Notes on Reverse Mathematics and Brouwer's Fixed Point Theorem

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Abstract

These notes include proofs of the results stated in the talk *Reverse mathematics and Brouwer's fixed point theorem,* given at the annual meeting of the ASL in Urbana-Champaign on June 4, 2000.

In [1], Arnold gives a short proof of the fundamental theorem of algebra using Brouwer's fixed point theorem. From a reverse mathematics viewpoint, the fundamental theorem of algebra is provable in $\mathbf{RCA}_0$ [4], while Brouwer's theorem is equivalent to the stronger system $\mathbf{WKL}_0$ [3]. In §1, a computable restriction of Brouwer's theorem will be presented. In §2, this computable restriction is formalized in $\mathbf{RCA}_0$, yielding a corresponding formalization of Arnold's proof using only $\mathbf{RCA}_0$.

1 A restriction of Brouwer's theorem

This section contains the definitions needed to state a computable restriction of Brouwer's fixed point theorem, and the statement and proof of the restricted theorem. We begin with the definitions.

The symbol I^2 denotes the unit square, $[0, 1] \times [0, 1]$, and ∂I^2 denotes the boundary of the square. We will encode computable reals using rapidly convergent Cauchy sequences, in the same fashion used in reverse mathematics [4]. Similarly, we will say that a function is a *computably coded continuous function* if it is represented by a set of 5-tuples in the same fashion as a continuous function code in reverse mathematics ([4], especially page 85).

Naïvely, each 5-tuple represents a pair of neighborhoods of rationals, where the first neighborhood is mapped into the second by the function. When f is a computably coded continuous function, the notation $f : I^2 \to I^2$ indicates that f is defined at each *computable* point in I^2.

Suppose that f is a computably coded continuous function defined on a closed domain X. A computable modulus of uniform continuity for f is a computable function $h : \mathbb{N} \to \mathbb{N}$ such that for each $n \in \mathbb{N}$ and each $x, y \in \mathbb{Q} \cap X$, if $|x - y| < 2^{-h(n)}$ then $|f(x) - f(y)| < 2^{-n}$.

Given a computably coded continuous function f on a domain X, we can define its extension $f^\star$ by setting $f^\star(a) = \lim_{x \to a} f(x)$ for each $a \in X$ where the limit exists, and saying that $f^\star$ is undefined at other points in X. Note that $f^\star$ is not a computable function, and will generally be defined at noncomputable points in X.

Now we have enough terminology to state our restricted version of Brouwer's theorem.

Theorem 1. *Suppose that*

- *$f : I^2 \to I^2$ is a computably coded continuous function,*
- *f has a computable modulus of uniform continuity, and*
- *$f^\star$ has finitely many fixed points.*

Then f has a computable fixed point.

Proof. If f has a computable modulus of uniform continuity, then $f^\star$ is continuous, total on I^2, and maps I^2 into I^2. By Brouwer's fixed point theorem, $f^\star$ has at least one fixed point. Since $f^\star$ has finitely many fixed points, it must have an isolated fixed point. It remains to show that each isolated fixed point of $f^\star$ is a computable fixed point of f. This is done in the following lemma. □

Lemma 2. *Suppose that*

- *$f : I^2 \to I^2$ is a computably coded continuous function, and*
- *f has a computable modulus of uniform continuity.*

Then every isolated fixed point of $f^\star$ is a computable fixed point of f.

Proof. Suppose f satisfies both hypotheses. Define $g : I^2 \to I^2$ by $g(z) = |f(z) - z|$. Note that g is a computably coded continuous function from I^2 to $\mathbb{R}$, g has a computable modulus of uniform continuity, and $g^\star$ has finitely many zeros corresponding to the fixed points of $f^\star$. Furthermore, these zeros are the minima of $g^\star$ on I^2.

