

Three Tales about Weak König's Lemma

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This talk contains work from collaboration with François Dorais, Damir Dzhafarov,
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These slides are available at: www.mathsci.appstate.edu/~jlh

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The first tale: Algebraic field extensions.

Defn: An algebraic extension of a countable field F is a pair $\langle K, \varphi \rangle$ where K is a countable field, φ is an embedding of F into K , and for every $a \in K$ there is a nonzero $f(x) \in F[x]$ such that $\varphi(f)(a) = 0$.

Theorem

(RCA₀) TFAE:

1. WKL₀
2. Let F be a field with algebraic closure \bar{F} . If $\alpha \in \bar{F}$ and $\varphi : F(\alpha) \rightarrow F(\alpha)$ is an F -automorphism of $F(\alpha)$, then φ extends to an F -automorphism of \bar{F} .

The second tale: Dichotomy on the reals.

Theorem

(RCA₀) *For every real α , either $\alpha \leq 0$ or $0 \leq \alpha$.*

Theorem

(RCA₀) *TFAE:*

1. WKL₀
2. *If $\langle \alpha_i \rangle_{i \in \mathbb{N}}$ is a sequence of real numbers, then there is a set I such that for all i ,*

$$(i \in I \rightarrow \alpha_i \leq 0) \wedge (i \notin I \rightarrow 0 \leq \alpha_i)$$

The third tale: Trees and forests.

Theorem

(RCA₀) TFAE:

1. WKL₀
2. seqWKL₀: *If $\langle T_i \rangle_{i \in \mathbb{N}}$ is a sequence of infinite 0-1 trees, then there is a sequence $\langle P_i \rangle_{i \in \mathbb{N}}$ of paths through them.*
3. seqWWKL₀: *If $\langle T_i \rangle_{i \in \mathbb{N}}$ is a sequence of infinite 0-1 trees of positive measure, then there is a sequence $\langle P_i \rangle_{i \in \mathbb{N}}$ of paths through them.*

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