Weihrauch analysis motivated by finite complexity

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Motivation

The following pattern was noted with Steffen Lempp [2]

	Finite	Infinite
Euler	P	ACA_0
Hamilton	NP complete	$\Pi^1_1{-}CA_0$

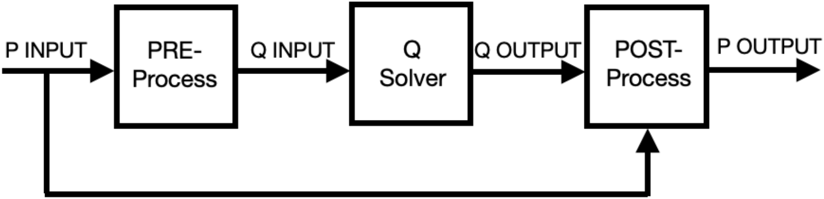
Does this pattern persist? No.

Does Weihrauch analysis tell us something different?

Weihrauch reductions

A problem P is weakly Weihrauch reducible to a problem Q (denoted $P \leqslant_W Q$) if we can find

- a computable pre-processor and
- a computable post-processor such that for every Q solver



is a P solver.

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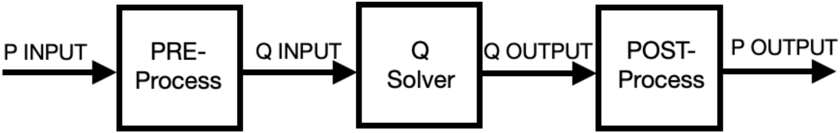

is a P solver.

Wiehrauch problems and Π^1_1–CA_0

Work with Zack BeMent

WF: Input: A tree T in $\mathbb{N}^{<\mathbb{N}}$
Output: 0 if T is not well founded, 1 otherwise.

S_L Input: A graph G
Output: 1 if the graph L is isomorphic to a subgraph of G,
0 otherwise.

The graph L:

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Prop: WF $\equiv_{\mathsf{sW}}$ S_L.

WF $\leqslant_{sW}$ S_L preprocessing: A tree is a graph.
postprocessing: Flip the output bit.

S_L $\leqslant_{sW}$ WF preprocessing: Build the tree of initial segments of isomorphisms from L into G.
postprocessing: Flip the output bit.

Wiehrauch problems and Π^1_1–CA_0

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Output: 0 if T is not well founded, 1 otherwise.

$\mathsf{S_L}$ Input: A graph G
Output: 1 if the graph L is isomorphic to a subgraph of G,
0 otherwise.

Prop: WF $\equiv_{\mathsf{sW}}$ $\mathsf{S_L}$.

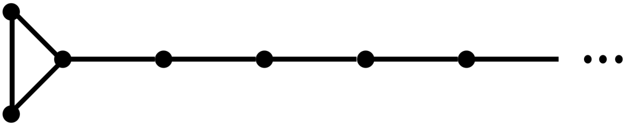
Coro: $\widehat{\mathsf{WF}} \equiv_{\mathsf{sW}} \widehat{\mathsf{S_L}}$.

$\widehat{\mathsf{WF}}$: Input: A sequence of trees $\langle T_i \rangle$ in $\mathbb{N}^{\mathbb{N}}$
Output: A function $f : \mathbb{N} \to 2$ such that $f(i) = 0$ if T_i is not well founded, 1 otherwise.

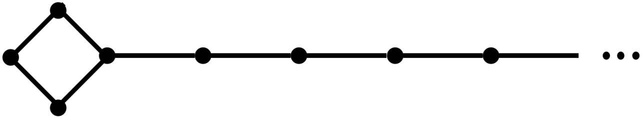
Wiehrauch problems and Π^1_1–CA_0

$S_{\widehat{L}}$: Input: A graph G
Output: A function f such that $f(n) = 1$ iff L_n is isomorphic to a subgraph of G

The graph L_0:



The graph L_1:

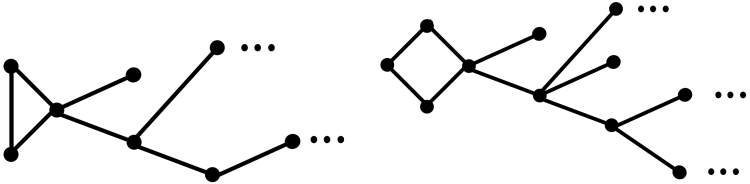


Wiehrauch problems and $\Pi^1_1\text{–}\mathsf{CA}_0$

$\mathsf{S}_{\widehat{\mathsf{L}}}$: Input: A graph G
Output: A function f such that $f(n) = 1$ iff L_n is isomorphic to a subgraph of G

Prop: $\widehat{\mathsf{WF}} \equiv_{sW} \mathsf{S}_{\widehat{\mathsf{L}}}$

A graph construction:



A question: Fixed graphs as subgraphs

LPO: Input: A function $f : \mathbb{N} \to \mathbb{N}$
Output: 0 if $0 \notin \mathit{Range}(f)$ and
$n + 1$ if n is least with $f(n) = 0$

Prop: If F is a finite graph with at least two vertices, then $\mathsf{S}_F \equiv_{\mathrm{W}}$ LPO.