Suppose that z_0 is an isolated fixed point of $f^\star$, and consequently an isolated zero of $g^\star$. Then there are rational numbers a_0, a_0', b_0, and b_0' such that z_0 is the only zero of $g^\star$ in the rectangle R defined by $a_0 \leq x \leq a_0'$ and $b_0 \leq y \leq b_0'$. Corresponding to each $k \in \mathbb{N}$, we can construct a rectangle R_k as follows. Set $\epsilon = 2^{-k}$ and using the modulus of uniform continuity for g, find a $\delta = 2^{-m}$ so that if $|z_1 - z_2| < \delta$, then $|g(z_1) - g(z_2)| < \epsilon/2$. Locate lattice points in R at integer multiples of δ from the corner point (a_0, b_0). For each lattice point z_1, use the function code for g to find an estimate for $g(z_1)$ of the form $s \pm \epsilon/2$. Combining the estimate with the information from the modulus of uniform continuity, we find that for each lattice point z_1, and for every point z such that $|z - z_1| < \delta$, $s - \epsilon < g(z) < s + \epsilon$. Mark those δ neighborhoods of lattice points where $s - \epsilon < 0 < s + \epsilon$. Let R_k be the smallest rectangle contained in R that contains all of the marked δ neighborhoods and has lattice points at its corners. Note that because the lattice points of R_j are included in those of R_{j+1}, R_j is a subset of R_{j+1}.

As k becomes large, the vertical and horizontal dimensions of the R_ks tend to 0. To see this, fix some $\epsilon > 0$, and suppose that for all k, R_k is not contained in the ϵ neighborhood centered at z_0, the isolated zero of g. Then for each $n \in \mathbb{N}$ there is a point z_n in R but outside this neighborhood that satisfies $|g(z)| < 2^{-n}$. By the compactness of R, this sequence has a convergent subsequence with some limit point p in R. By the continuity of $g^\star$ on I^2, we must have $g(p) = 0$, contradicting the claim that z_0 is the only zero of $g^\star$ in R.

Let a_k, a_k', b_k and b_k' denote the corner points of R_k. By nested interval convergence, the reals $p_1 = \lim_{k\to\infty} a_k = \lim_{k\to\infty} a_k'$ and $p_2 = \lim_{k\to\infty} b_k = \lim_{k\to\infty} b_k'$ are computable. Thus $z_0 = (p_1, p_2)$ is a computable zero of g and a computable fixed point of f. $\square$

The previous result leaves us with a number of computability-theoretic questions:

- Does every computably coded continuous function $f : I^2 \to I^2$ such that $f^\star$ is total have a computable fixed point?

- Does every computably coded continuous function from I^2 to I^2 with a computable modulus of continuity have a computable fixed point? Since a computable modulus of continuity guarantees the totality of the extension, this is a special case of the preceding question.

- Given an infinite computable $0-1$ tree T with no computable infinite paths, is there a computably coded continuous function f such that $f^\star$ is total and there is a degree preserving isomorphism between the fixed points of $f^\star$ and the paths through T?

In [2], Orevkov constructs a computably coded continuous function $f : I^2 \to I^2$ such that f has no computable fixed point. (This construction also appears in [3].) This map consists of a retract of the computable elements of I^2 to ∂I^2 followed by a rotation of the boundary of the square. Since there is no continuous retract of all of I^2 to ∂I^2, Orevkov's map does not have a total extension. This can also be shown by the following argument. Suppose, by way of contradiction, that Orevkov's function has a total extension $f^\star$. Then $f^\star$ is a continuous function from (all of) I^2 to ∂I^2. Clearly, any fixed points of $f^\star$ must occur on ∂I^2. However, because of the rotation used in defining Orevkov's function, and since $f^\star$ is a continuous extension of that function, $f^\star$ has no fixed points on ∂I^2. Summarizing, $f^\star$ is a continuous function from ∂I^2 to ∂I^2 with no fixed points, contradicting Brouwer's theorem. Thus, $f^\star$ is not total. The preceding argument also shows that the extension of Orevkov's function has no fixed points, computable or noncomputable.

If we consider only those functions mapping I^2 to ∂I^2, then we can guarantee the existence of computable fixed points. To simplify the proof of the following theorem, we will state it for D, the closed unit disk. The theorem holds with D replaced by I^2.

Theorem 3. *Suppose that*

- $f : D \to \partial D$ *is a computably coded continuous function, and*

- $f^\star$ *is total.*

Then f has a computable fixed point.

