Representations of reals in reverse mathematics

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Four representations of reals suitable for computable analysis or reverse mathematics

1. *Rapidly converging Cauchy sequences:* $\rho : \mathbb{N} \to \mathbb{Q}$ such that
$$\forall k \forall i \ |\rho(k) - \rho(k+i)| \leq 2^{-k}$$

2. *Decimal expansions:* a Cauchy sequence $\delta : \mathbb{N} \to \mathbb{Q}$ in which each term adds one new correct decimal place.

3. *Lower Dedekind cuts:* a set $\emptyset \subsetneq \lambda \subsetneq \mathbb{Q}$ such that $\forall s \in \mathbb{Q} \forall s' \in \mathbb{Q} \left((s \in \lambda \wedge s' \notin \lambda) \to s < s' \right)$.

4. *Open Dedekind cuts:* a lower Dedekind cut σ with no greatest element.

Representations of 1

ρ: $0,\ \frac{1}{2},\ \frac{3}{4},\ \frac{7}{8},\ \frac{15}{16},\ \ldots$

δ: $1,\ 1.0,\ 1.00,\ 1.000,\ 1.0000,\ \ldots$

δ: $.9,\ .99,\ .999,\ .9999,\ .99999,\ \ldots$

λ: $\mathbb{Q} \cap (-\infty, 1]$

σ: $\mathbb{Q} \cap (-\infty, 1)$

An early theorem of computable analysis

Theorem 1. *Any computable real in any of the preceding representations can be converted to a computable real in any of the other representations.*

This was observed by: Raphael Robinson [6], John Myhill [4], and H.G. Rice [5]

To formalize this result in reverse mathematics, it's good to specify what we mean by "conversion."

Subsystems of second order arithmetic for reverse math

RCA_0: First order arithmetic with
induction restricted to Σ^0_1 formulas
plus set comprehension for Δ^0_1 sets.

WKL_0: All axioms of RCA_0
plus Weak König's Lemma:
"Every infinite 0–1 tree has an infinite path."

ACA_0: All axioms of RCA_0
plus comprehension for arithmetically definable sets.

Equality between representations of reals

Between Cauchy sequences / decimal expansions:

$\rho_1 = \rho_2$ means: $\forall k \; |\rho_1(k) - \rho_2(k)| \leq 2^{-k+1}$

Also, $\rho_1 < \rho_2$ means $\exists k \; (\rho_1(k) + 2^{-k+1} < \rho_2(k))$

Between pairs of cuts:

$\lambda_1 = \lambda_2$ means these sets differ in at most one element.

Between Cauchy sequences and cuts:

$\lambda = \rho$ means:

$\forall k \forall s \forall s' \left((s \in \lambda \wedge s' \notin \lambda) \to [s, s'] \cap [\rho(k) - 2^{-k}, \rho(k) + 2^{-k}] \neq \emptyset \right).$

(Every closed interval containing λ meets every closed ball containing ρ.)

Reverse math of converting single reals

Theorem 2 (RCA_0)**.** *Given any two forms of representations, if α is a real in the first representation, then there is a real β in the second representation such that $\alpha = \beta$.*

Comments on the proof:

A good strategy is: open cuts $\rightarrow$ Dedekind cuts $\rightarrow$ decimal expansions $\rightarrow$ rapidly converging Cauchy sequences $\rightarrow$ open cuts. The first four steps are "uniform," but the last step is different.

The last step: Suppose ρ is a rapidly converging Cauchy sequence. If ρ is rational, let σ be the rationals less than ρ. Otherwise, for every rational q, either $q < \rho$ or $\rho < q$. For each q, find the witness and build σ appropriately.

Capitalizing on the uniformity
Sequences!

Theorem 3 (RCA_0). *If* $\langle \mu_i \rangle_{i\in\mathbb{N}}$ *is a sequence of reals in a representation in the following list, then for any representation appearing lower in the list there is a sequence* $\langle \tau_i \rangle_{i\in\mathbb{N}}$ *in that representation such that for all* $i \in \mathbb{N}$, $\mu_i = \tau_i$.

open cuts,

Dedekind cuts,

decimal expansions,

rapidly converging Cauchy sequences.

What about the last step?
Nonuniformity demands additional comprehension…

Theorem 4 (RCA_0)**.** *The following is equivalent to* ACA_0*:*

> *If* $\langle \rho_i \rangle_{i \in \mathbb{N}}$ *is a sequence of rapidly converging Cauchy sequences, then there is a sequence* $\langle \sigma_i \rangle_{i \in \mathbb{N}}$ *of open cuts such that for every* $i \in \mathbb{N}$, $\rho_i = \sigma_i$.

The preceding theorem follows easily from:

Theorem 5 (RCA_0)**.** *The following is equivalent to* ACA_0*:*

> *If* $\langle \lambda_i \rangle_{i \in \mathbb{N}}$ *is a sequence of Dedekind cuts, then there is a sequence* $\langle \sigma_i \rangle_{i \in \mathbb{N}}$ *of open cuts such that for every* $i \in \mathbb{N}$, $\lambda_i = \sigma_i$.

