Reverse Mathematics of Transfinite Triangular Numbers

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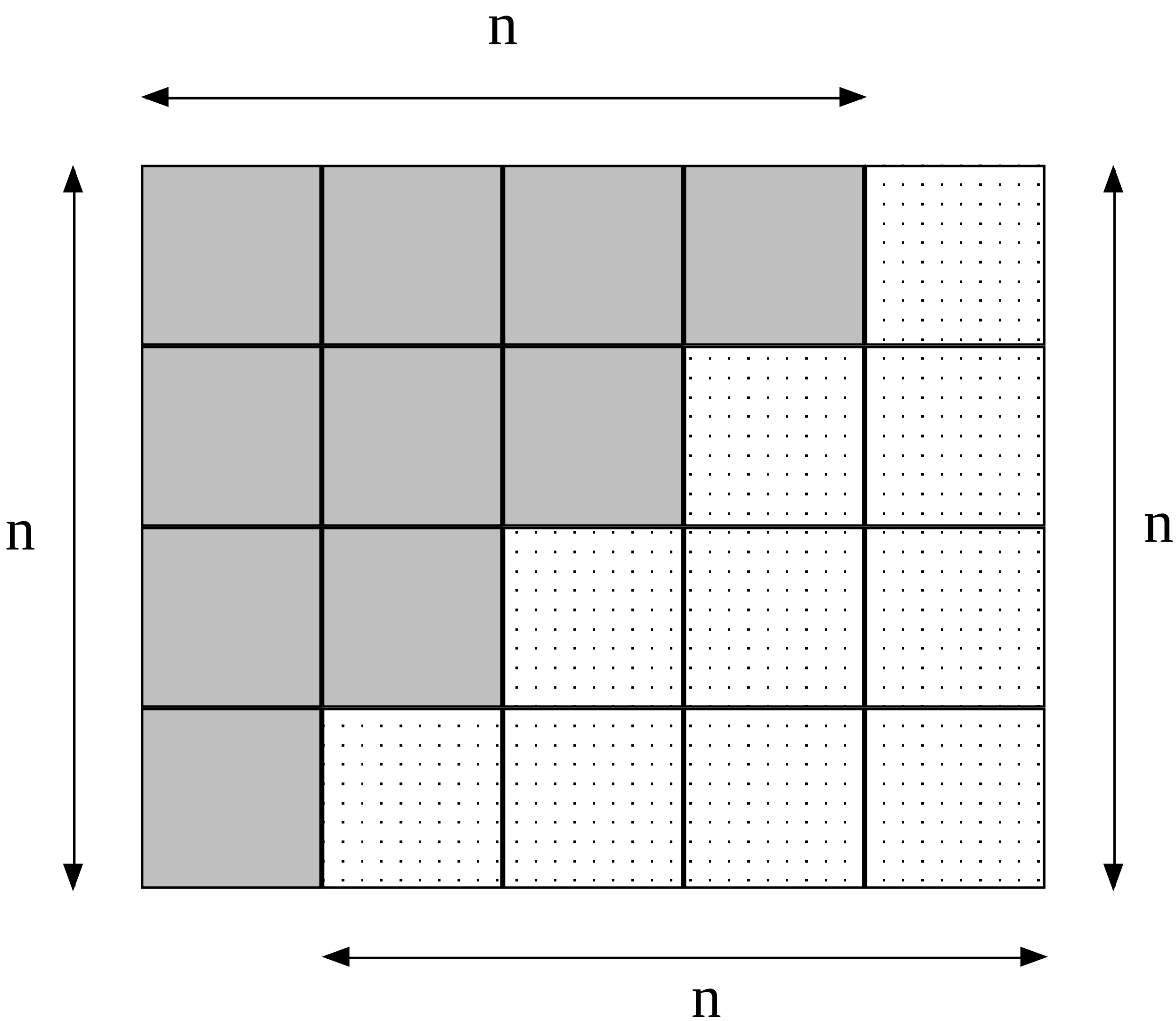
An elementary exercise of Gauss:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$



An elementary exercise of Sierpiński:

For each positive natural number n, we have

$$\sum_{\alpha<\omega^n} \alpha = \omega^{2n-1}.$$

Sample cases:

$$\sum_{\alpha<\omega^1} \alpha = 1 + 2 + 3 + \cdots = \omega = \omega^{2\cdot 1-1}$$

$$\begin{aligned}
\sum_{\alpha<\omega^2} \alpha &= 0 + 1 + 2 + \cdots + \omega + \cdots + \omega\cdot 2 + \cdots + \omega\cdot 3 + \cdots \\
&= \omega + (\omega + 0) + (\omega + 1) + (\omega + 2) + \cdots \\
&\quad + (\omega\cdot 2 + 0) + (\omega\cdot 2 + 1) + (\omega\cdot 2 + 2) + \cdots \\
&\quad \cdots \\
&= \omega + \omega + (0 + \omega) + (1 + \omega) + (2 + \cdots \\
&\quad + \omega\cdot 2 + (0 + \omega\cdot 2) + (1 + \omega\cdot 2) + (2 + \cdots \\
&\quad \cdots \\
&= \omega + \omega + \omega + \omega + \cdots \\
&\quad + \omega\cdot 2 + \omega\cdot 2 + \omega\cdot 2 + \cdots \\
&\quad \cdots \\
&= \omega\cdot\omega + \omega\cdot\omega + \cdots \\
&= (\omega\cdot\omega)\cdot\omega = \omega^3 = \omega^{2\cdot 2-1}
\end{aligned}$$

The proof of Sierpiński's exercise relies on the fact that ω^n is *indecomposable.* That is, whenever $\alpha < \omega^n$, we have $\alpha + \omega^n = \omega^n$.

