Reverse Mathematics and the Heine/Borel Theorem

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Thm: ($\mathbf{RCA}_0$) The following are equivalent:

1) $\mathbf{WKL}_0$: Every infinite 0-1 tree contains an infinite path.
2) Heine/Borel Theorem for $[0, 1]$:
   If $\langle (c_i, d_i) \mid i \in \mathbb{N} \rangle$ is a sequence of open intervals that cover $[0, 1]$, then some finite subsequence covers $[0, 1]$.

This theorem appears in Friedman's 1976 abstracts [1].

For the proof, see page 127 of Simpson's book [3].

Picture-sketch of the proof that $\mathbf{WKL}_0$ implies Heine/Borel.

Strategy: Given a cover, build a tree.

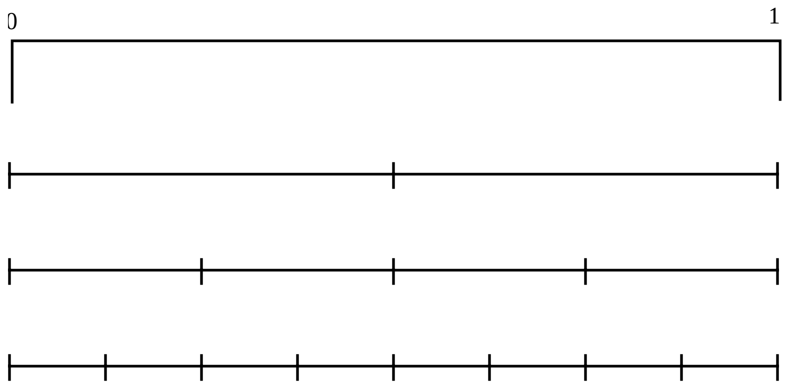

Figure 1: Subintervals of [0,1].

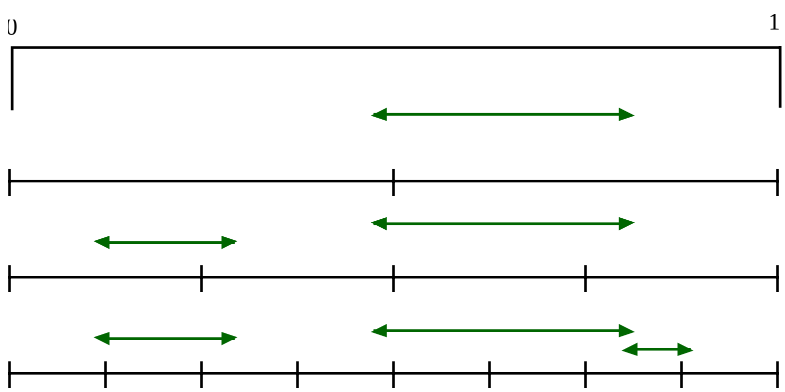

Figure 2: Subintervals with successive elements of the cover.

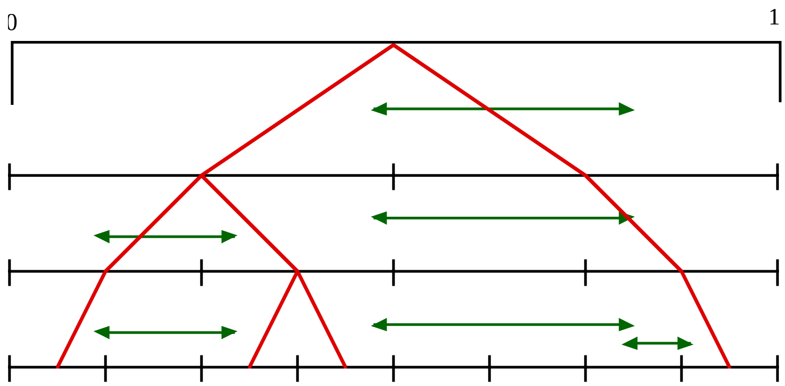

Figure 3: Elements of the cover and the associated 0-1 tree.

Picture-sketch of the proof that Heine/Borel implies $\mathbf{WKL}_0$.

Strategy: Given a tree, build a cover.

