Minima of initial segments of sequences of reals

Jeffry L. Hirst
Appalachian State University
Boone, North Carolina

2004 Winter ASL Meeting in Phoenix

These slides appear at: `www.mathsci.appstate.edu/~jlh`

Subsystems of second order arithmetic for reverse math

RCA_0: First order arithmetic with

induction restricted to Σ^0_1 formulas

plus set comprehension for Δ^0_1 sets.

WKL_0: All axioms of RCA_0

plus Weak König's Lemma:

"Every infinite 0–1 tree has an infinite path."

Models

ω with the computable sets models RCA_0.

Every ω model of WKL_0 includes noncomputable sets.

Reverse Math

Theorem 1. (RCA_0) *The following are equivalent:*

1. WKL_0.
2. *If f and g are injections with disjoint ranges, then there is a set Y such that for all $t \in \mathbb{N}$,*

$$f(t) \in Y \text{ and } g(t) \notin Y.$$

Encoding the reals

A *real number* is a function $x : \mathbb{N} \to \mathbb{Q}$ such that

$$\forall k \forall i \ |x(k) - x(k+i)| \leq 2^{-k}$$

(that is, $\langle x(i) \rangle_{i \in \mathbb{N}}$ is a rapidly converging Cauchy sequence of rationals.)

Examples of reals

$\sqrt{2}$: 1, 1.4, 1.41, 1.414, 1.4142, ...

π : 3, 3.1, 3.14, 3.141, 3.1415, ...

0 : $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

0 : 0, 0, 0, 0, 0, ...

Relationships between reals

$x = y$ means: $\forall k \ |x(k) - y(k)| \leq 2^{-k+1}$

$x \leq y$ means: $\forall k \ (x(k) \leq y(k) + 2^{-k+1})$

$y < x$ means $x \not\leq y$,
which is $\exists k \ (y(k) + 2^{-k+1} < x(k))$

Sequences of minima

Theorem 2. (RCA_0) *Let* $\langle x_k \rangle_{k \in \mathbb{N}}$ *be a sequence of reals. Then there is a sequence of reals* $\langle u_k \rangle_{k \in \mathbb{N}}$ *such that for each* k, $u_k = \min\{x_j \mid j \leq k\}$. *That is, for each* k *we have:*

1. $\forall j \leq k \ (u_k \leq x_j)$ *and*
2. $\exists j \leq k \ (u_k = x_j)$.

Corollary 3. *Let* $\langle x_k \rangle_{k \in \mathbb{N}}$ *be a computable sequence of computable real numbers. Then there is a computable sequence of computable reals* $\langle u_k \rangle_{k \in \mathbb{N}}$ *such that for each* k *we have* $u_k = \min\{x_j \mid j \leq k\}$.

Picking the minima

Let $u_k(j) = \min\{x_n(j) \mid n \leq k\}$.

Example:

x_0:	1	1.4	1.41	1.414	1.4142	...	$(\sqrt{2})$
x_1:	3	3.1	3.14	3.141	3.1415	...	(π)
x_2:	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	...	(0)
x_3:	0	$\frac{1}{8}$	$\frac{3}{16}$	$\frac{7}{32}$	$\frac{15}{64}$	...	$(\frac{1}{4})$

Picking the indices

Rather than constructing real number codes for the minima of initial segments, suppose we want to pick the (index of the) actual sequence element…

Theorem 4. (RCA_0) *The following are equivalent:*

1. WKL_0
2. *If $\langle x_k \rangle_{k\in\mathbb{N}}$ is a sequence of reals, then there is a sequence of integers $\langle \mu_k \rangle_{k\in\mathbb{N}}$ such that*

$$\forall k (x_{\mu_k} = \min\{x_j \mid j \leq k\}).$$

Sketch of (2) implies WKL_0

Suppose f and g are injections with disjoint ranges. Use a sequence of indices of minima to construct a separating set.

If $f(3) = 0$, g(2)=1, and $2 \notin \mathsf{Ran} f \cup \mathsf{Ran} g$, build:

$x_{0,f}$:	0	0	0	-.9999	...
$x_{0,g}$:	0	0	0	0	...
$x_{1,f}$:	-1	-1	-1	-1	...
$x_{1,g}$:	-1	-1	-1.001	-1.001	...
$x_{2,f}$:	-2	-2	-2	-2	...
$x_{2,g}$:	-2	-2	-2	-2	...

Computability theoretic corollaries

Corollary 5. *There is a computable sequence of computable reals such that no sequence of indices of minima of initial segments is computable.*

Corollary 6. *Given any computable sequence of computable reals, we can find a sequence of indices of minima of initial segments that is of low degree. (That is, the degree of the sequence of indices is* $\mathbf{a}$ *where* $\mathbf{a}' \leq_{\mathbf{T}} \mathbf{0}'$*.)*

Question: Is there ever a computable sequence of indices?

Answer: Sometimes.

Theorem 7. (RCA$_0$) *If* $\langle x_k \rangle_{k \in \mathbb{N}}$ *is a* **nonrepeating** *sequence of reals, then there is a sequence of integers* $\langle \mu_k \rangle_{k \in \mathbb{N}}$ *such that* $\forall k (x_{\mu_k} = \min\{x_j \mid j \leq k\})$.

Corollary 8. *If* $\langle x_k \rangle_{k \in \mathbb{N}}$ *is a* **nonrepeating** *computable sequence of computable reals, then there is a computable sequence of integers* $\langle \mu_k \rangle_{k \in \mathbb{N}}$ *such that*

$$\forall k (x_{\mu_k} = \min\{x_j \mid j \leq k\}).$$

Picking μ_k

Example (of a nonrepeating sequence):

x_0:	1	1.4	1.41	1.414	1.4142	...	$(\sqrt{2})$
x_1:	3	3.1	3.14	3.141	3.1415	...	(π)
x_2:	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	...	(0)
x_3:	0	$\frac{1}{8}$	$\frac{3}{16}$	$\frac{7}{32}$	$\frac{15}{64}$	...	$(\frac{1}{4})$

Constructive Analysis vs. Computable Analysis

Computable Analyst:

We can select minima of initial segments.

We can't select the indices of the minima.

Constructive Analyst:

We can't select the indices, so we can't select the minima.

We can select the minima of nonrepeating sequences.

Reverse Mathematics

Jeffry L. Hirst. *Minima of initial segments of infinite sequences of reals,* Math. Logic Quarterly, **50** (2004).

Stephen G. Simpson. *Subsystems of second order arithmetic,* Springer-Verlag, Berlin, 1999.

Computable Analysis

Marian B. Pour-El and J. Ian Richards. *Computability in analysis and physics,* Springer-Verlag, Berlin, 1989.

Constructive Analysis

Errett Bishop and Douglas Bridges. *Constructive analysis,* Springer-Verlag, Berlin, 1985.