A reverse mathematics demonstration

Jeff Hirst
Appalachian State University
Boone, North Carolina, USA

August 16, 2016

For more information on
computability theory and reverse mathematics,
enroll in **MAT 5530: Noncomputability** in Spring 2017

The method

Reverse Mathematics measures the strength of theorems by proving equivalence results over. . .

The base theory RCA_0

The method

Reverse Mathematics measures the strength of theorems by proving equivalence results over...

The base theory RCA_0:

Variables for natural numbers and sets of natural numbers

Axioms

Arithmetic axioms

(e.g. $n + 0 = n$ and $n + (m') = (n + m)'$)

Induction for particularly simple formulas

Recursive comprehension:

If you can compute a set, then it exists.

A theorem of RCA_0

Thm: (RCA_0) For any $X \subset \mathbb{N}$, the set $Y = \{n+1 \mid n \in X\}$ exists.

An example:

n	0	1	2	3	4	5	6
χ_x	1	0	0	1	1	0	0
χ_y	0	1	0	0	1	1	0

A theorem of RCA_0

Thm: (RCA_0) For any $X \subset \mathbb{N}$, the set $Y = \{n+1 \mid n \in X\}$ exists.

An example:

n	0	1	2	3	4	5	6
χ_x	1	0	0	1	1	0	0
χ_y	0	1	0	0	1	1	0

A proof sketch: Given χ_x, define

$$\chi_y(n) = \begin{cases} 0 & \text{if } n = 0, \\ \chi_x(n-1) & \text{if } n \neq 0. \end{cases}$$

An equivalence theorem!

Thm: (RCA_0) The following are equivalent:

(1) WKL_0: Every infinite 0-1 tree has an infinite path.

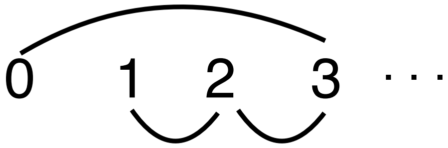
(2) If every finite subgraph of G can be 2-colored, then G can be 2-colored.

Proof sketch:

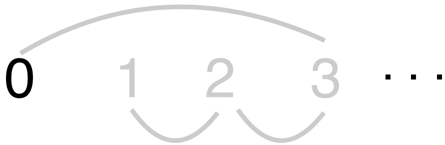
(1)$\rightarrow$(2) Given a graph, build a tree such that every path computes a coloring.

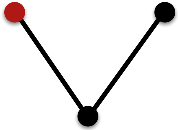
(2)$\rightarrow$(1) Given a tree, build a graph such that every 2-coloring computes a path.