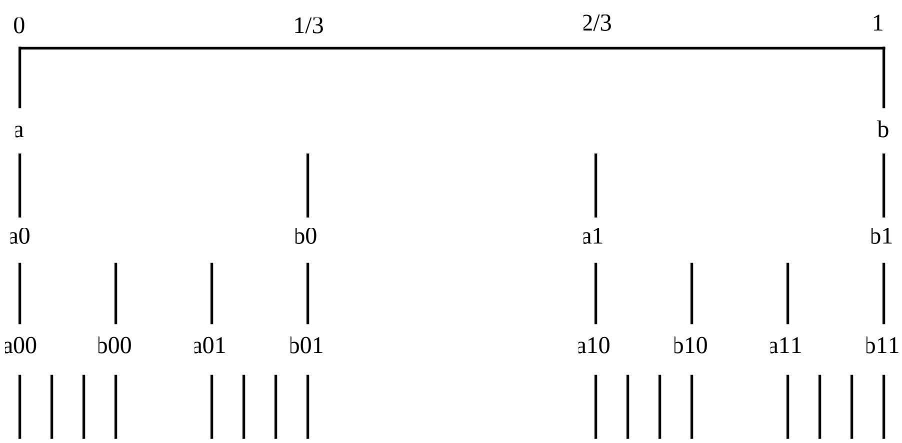

Figure 1: Points in [0,1] associated with the Cantor set.

Picture-sketch of H/B implies $\mathbf{WKL}_0$, continued.

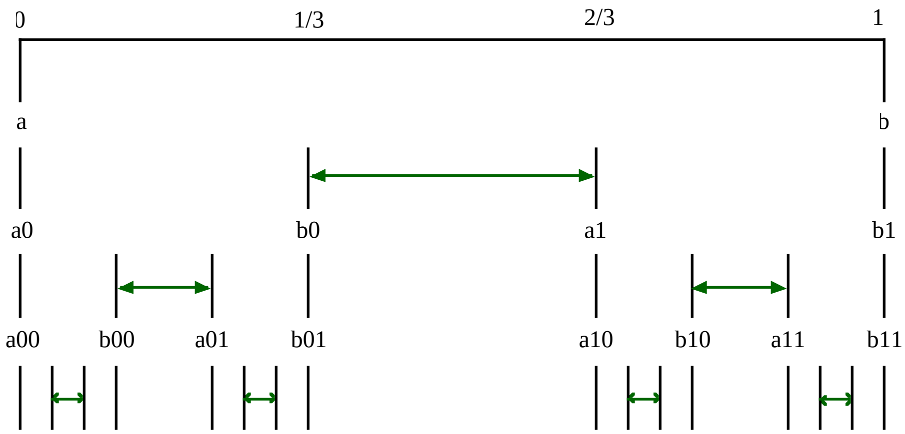

Figure 2: The "middle third" open intervals

(b0,a1), (b00,a01), (b10,a11), etc.

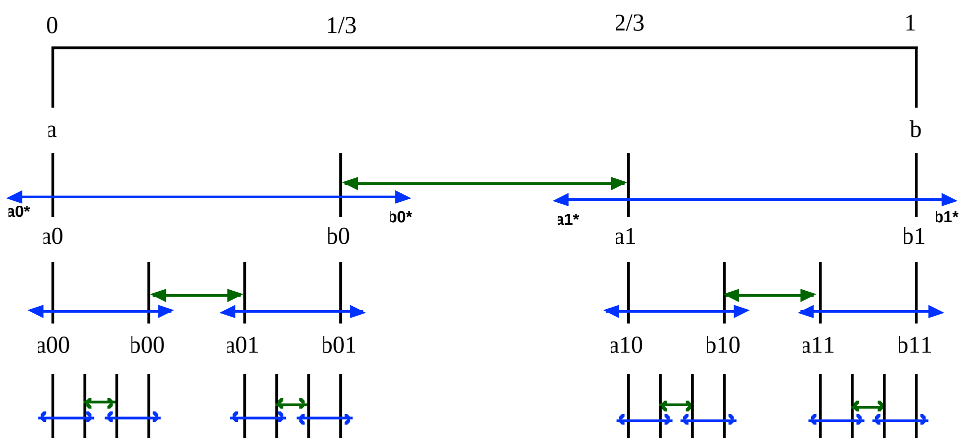

Figure 3: Intervals covering the Cantor set:

(a0*,b0*), (a1*,b1*), (a00*,b00*), etc.

Picture-sketch of H/B implies $\mathbf{WKL_0}$, continued.

