Two Familiar Principles in Disguise

Jeff Hirst[1]
Appalachian State University
Boone, NC

These slides are available at: www.mathsci.appstate.edu/~jlh

October 27, 2011

Notre Dame Logic Seminar

[1] Jeff Hirst's research is partially supported by the John Templeton Foundation. Any opinions expressed here are those of the author and do not necessarily reflect the views of the John Templeton Foundation.

A weak form of Hindman's theorem

HIL: Suppose $f : \mathbb{N}^{<\mathbb{N}} \to k$ is a finite coloring of the finite subsets of the natural numbers. Then there is a an infinite sequence $\langle X_i \rangle_{i \in \mathbb{N}}$ of distinct finite sets and a color $c < k$ such that for every finite set $F \subset \mathbb{N}$ we have $f(\cup_{i \in F} X_i) = c$.

HTU: Suppose $f : \mathbb{N}^{<\mathbb{N}} \to k$ is a finite coloring of the finite subsets of the natural numbers. Then there is a an infinite sequence $\langle X_i \rangle_{i \in \mathbb{N}}$ of increasing finite sets and a color $c < k$ such that for every finite set $F \subset \mathbb{N}$ we have $f(\cup_{i \in F} X_i) = c$.

$X_i < X_j$ means $\max(X_i) < \min(X_j)$

Theorem

(RCA_0) *The following are equivalent:*

1. HIL.
2. $\mathrm{RT}(1)$: *If $f : \mathbb{N} \to k$ then there is a $c < k$ such that $\{n \mid f(n) = c\}$ is infinite.*

Theorem

(RCA_0) *The following are equivalent:*

1. HIL.
2. $\mathrm{RT}(1)$*: If* $f : \mathbb{N} \to k$ *then there is a* $c < k$ *such that* $\{n \mid f(n) = c\}$ *is infinite.*

Sketch.

$(1) \to (2)$. Given $f : \mathbb{N} \to k$, define $g(x) = f(\max(X))$. Apply HIL. Thin so $\max(X_i) < \max(X_{i+1})$. f is constant on the maxima.

Theorem

(RCA_0) *The following are equivalent:*

1. HIL.
2. RT(1)*: If $f : \mathbb{N} \to k$ then there is a $c < k$ such that $\{n \mid f(n) = c\}$ is infinite.*

Sketch.

$(1) \to (2)$. Given $f : \mathbb{N} \to k$, define $g(x) = f(\max(X))$. Apply HIL. Thin so $\max(X_i) < \max(X_{i+1})$. f is constant on the maxima.

$(2) \to (1)$. Fiven $f : \mathbb{N}^{<\mathbb{N}} \to k$, define $g(n) = f([0, n])$. Apply RT(1) to find $n_0, n_1, \ldots$ monochromatic. Let $X_i = [0, n_i]$. □

Why bother?

Based on Tait's work, Simpson [6] says that a theorem is *finitistically reducible* if it is provable in a theory which is a conservative extension of PRA (primitive recursive arithmetic) for Π^0_1 sentences.

$\mathsf{WKL}_0 + \mathsf{RT}(1)$ is conservative over PRA for Π^0_2 formulas.

Since $\mathsf{WKL}_0 + \mathsf{RT}(1)$ proves $\mathsf{RCA}_0 + \mathsf{HIL}$, we know HIL is finitistically reducible.

Why bother?

Based on Tait's work, Simpson [6] says that a theorem is *finitistically reducible* if it is provable in a theory which is a conservative extension of PRA (primitive recursive arithmetic) for Π^0_1 sentences.

$\mathsf{WKL}_0 + \mathsf{RT}(1)$ is conservative over PRA for Π^0_2 formulas.

Since $\mathsf{WKL}_0 + \mathsf{RT}(1)$ proves $\mathsf{RCA}_0 + \mathsf{HIL}$, we know HIL is finitistically reducible.