Part of the proof of Theorem 5

We want to show: *Dedekind cuts* $\to$ *open cuts* implies ACA_0.

Suppose $f : \mathbb{N}^+ \to \mathbb{N}$ is an injection. We'll find its range.

Define the sequence $\langle \lambda_i \rangle_{i \in \mathbb{N}}$ of Dedekind cuts by putting $q \in \mathbb{Q}$ in λ_i if and only if:

$$q \leq 0 \quad \text{or} \quad q > 0 \text{ and } (\exists t < 1/q)(f(t) = i).$$

Informally,

$$\text{if } i \notin \text{Range}(f), \text{ then } \lambda_i = (-\infty, 0] \cap \mathbb{Q}, \text{ and}$$

$$\text{if } f(t) = i, \text{ then } \lambda_i = (-\infty, 1/t) \cap \mathbb{Q}.$$

If σ_i is an open cut with $\sigma_i = \lambda_i$, then $i \in \text{Range}(f)$ if and only if $0 \in \sigma_i$.

Summary of conversion results for sequences

from \ to	ρ	δ	λ	σ
ρ	RCA_0	WKL_0	WKL_0	ACA_0
δ	RCA_0	RCA_0	WKL_0	ACA_0
λ	RCA_0	RCA_0	RCA_0	ACA_0
σ	RCA_0	RCA_0	RCA_0	RCA_0

Legend:

ρ: Rapidly converging Cauchy sequences

δ: Decimal expansions

λ: Dedekind cuts

σ: Open cuts

Mostowski [3] gave computable counterexamples corresponding to the reversals for $(\delta \to \sigma)$ and $(\rho \to \delta)$.

Another example

Theorem 6 (RCA_0)**.** *The following are equivalent:*

1. WKL_0.
2. *If* $\langle \rho_i \rangle_{i \in \mathbb{N}}$ *is a sequence of rapidly converging Cauchy sequences then there is a sequence* $\langle \delta_i \rangle_{i \in \mathbb{N}}$ *of decimal expansions such that for each* $i \in \mathbb{N}$, $\rho_i = \delta_i$.

Idea behind reversal: Separate ranges of f and g

If $f(t) = i$, let $\rho_i(t) = 1 + 10^{-t}$.

If $g(t) = i$, let $\rho_i(t) = 1 - 10^{-t}$.

The set $S = \{i \mid \delta_i(0) = 1\}$ separates the ranges.

Pesky rationals

Conversions of sequences of reals

from \ to	ρ	δ	λ	σ
ρ	RCA_0	WKL_0	WKL_0	ACA_0
δ	RCA_0	RCA_0	WKL_0	ACA_0
λ	RCA_0	RCA_0	RCA_0	ACA_0
σ	RCA_0	RCA_0	RCA_0	RCA_0

All the reversals shown in the chart hold even when the sequences are restricted to sequences of rationals.

A strictly irrational sequence in any format can be converted to a sequence in any other format, using only RCA_0.

Mostowski and change of basis

Theorem 7 (**RCA_0**)**.** *If $c \mid b^n$ for some n, then for every sequence $\langle \beta_i \rangle_{i \in \mathbb{N}}$ of base b expansions there is a sequence $\langle \gamma_i \rangle_{i \in \mathbb{N}}$ of base c expansions such that for all $i \in \mathbb{N}$, $\beta_i = \gamma_i$.*

Theorem 8 (**RCA_0**)**.** *If for all n we have $c \nmid b^n$, then the following are equivalent:*

1. WKL_0.
2. *For every sequence $\langle \beta_i \rangle_{i \in \mathbb{N}}$ of base b expansions there is a sequence $\langle \gamma_i \rangle_{i \in \mathbb{N}}$ of base c expansions such that for all $i \in \mathbb{N}$, $\beta_i = \gamma_i$.*

Concrete examples

Theorem 9 (RCA_0)**.** *For every sequence* $\langle \beta_i \rangle_{i \in \mathbb{N}}$ *of base* 10 *expansions there is a sequence* $\langle \gamma_i \rangle_{i \in \mathbb{N}}$ *of base* 2 *expansions such that for all* $i \in \mathbb{N}$, $\beta_i = \gamma_i$.

Theorem 10 (RCA_0)**.** *The following are equivalent:*

1. WKL_0.
2. *For every sequence* $\langle \beta_i \rangle_{i \in \mathbb{N}}$ *of base* 2 *expansions there is a sequence* $\langle \gamma_i \rangle_{i \in \mathbb{N}}$ *of base* 10 *expansions such that for all* $i \in \mathbb{N}$, $\beta_i = \gamma_i$.

Underlying cause of this behavior:

$\frac{1}{2} = .5$ (base 10), but $\frac{1}{10} = .000\overline{1100}$ (base 2).

Computable analysis and constructive analysis

Examples of related results:

1. Every computable sequence of base 10 expansions can be converted to a computable sequence of base 2 expansions.
2. There is a computable sequence of base 2 expansions with no termwise equal computable sequence of base 10 expansions.
3. Every computable sequence of base 2 expansions can be converted to a low sequence of base 10 expansions.
4. There is a base 2 expansion that cannot be constructively converted into a base 10 expansion.

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