Reverse Mathematics of Ordinal Arithmetic

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From Cantor's *Contributions to the founding of the theory of transfinite numbers* [1]:

We call a simply ordered aggregate F "well-ordered" if its elements f ascend in a definite succession from a lowest f_1 in such a way that:

I. There is in F an element f_1 which is lowest in rank.

II. If F' is any part of F and if F has one or many elements of higher rank than all elements of F', then there is an element f' of F which follows immediately after the totality of F', so that no elements in rank between f' and F' occur in F.

Reverse Mathematics of Cantor's definition

Thm: (RCA_0) The following are equivalent:

1. ACA_0
2. ([1] §12 Thm A) Let X be a linear order. If every subset of X with a strict upper bound has a least strict upper bound, then every nonempty subset of X has a least element.
3. ([1] §12 Thm B) Let X be a linear order. If every nonempty subset of X has a least element, then every subset of X with a strict upper bound has a least strict upper bound.

Thm: (RCA_0) Let X be a linear order. Then every nonempty subset of X has a least element if and only if X contains no infinite descending sequences.

Notes on reversals:

(least strict u. b. $\rightarrow$ least elt.) $\rightarrow \mathsf{ACA}_0$

Assume $\neg\mathsf{ACA}_0$.

Use Friedman's [2] well ordering β satisfying

$$\omega \leq_W \beta \qquad \beta \not\leq_W \omega \qquad \omega+1 \not\leq_W \beta$$

Invert β; call the resulting order B.

Claim: Every subset of B with a strict upper bound has a least strict upper bound.

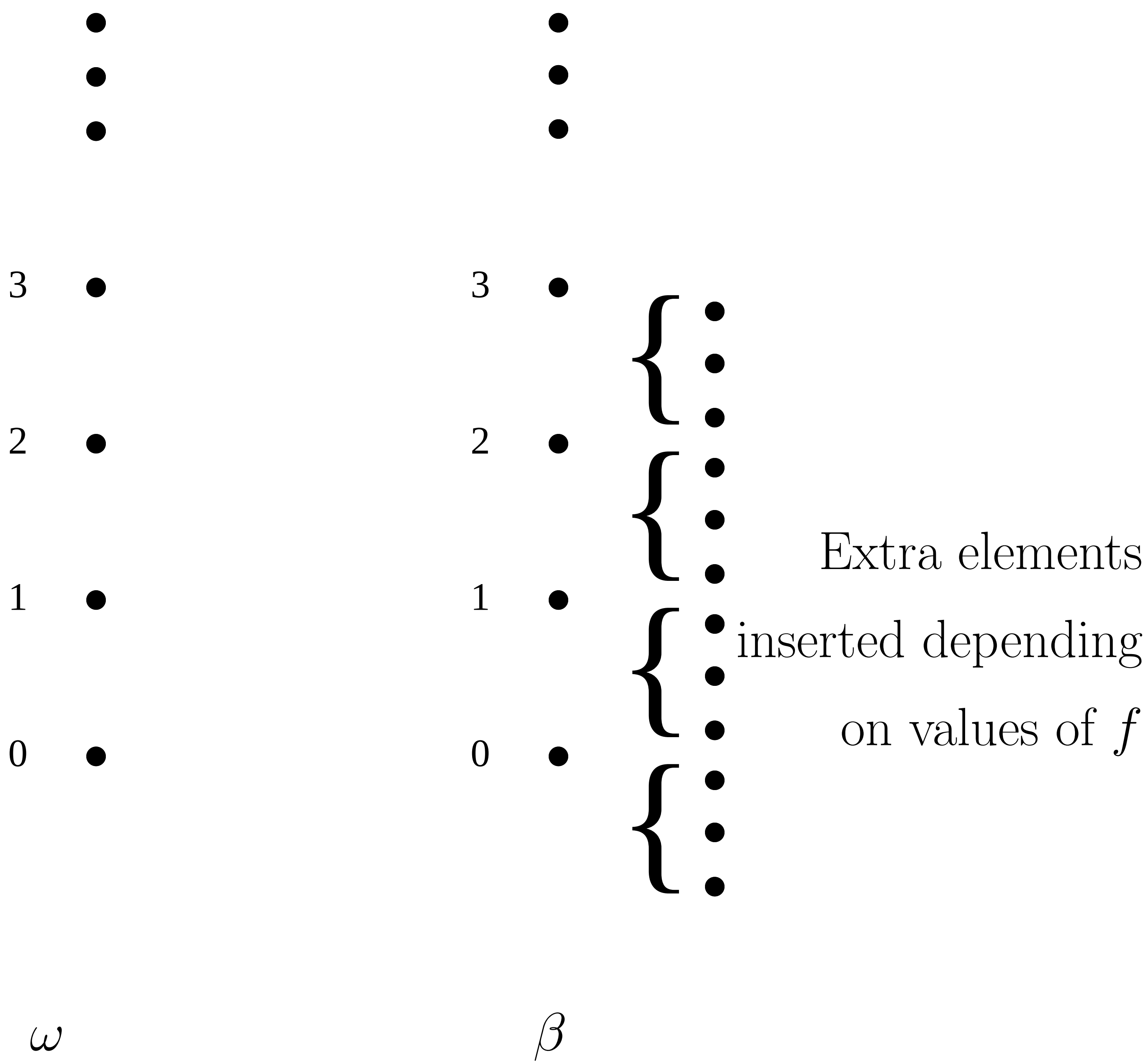
Let X be a subset of B with a strict upper bound. X has a largest element, μ. Because $\omega + 1 \not\leq_W \beta$, $\{x \in X \mid x > \mu\}$ has a least element, $\mu + 1$. This element is the least strict upper bound for X.