$\mathsf{RCA}_0 + \mathsf{HTU}$ proves ACA_0 [1], so $\mathsf{RCA}_0 + \mathsf{HTU}$ proves Π^0_1 formulas that PRA can't.

The consistency of PRA is a Π^0_1 formula.
HTU is not finitistically redicible.

Decomposing graphs

Two vertices of a graph lie in the same connected component if there is a path between them.

A *decomposition* of a graph into connected components is a function f mapping vertices into $\mathbb{N}$ such that v_1 and v_2 lie in the same connected component if and only if $f(v_1) = f(v_2)$.

Theorem

(RCA_0) *The following are equivalent:*

1. ACA_0*.*
2. *Every graph can be decomposed into its connected components.*

Finitely many components

A graph has *at most k connected components* if every collection of $k+1$ vertices has at least one pair that is connected by a path.

DkG: For every k, if G has at most k connected components, then G can be decomposed into its connected components.

Finitely many components

A graph has *at most k connected components* if every collection of $k+1$ vertices has at least one pair that is connected by a path.

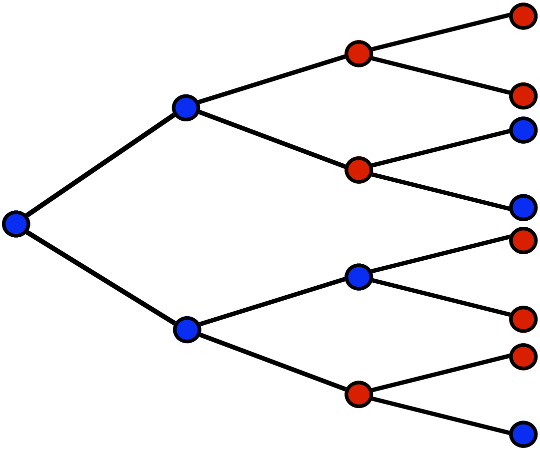
DkG: For every k, if G has at most k connected components, then G can be decomposed into its connected components.

Theorem
RCA_0 *proves that* Σ^0_2-IND *implies* DkG.

Sketch.
Σ^0_2-IND (in the form of Π^0_2-LE) proves that there is a least code for a sequence of vertices such that every vertex is path connected to some sequence element. □

Another pigeonhole principle

TT(1): For any finite coloring of $2^{<\mathbb{N}}$, there is a monochromatic subtree order-isomorphic to $2^{<\mathbb{N}}$.



Not your garden variety pigeonhole principle

$\mathsf{TT}(1)$: For any finite coloring of $2^{<\mathbb{N}}$, there is a monochromatic subtree order-isomorphic to $2^{<\mathbb{N}}$.

$\mathsf{RT}(1)$: If $f : \mathbb{N} \to k$ then there is a $c < k$ such that $\{n \mid f(n) = c\}$ is infinite.

A theorem of Corduan, Groszek, and Mileti [2]:

Theorem
$\mathsf{RCA}_0 + \mathsf{RT}(1)$ *does not prove* $\mathsf{TT}(1)$.

Their proof shows how to extend any model where Σ^0_2-IND fails to a model where $\mathsf{TT}(1)$ fails.

Theorem

RCA_0 *proves that* DkG *implies* $\mathsf{TT}(1)$.

Ideas for the proof:

Given $f : 2^{<\mathbb{N}} \to k$, we want to build some new graph G with finitely many connected components. We'll use the decomposition of G to find a monochromatic subtree for f.

We can enumerate the nodes in $2^{<\mathbb{N}}$.

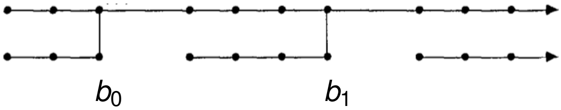
For any node n, let T_n denote all the nodes extending it (including n).

Let $\mathrm{Sp(T_n)}$ be shorthand for the *spectrum* above n, that is, the range of f on T_n.

Constructing the graph

Construct G from subgraphs G_X for each non-empty $X \subset [0, k)$.

