ACA_0^+ (Mileti).