Claim: Not every subset of B has a least element.

$\omega \leq_W \beta$, so B contains an infinite descending sequence.

Friedman's ordering:

Assume $\neg\mathsf{ACA}_0$. Let f be a function whose range doesn't exist.



Notes on reversals:

(least elt. $\rightarrow$ least strict u. b.) $\rightarrow$ $\mathsf{ACA_0}$

Assume $\neg\mathsf{ACA_0}$.

Use Hirst's [4] well ordering β satisfying

$$\omega \leq_s \beta \qquad \beta \not\leq_s \omega \qquad \omega + 1 \not\leq_s \beta$$

Let f witness $\omega \leq_s \beta$.

$\mathsf{RCA_0}$ proves that there is a set $Y \subset \beta$ such that b is an upper bound for Y if and only if b is an upper bound for the range of f. Since $\omega + 1 \not\leq_s \beta$, Y has no least strict upper bound.

Since β is a well ordering, $\mathsf{RCA_0}$ proves that every nonempty subset of β has a least element.

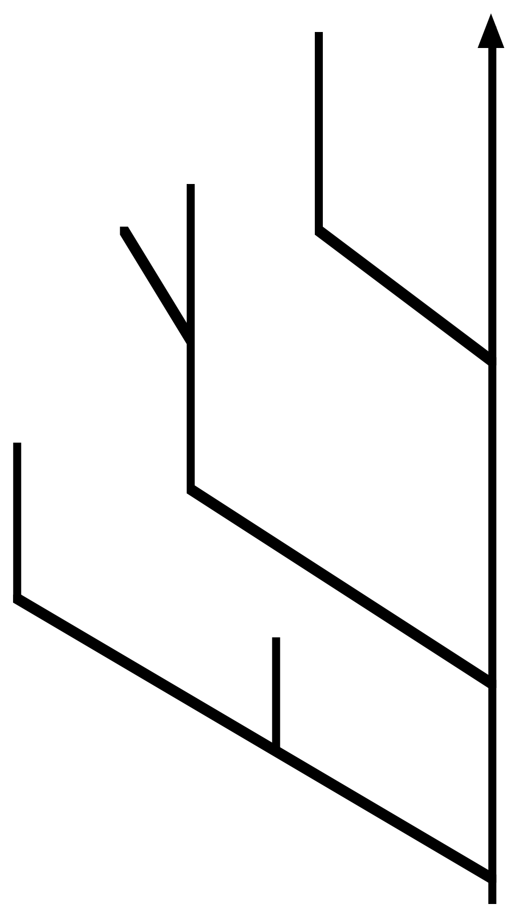
Hirst's ordering:

Assume $\neg\mathsf{ACA_0}$. Let f be a function whose range doesn't exist.

Consider the notation:

$$f^{-1}(y) = \begin{cases} x+1 & \text{if } f(x) = y, \\ 0 & \text{if } y \notin \text{Range}(f). \end{cases}$$

Let β be the Kleene-Brouwer ordering on the tree T of approximations to f^{-1}.



Statements about ordinal arithmetic that are equivalent to ACA_0 are relatively rare.