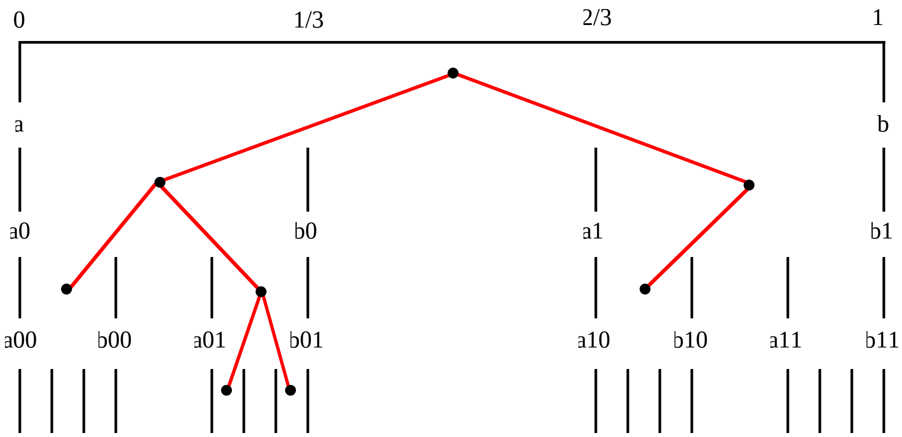

Figure 4: Embedded tree

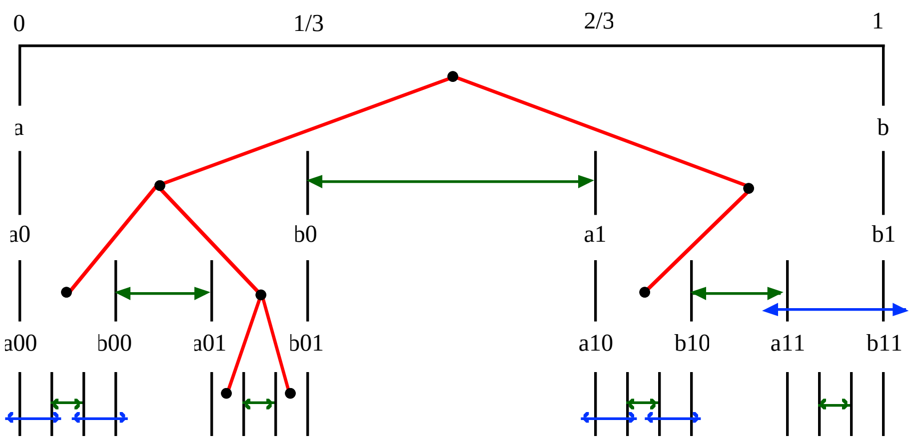

Figure 5: Embedded tree with associated cover of [0,1]

Thm: ($\mathbf{RCA}_0$) The following are equivalent:

1) $\mathbf{WKL}_0$.
2) Countable Heine/Borel theorem:
   If X is a closed subset of the rationals in $[0, 1]$ and $\langle (c_i, d_i) \mid i \in \mathbb{N} \rangle$ is a sequence of open intervals that cover X, then some finite subsequence covers X.

Note: A closed set is the complement of a union of open intervals.

This result is tangentially related to Friedman's new results on comparability.

The proof will appear in Hirst's *A note on compactness of countable sets* in Simpson's forthcoming book [2].

Picture-sketch of the proof that countable Heine/Borel implies $\mathbf{WKL}_0$.

Strategy: Given a tree, build a closed set and a cover.