Is there a graph H satisfying the following inequality?

$$\mathrm{LPO} \equiv_{\mathrm{W}} \mathsf{S}_F <_{\mathrm{W}} \mathsf{S}_H <_{\mathrm{W}} \mathsf{S}_L \equiv_{\mathrm{W}} \mathrm{WF}$$

Subgraphs of a fixed graph

S^G: Input: A graph H
Output: 1 if H is isomorphic to a subgraph of G,
0 otherwise.

Suppose K is the complete (countable) graph and D is the totally disconnected graph.

Prop: $\mathsf{S}^K \equiv_{\mathrm{W}} \mathsf{S}^D \equiv_{\mathrm{W}} \mathsf{LPO}$.

Is there a graph G with $\mathsf{S}^G \equiv_{\mathrm{W}} \mathsf{WF}$?

There is a computable graph G such that every S^G solver computes a Σ^1_1-complete set. [2]

Formal Weihrauch reducibility

Work with Asuka Wallace

We can carry out Weihrauch analysis in higher order arithmetic. The following is a consequence of work with Carl Mummert [3].

Thm: If $p(x, y)$ formalizes "y is a solution of the instance x of the problem P" and $q(x, y)$ formalizes the problem Q, then

$$\mathrm{iRCA}_0^\omega \vdash Q \leqslant_W P$$

if and only if

$$\mathrm{iRCA}_0^\omega \vdash \forall u \exists x \forall y \exists v (p(x, y) \to q(u, v)),$$

provided P and Q are total and $p(x, y) \to q(u, v)$ is in Γ_1.

- iRCA_0^ω is an intuitionistic version of RCA_0 with higher types.
- Γ_1 limits the use of $\exists$.

Versions of LPO

LPO: Input: A function $f : \mathbb{N} \to \mathbb{N}$
Output: 0 if $0 \notin \mathit{Range}(f)$ and
$n + 1$ if n is least with $f(n) = 0$

LPO$_t$: Input: A function $f : \mathbb{N} \to \mathbb{N}$
Output: 0 if $0 \notin \mathit{Range}(f)$ and 1 otherwise.

Note: LPO$_t \equiv_{\mathrm{W}}$ LPO but LPO$_t <_{\mathrm{sW}}$ LPO.

Versions of LPO

LPO: Input: A function $f : \mathbb{N} \to \mathbb{N}$
Output: 0 if $0 \notin Range(f)$ and
$n+1$ if n is least with $f(n) = 0$

LPO$(f) = n$ is: $(n = 0 \wedge \forall x(f(x) \neq 0)) \vee$
$(n \neq 0 \wedge f(n-1) = 0 \wedge \forall j < n-1(f(j) \neq 0))$

LPO$_t$: Input: A function $f : \mathbb{N} \to \mathbb{N}$
Output: 0 if $0 \notin Range(f)$ and 1 otherwise.

LPO$_t(f) = n$ is:
$(n = 0 \wedge \forall x(f(x) \neq 0)) \vee (n \neq 0 \wedge \exists x(f(x) = 0))$

Note: LPO$_t \equiv_{\mathrm{W}}$ LPO but LPO$_t <_{\mathrm{sW}}$ LPO.

Colorability of initial subgraphs

If G is a graph with vertices $\{v_0, v_1, v_2, \ldots\}$, let G_n be the induced subgraph with vertices $\{v_0, v_1, v_2, \ldots v_n\}$.

LG2 Input: A graph G.
Output: 0 if every G_k has a 2-coloring, $n \neq 0$ is G_n is the first non-2-colorable initial subgraph.

Prop: iRCA_0^ω proves the implications associated with LG2 $\leqslant_W$ LPO and LPO $\leqslant_W$ LG2.

Coro: iRCA_0^ω proves LG2 $\equiv_W$ LPO

Parallelization

Parallelization preserves Weihrauch equivalence in the formal setting. We know iRCA_0^ω proves $\mathrm{LG2} \equiv_W \mathrm{LPO}$.

Coro: iRCA_0^ω proves $\widehat{\mathrm{LG2}} \equiv_W \widehat{\mathrm{LPO}}$

Coro: Over iRCA_0, the following are equivalent:

(1) $\widehat{\mathrm{LG2}}$

(2) $\widehat{\mathrm{LPO}}$

(3) ACA_0

In the last corollary we are conflating (for example) $\widehat{\mathrm{LPO}}$ with the second order arithmetic statement that for every sequence of functions there is a sequence of natural numbers such that for all n, the n^{th} sequence element is the solution to LPO for f_n.

Why formalize?

It is messy to do even simple Weihrauch reductions in the formal setting. However...

- Formalization allows us to use Weihrauch methodology (e.g. parallelization) to prove results of reverse mathematics.
- In intuitionistic systems, proofs of reductions could be mined to extract pre/post processing algorithms. If the proofs act as verifications for the algorithms, this is a framework for deriving new verified problem solvers from a trusted library.

References

[1] Vasco Brattka, Guido Gherardi, and Arno Pauly, *Weihrauch Complexity in computable analysis* (2018), 1–61. arXiv:1707.03202.

[2] Jeffry L. Hirst and Steffen Lempp, *Infinite versions of some problems from finite complexity theory*, Notre Dame J. Formal Logic **37** (1996), no. 4, 545–553, DOI 10.1305/ndjfl/1040046141. MR1446228

[3] Jeffry L. Hirst and Carl Mummert, *Using Ramsey's theorem once*, Arch. Math. Logic **58** (2019), no. 7-8, 857–866, DOI 10.1007/s00153-019-00664-z. MR4003638