Proof. By the Brouwer fixed point theorem, $f^\star$ must have a fixed point. Since $f^\star : D \to \partial D$, this fixed point must occur on the boundary of D. Let $\theta : \partial D \to [0, 2\pi)$ be an isomorphism of D onto radian angles associated with the elements of D. Let $g(x) = \theta(f(\theta^{-1}(x)))$. Intuitively, $g(x)$ is the restriction of f to ∂D, with the domain and range expressed as radian measure. Since $f^\star$ has a fixed point on ∂D, $g^\star$ has a fixed point. If the graph of $g^\star(x)$ never crosses the line $y = x$, then there is a continuous perturbation $h : [0, 2\pi) \to [-1, 1]$ such that $(g^\star + h)(x)$ has no fixed points. In this case, the function $f^\star(z) + h(\theta(f^\star(z)))$ is a continuous map of D into ∂D with no fixed points, contradicting Brouwer's fixed point theorem. Thus, the graph of $g^\star(x)$ must cross the line $y = x$. Applying the computable version of the intermediate value theorem to g in some small neighborhood where $g^\star(x) = x$ yields a computable fixed point for g which is also a computable fixed point for f. □

The preceding theorem shows that the single computable fixed point cannot be eliminated from the following example.

Theorem 4. *Given any infinite computable tree T with no computable paths, there is a computably coded continuous function $f : I^2 \to \partial I^2$ such that $f^\star$ is total, the only computable fixed point of $f^\star$ is $(0, 0)$, and there is a degree preserving isomorphism between the noncomputable fixed points of $f^\star$ and the infinite paths through T.*

Sketch of proof. Fix T. Construct a computably coded continuous function $g : [0, 1] \to [0, 1]$ such that the maximum of g is 1, and there is a degree preserving isomorphism between $\{x \mid g^\star(x) = 1\}$ and the infinite paths through T. Define $f(x, y) = (x \cdot g(x), 0)$. The only fixed points of f occur where $y = 0$ and either $x = 0$ or $g(x) = 1$. □

2 Formalizing the restriction

Our original goal was to find a restricted version of the Brouwer fixed point theorem that would allow us to carry out Arnold's proof of the fundamental theorem of algebra in $\mathbf{RCA}_0$. Because Theorem 1 makes reference to $f^\star$, we cannot use it directly in $\mathbf{RCA}_0$. However, the following statement is provable in $\mathbf{RCA}_0$ and captures the content of 1.

Theorem 5. ($\mathbf{RCA}_0$) *Suppose that*

- *$f : I^2 \to I^2$ is a total continuous function,*
- *f has a modulus of uniform continuity, and*

- *there is an integer m and sequences $\langle n_k \rangle_{n\in\mathbb{N}}$ and $\langle\langle B_{k,i}\rangle_{i<m_k}\rangle_{k\in\mathbb{N}}$ such that for each k, $m_k < m$, each $B_{k,i}$ is an open ball of radius at most 2^{-k} contained in exactly one ball in the list $\langle B_{k-1,i}\rangle_{i<m_{k-1}}$, and for every rational point z exterior to $\cup_{i<m_k} B_{k,i}$ we have $|f(z)-z| > 2^{-n_k}$.*

Then f has a fixed point in I^2.

Sketch of proof. Applying nested interval convergence to the balls, locate potential fixed points of f. Suppose that none of these potential fixed points are actual fixed points. Then there is a value M such that for all $z \in I^2$, $|f(z)-z| > 2^{-m}$, contradicting an approximation result of Orevkov [2] that can be proved in $\mathbf{RCA}_0$. □

Note that Arnold's proof of the fundamental theorem of algebra can be carried out in $\mathbf{RCA}_0$ using Theorem 5. Given a polynomial, a continuous mapping from a disk to itself is defined in exactly the same fashion as in [1]. Sequences as described in Theorem 5 can be found from the polynomial. Applying the theorem yields a fixed point, which is also a root of the original polynomial.

Of course, no one would actually want to prove the fundamental theorem of algebra in this fashion. Indeed, the most obvious way to prove the existence of the sequences for the third hypothesis of Theorem 5 is to prove and use the fundamental theorem of algebra.

3 Bibliography

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