Graph→tree and path→coloring

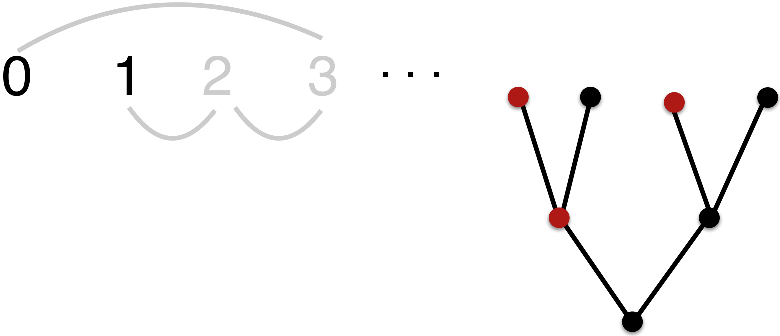


Graph→tree and path→coloring

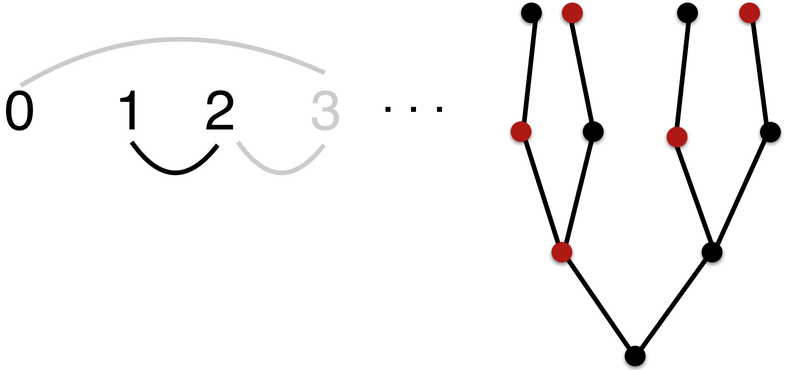




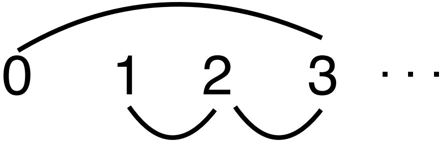
Graph→tree and path→coloring

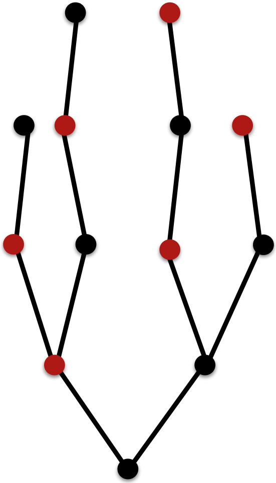


Graph→tree and path→coloring

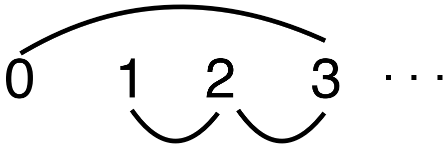


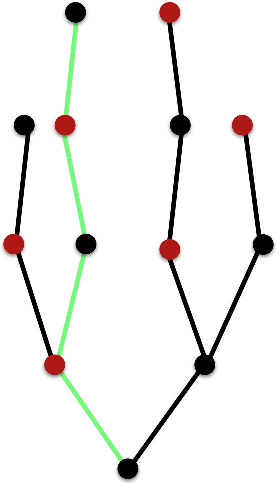
Graph→tree and path→coloring



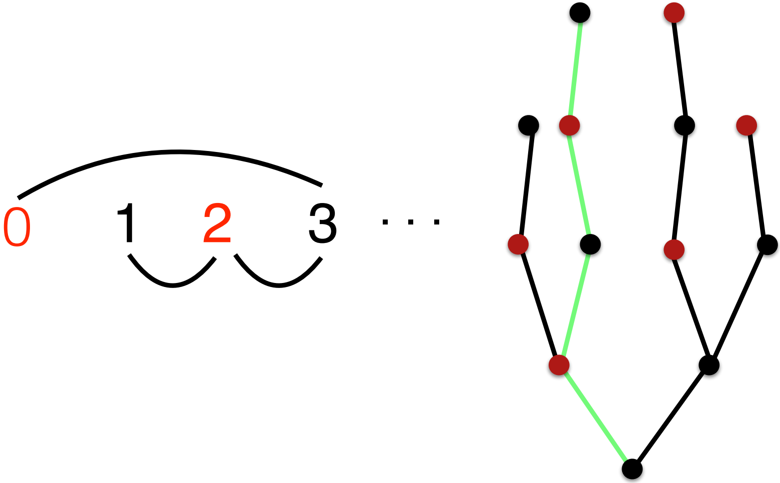


Graph→tree and path→coloring

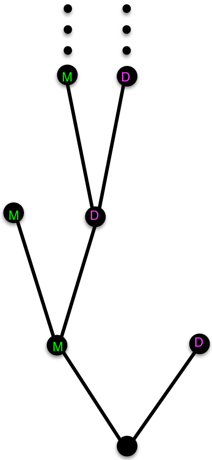




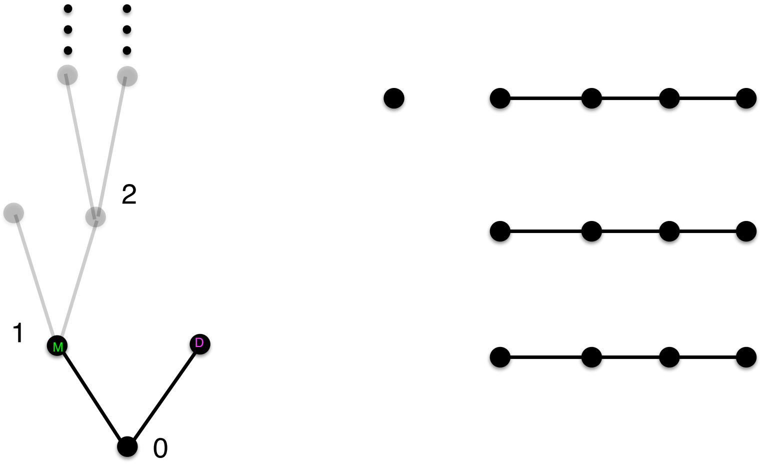
Graph→tree and path→coloring



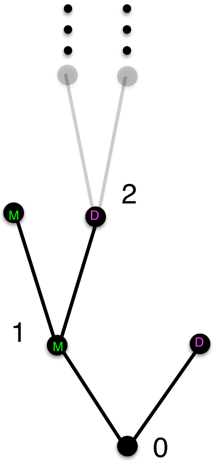
Tree→graph and coloring→path

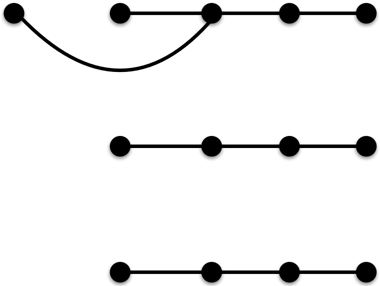


Tree→graph and coloring→path

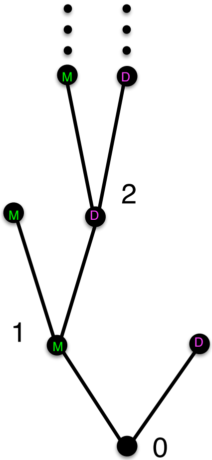


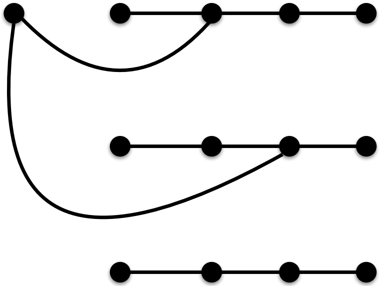
Tree→graph and coloring→path



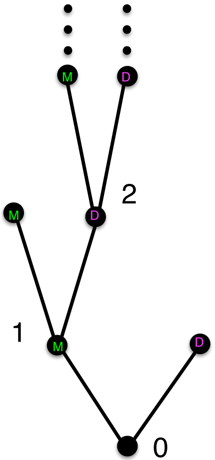


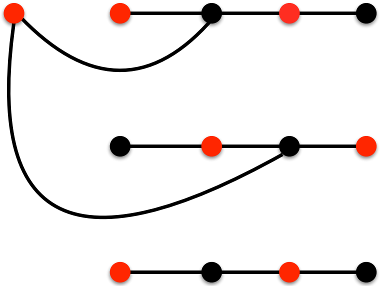
Tree→graph and coloring→path



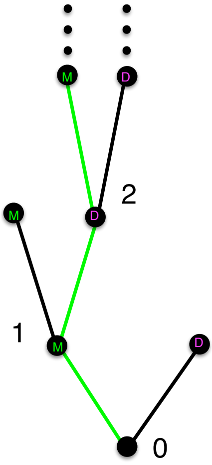


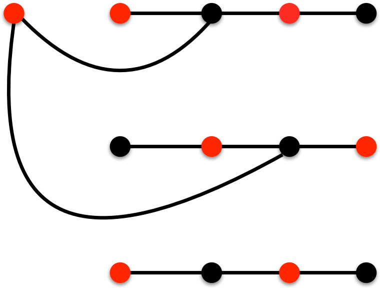
Tree→graph and coloring→path





Tree→graph and coloring→path





More things equivalent to WKL_0

Thm: (RCA_0) The following are equivalent:

1. WKL_0
2. Every continuous function on $[0, 1]$ is bounded. [5]
3. Every continuous function on $[0, 1]$ is Riemann integrable. [5] [4]