Sierpiński's exercise (and the proofs) can be formalized in reverse mathematics, yielding:

Thm: For each positive natural number n, RCA_0 proves

$$\sum_{\alpha<\omega^n} \alpha = \omega^{2n-1}.$$

Notes:

- RCA_0 is an axiom system for natural numbers and sets of natural numbers that consists of PA with induction restricted to Σ^0_1 formulas and the **r**ecursive **c**omprehension **a**xiom.
- In RCA_0, countable well ordered sets (like $\sum_{\alpha<\omega^n} \alpha$) can be represented by subsets of $\mathbb{N}$.
- We say RCA_0 proves $\alpha = \beta$ if RCA_0 proves that there is an order preserving bijection between α and β.

For each positive natural number n, RCA_0 can prove that ω^n is indecomposable. A complete analysis of indecomposable countable well orderings requires additional axiomatic strength.

Thm: RCA_0 proves these are equivalent:

1. ATR_0
2. If α is a countable well ordering, then α is indecomposable if and only if $\alpha = \omega^\gamma$ for some choice of γ.

Notes:

- The axiom system ATR_0 consists of RCA_0 plus the **a**rithmetical **t**ransfinite **r**ecursion scheme.
- ATR_0 is also equivalent to the statement: " if α and β are well orderings, then $\alpha \leq \beta$ or $\beta \leq \alpha$." (Friedman)
- Cantor used the term *γ-number* to denote numbers of the form ω^γ.

A generalization of Sierpiński's exercise

In *On Series of Ordinals and Combinatorics* (MLQ), Jones, Levitz and Nichols prove the following

γ **lemma**: Suppose γ is an ordinal and f is a non-decreasing function from ω^γ into the ordinals. Then

$$\sum_{\alpha<\omega^\gamma} f(\alpha) = \sup\{f(\alpha)\cdot\omega^\gamma | \alpha < \omega^\gamma\}.$$

Notes:

- Using $f(\alpha) = \alpha$, the γ lemma computes all of Sierpiński's triangular numbers, plus extras.

$$\sum_{\alpha<\omega^\omega} \alpha = \sup\{\alpha\cdot\omega^\omega | \alpha < \omega^\omega\}$$

$$= \sup\{\omega^j\cdot\omega^\omega | j < \omega\}$$

$$= \sup\{\omega^\omega | j < \omega\} = \omega^\omega$$

- We can use reverse math to show that the γ lemma is strictly stronger than Sierpiński's exercise.

- We have to decide what "=" means in the γ lemma.

Suprema of well orderings

Thm: RCA_0 proves these are equivalent:

1. ATR_0
2. Suppose $\langle \alpha_x \mid x \in \beta \rangle$ is a well ordered sequence of well orderings. Then $\sup\langle \alpha_x \mid x \in \beta \rangle$ exists. That is, there is a well ordering α unique up to order isomorphism satisfying
   - $\forall x \in \beta(\alpha_x \leq \alpha)$, and
   - $\forall \gamma(\gamma + 1 \leq \alpha \rightarrow \exists x \in \beta(\alpha_x \not\leq \gamma))$.

Notes:

- Suppose $\alpha \leq_s \beta$ means there's an order preserving bijection between α and an initial segment of β.
- Suppose $\alpha \leq_w \beta$ means there's an order preserving map of α into β.
- The theorem holds if $\leq$ is either $\leq_s$ or $\leq_w$.
- If $\leq$ is $\leq_s$, then the theorem holds when uniqueness is omitted.
- Question: Does 2 imply 1 when $\leq$ is $\leq_w$ and uniqueness is omitted?

Analysis of the γ lemma

γ lemma: If γ is an ordinal and f is non-decreasing,

$$\sum_{\alpha<\omega^\gamma} f(\alpha) = \sup\{f(\alpha)\cdot\omega^\gamma | \alpha < \omega^\gamma\}.$$

Thm: RCA_0 proves these are equivalent:

1. ATR_0.
2. (γ-lemma) Suppose that ω^γ is well ordered and f assigns a well ordered set to each $\alpha < \omega^\gamma$ in such a way that if $\alpha < \beta < \omega^\gamma$ then $f(\beta) + 1 \not\leq f(\alpha)$. Then
   - For all $\alpha < \omega^\gamma$, $f(\alpha)\cdot\omega^\gamma \leq \sum_{\alpha<\omega^\gamma} f(\alpha)$, and
   - If $\delta + 1 \leq \sum_{\alpha<\omega^\gamma} f(\alpha)$, then there is an $\alpha < \omega^\gamma$ such that $f(\alpha)\cdot\omega^\gamma \not\leq \delta$.

Sketch of 2 $\implies$ 1: Assume RCA_0 and $\neg\mathsf{ATR}_0$.

Suppose α and β are incomparable indecomposable wos.

Define $f(0) = \alpha$ and $f(n) = \beta$ for $n > 0$.

$f(0)\cdot\omega = \alpha + \alpha + \cdots \not\leq \alpha + \beta + \beta + \cdots = \sum_{n<\omega} f(n)$

Question: If $\leq$ means $\leq_w$ and $f(\beta) + 1 \not\leq f(\alpha)$ is replaced by $f(\alpha) \leq f(\beta)$, does 2 still imply 1?

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