G_X will look something like this:



where b_0 witnesses $\mathrm{Sp}(\mathrm{T}_0) \not\subset \mathrm{X}$,
b_1 witnesses $\mathrm{Sp}(\mathrm{T}_1) \not\subset \mathrm{X}$
and so on. . . .

Note that there is an n such that $\mathrm{Sp}(\mathrm{T}_\mathrm{n}) \subset \mathrm{X}$ if and only if G_X has two components.

Conclusion of the proof that DkG implies TT(1)

Suppose g is the decomposition of G.
WLOG suppose the range of g is an initial segment of $\mathbb{N}$.

We can calculate

- the exact size of the range of g.
- the exact number of components of G.
- the first vertex in each component.
- which subgraphs G_X have two components and which have one.

Conclusion of the proof that DkG implies TT(1)

Suppose g is the decomposition of G.
WLOG suppose the range of g is an initial segment of $\mathbb{N}$.

We can calculate

- the exact size of the range of g.
- the exact number of components of G.
- the first vertex in each component.
- which subgraphs G_X have two components and which have one.

Pick the first set $X_0 \subset [0, k)$ such that

G_{X_0} has two components, and

for every proper subset Y of X_0, G_Y has one component.

If $\mathrm{Sp}(\mathrm{T_n}) = \mathrm{X_0}$, then every extension of node n also has X_0 as its spectrum. Build the monochromatic subtree.

DkG and Σ^0_2-IND

We've shown that $\mathsf{RCA}_0 + \mathsf{DkG}$ implies $\mathsf{TT}(1)$.

RCA_0 also proves that DkG is equivalent to a Π^1_1 formula. (Every graph with at most *k* components has a minimal list of vertices such that every vertex can be reached from a list member.)

DkG and Σ^0_2-IND

We've shown that $\mathrm{RCA}_0 + \mathrm{DkG}$ implies $\mathrm{TT}(1)$.

RCA_0 also proves that DkG is equivalent to a Π^1_1 formula. (Every graph with at most k components has a minimal list of vertices such that every vertex can be reached from a list member.)

Corduan, Groszek, and Mileti [2] show that whenever θ is Π^1_1, $\mathrm{RCA}_0 + \theta \vdash \mathrm{TT}(1)$ if and only if $\mathrm{RCA}_0 + \theta \vdash \Sigma^0_2$-IND.

Consequently, $\mathrm{RCA}_0 \vdash \mathrm{DkG} \leftrightarrow \Sigma^0_2$-IND.

Decomposition of graphs with a finite number of connected components is equivalent to Σ^0_2-IND.

Bibliography

[1] Andreas R. Blass, Jeffry L. Hirst, and Stephen G. Simpson, *Logical analysis of some theorems of combinatorics and topological dynamics*, Logic and combinatorics (Arcata, Calif., 1985), Contemp. Math., vol. 65, Amer. Math. Soc., Providence, RI, 1987, pp. 125–156. MR891245 (88d:03113).

[2] Jared Corduan, Marcia J. Groszek, and Joseph R. Mileti, *Reverse mathematics and Ramsey's property for trees*, J. Symbolic Logic **75** (2010), no. 3, 945–954. DOI 10.2178/jsl/1278682209 MR2723776 (2011m:03025).

[3] Jeffry L. Hirst, *Connected components of graphs and reverse mathematics*, Arch. Math. Logic **31** (1992), no. 3, 183–192. DOI 10.1007/BF01269946 MR1147740 (92j:03052).

[4] ______, *Disguising induction: Proofs of the pigeonhole principle for trees*. To appear in the proceedings of Foundational Adventures, a conference in honor of Harvey Friedman. Draft available.

[5] ______, *Hilbert versus Hindman*. Submitted. Draft available.

[6] Stephen G. Simpson, *Subsystems of second order arithmetic*, 2nd ed., Perspectives in Logic, Cambridge University Press, Cambridge, 2009. DOI 10.1017/CBO9780511581007 MR2517689.