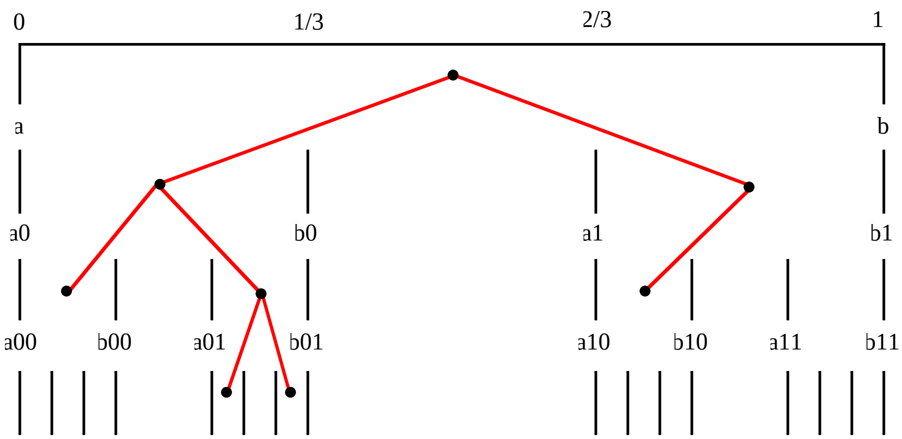

Figure 1: Embedded tree

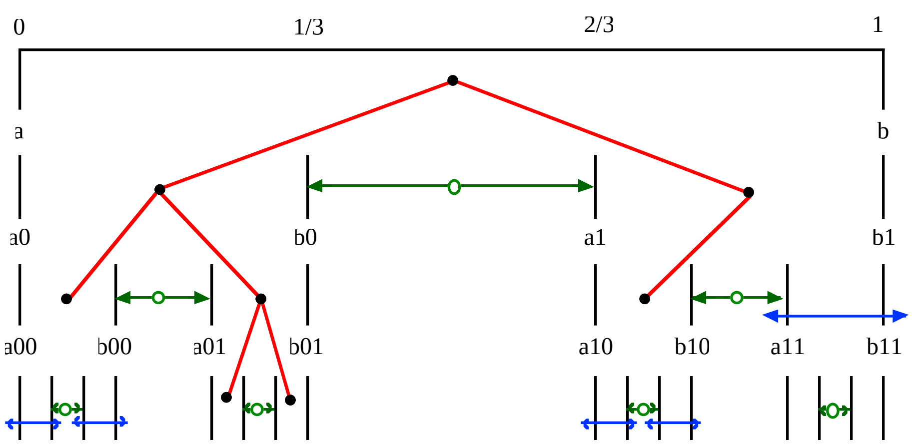

Figure 2: The embedded tree is the complement of a union of open sets.

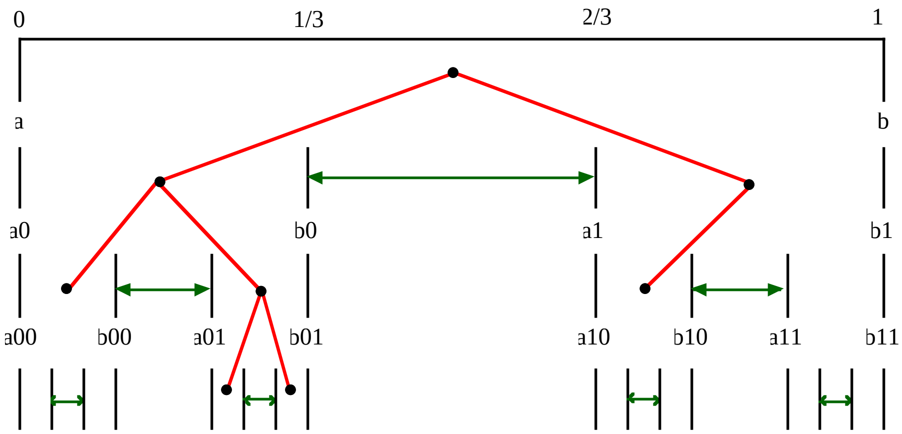

Figure 3: An open cover of the embedded tree.

The preceding theorem is sensitive to the definition of closed set.

We say a set $\bar{X}$ is separably closed if there is a countable set X such that $\bar{X}$ is collection of limits of sequences of elements of X.

Thm: ($\mathbf{RCA}_0$) The following are equivalent.

1) $\mathbf{ACA}_0$, arithmetical comprehension.
2) Every separably closed set of rationals in $[0, 1]$ is Heine/Borel compact.

Proof appears in [2].

Veblen ([4] 1905) said " We may note in passing, what seems to be a new result, that, ... the Heine/Borel theorem is a true theorem of any well-ordered set."

Thm: ($\mathbf{RCA}_0$) If α is a countable well ordered set with a largest element and $\langle (c_i, d_i) \mid i \in \mathbb{N} \rangle$ is a sequence of open intervals covering α, then some finite subsequence covers α.

Sketch of proof: The sequence defined by

$$x_i = \max\Big(\{\min(\alpha)\} \\ \cup \{\max(\alpha), c_j \mid j < i\} - \cup_{j<i}(c_i, d_i)\Big)$$

is a descending sequence.

References

1 H. Friedman, *Systems of second order arithmetic with restricted induction, I, II* (abstracts), Journal of Symbolic Logic, **41** (1976) 557–559.

2 J. Hirst, *A note on compactness of countable sets,* to appear in Simpson's forthcoming volume on reverse mathematics.

3 S. Simpson, *Subsystems of second order arithmetic,* Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1999.

4 O. Veblen, *Definition in terms of order alone in the linear continuum and in well-ordered sets,* Trans. of the AMS, **6** (1905) 165–171.