4. Every open cover of $[0, 1]$ has a finite subcover. [1]
5. Every countable commutative ring has a prime ideal. [2]
6. Every bounded marriage problem with a unique solution has an an enumeration of the boys such that the first n boys know exactly n girls.[3]

Many theorems of mathematics are either provable in RCA_0 or equivalent to one of: WKL_0, ACA_0, ATR_0, and Π^1_1-CA_0

Some references

[1] Harvey Friedman, *Abstracts: Systems of second order arithmetic with restricted induction, I and II*, J. Symbolic Logic **41** (1976), 557–559. http://www.jstor.org/stable/2272259.

[2] Harvey M. Friedman, Stephen G. Simpson, and Rick L. Smith, *Countable algebra and set existence axioms*, Ann. Pure Appl. Logic **25** (1983), no. 2, 141–181. DOI 10.1016/0168-0072(83)90012-X MR725732.

[3] Jeffry L. Hirst and Noah A. Hughes, *Reverse mathematics and marriage problems with unique solutions*, Arch. Math. Logic **54** (2015), no. 1-2, 49–57. DOI 10.1007/s00153-014-0401-z MR3304736.

[4] Marian B. Pour-El and J. Ian Richards, *Computability in analysis and physics*, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1989. https://projecteuclid.org/euclid.pl/1235422916 MR1005942.

[5] Stephen G. Simpson, *Subsystems of second order arithmetic*, 2nd ed., Perspectives in Logic, Cambridge University Press, Cambridge, 2009. DOI 10.1017/CBO9780511581007 MR2517689.