A survey of the reverse mathematics of ordinal arithmetic† lists a number of results on ordinal arithmetic:

29 statements are provable in RCA_0

7 statements are equivalent to ACA_0

28 statements are equivalent to ATR_0

†preprint available at
www.mathsci.appstate.edu/ ~jlh/bib.html
to appear in Simpson's *Reverse Math 2001*
http://www.math.psu.edu/simpson/revmath/

Statements of ordinal arithmetic that are equivalent to $\mathsf{ACA_0}$.

Thm: ($\mathsf{RCA_0}$) The following are equivalent:

1. If α and β are well orderings with $\alpha \leq_s \beta$ and $\beta \not\leq_s \alpha$, then $\alpha + 1 \leq_s \beta$.
2. If β is a well ordering such that $\omega \leq_w \beta$ and $\beta \not\leq_w \omega$, then $\omega + 1 \leq_w \beta$. (Friedman)
3. If β is a well ordering such that $\omega \leq_w \beta$ and $\beta \leq_w \omega$, then $\omega \equiv_s \beta$.
4. If α and β are well ordered, then so is α^β. (Girard)
5. If α is well ordered, then so is 2^α. (Girard)
6. $\mathsf{ACA_0}$.

Existence of suprema of sequences of well orderings

Thm: ($\mathsf{RCA_0}$) The following are equivalent:

1. $\mathsf{ATR_0}$.
2. Suppose $\langle \alpha_x \mid x \in \beta \rangle$ is a well ordered sequence of well orderings. Then $\sup\langle \alpha_x \mid x \in \beta \rangle$ exists. That is, there is a well ordering α unique up to order isomorphism satisfying
   - $\forall x \in \beta(\alpha_x \leq \alpha)$, and
   - $\forall \gamma(\gamma + 1 \leq \alpha \rightarrow \exists x \in \beta(\alpha_x \not\leq \gamma))$.

This result holds for both strong and weak comparability.

Thm: (Sierpiński's exercise [6]) For each positive natural number n, RCA_0 proves

$$\sum_{\alpha<\omega^n} \alpha \equiv_s \omega^{2n-1}.$$

Informally, $\sum_{\alpha<\omega^n} \alpha = \sup\langle \alpha\omega^n \mid \alpha \in \omega^n\rangle$

Sierpiński's exercise was generalized by Jones, Levitz, and Nichols [5]. Their γ-lemma is equivalent to ATR_0.

Thm: (RCA_0) The following are equivalent:

1. ATR_0.
2. (γ-lemma) Suppose that ω^γ is well ordered and f assigns a well ordered set to each $\alpha < \omega^\gamma$ in such a way that if $\alpha < \beta < \omega^\gamma$ then $f(\beta) + 1 \not\leq f(\alpha)$. Then
   - For all $\alpha < \omega^\gamma$, $f(\alpha) \cdot \omega^\gamma \leq \sum_{\alpha<\omega^\gamma} f(\alpha)$, and
   - If $\delta < \sum_{\alpha<\omega^\gamma} f(\alpha)$, then there is an $\alpha < \omega^\gamma$ such that $f(\alpha) \cdot \omega^\gamma \not\leq \delta$.

Informally, $\sum_{\alpha<\omega^\gamma} f(\alpha) = \sup\langle f(\alpha)\omega^\gamma \mid \alpha \in \omega^\gamma\rangle$

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REFERENCES

[1] Georg Cantor, *Beiträge zur Begründung der transfiniten Mengenlehre*, ***Math. Ann.***, vol. 49 (1897), pp. 207–246, English translation by P. Jourdain published as ***Contributions to the founding of the theory of transfinite numbers***, Dover, New York, 1955.

[2] Harvey M. Friedman and Jeffry L. Hirst, *Weak comparability of well orderings and reverse mathematics*, ***Ann. Pure Appl. Logic***, vol. 47 (1990), pp. 11–29.

[3] Jeffry L. Hirst, *A survey of the reverse mathematics of ordinal arithmetic*, to appear in Simpson's *Reverse Mathematics 2001*.

[4] ———, *Ordinal inequalities, transfinite induction, and reverse mathematics*, ***J. Symbolic Logic***, vol. 64 (1999), pp. 769–774.

[5] James P. Jones, Hilbert Levitz, and Warren D. Nichols, *On series of ordinals and combinatorics*, ***Math. Logic Quart.***, vol. 43 (1997), pp. 121–133.

[6] Wacław Sierpiński, ***Cardinal and ordinal numbers***, Polska Akademia Nauk, Monografie Matematyczne, Państwowe Wydawnictwo Naukowe, Warszawa, 1958.