Reverse Mathematics of Two Theorems of Graph Theory

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2-coloring graphs

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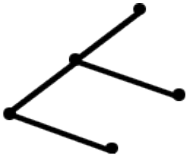

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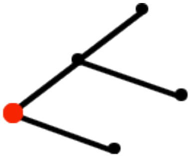

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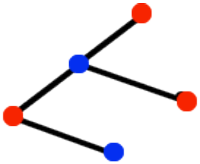

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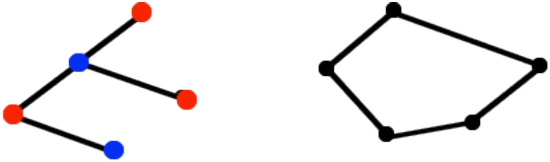

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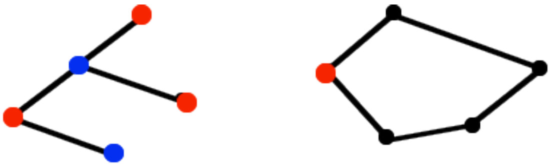

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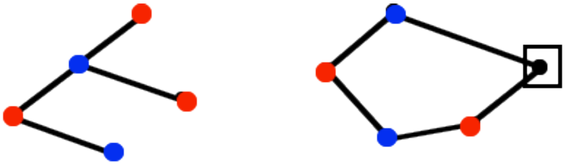

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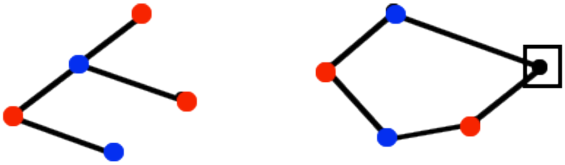

Theorem

Every graph with no cycles of odd length can be 2-colored.

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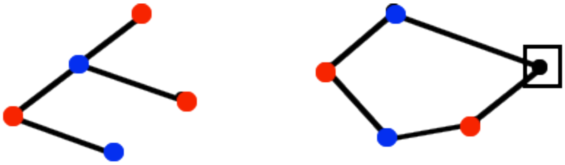

Theorem

Every graph with no cycles of odd length can be 2-colored.

What is the logical strength of this statement?

Reverse Mathematics

Goal: Determine exactly which set existence axioms are needed to prove familiar theorems.

Method: Prove results of the form

$$\mathrm{RCA}_0 \vdash \textbf{AX} \leftrightarrow \textbf{THM}$$

where:

- RCA_0 is a weak axiom system,
- **AX** is a set existence axiom selected from a small hierarchy of axioms, and
- **THM** is a familiar theorem.

Why bother?

Work in reverse mathematics can:

- precisely categorize the logical strength of theorems.
- differentiate between different proofs of theorems.
- provide insight into the foundations of mathematics.
- utilize and contribute to work in many subdisciplines of mathematical logic – including proof theory, computability theory, models of arithmetic, etc.

RCA_0

Language:

Integer variables: x, y, z Set variables: X, Y, Z

Axioms:

basic arithmetic axioms

(0, 1, +, $\times$, =, and $<$ behave as usual.)

Restricted induction

$(\psi(0) \wedge \forall n(\psi(n) \to \psi(n+1))) \to \forall n \psi(n)$

where $\psi(n)$ has (at most) one number quantifier.

Recursive set comprehension

If $\theta \in \Sigma^0_1$ and $\psi \in \Pi^0_1$, and $\forall n(\theta(n) \leftrightarrow \psi(n))$,

then there is a set X such that $\forall n(n \in X \leftrightarrow \theta(n))$

Models and coding

- The smallest ω-model of RCA_0 consists of the usual natural numbers and the computable sets of natural numbers. We write $\mathfrak{M} = \langle \omega, \mathsf{REC} \rangle$.

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- Sets of pairs correspond to functions and/or countable sequences.

- Many mathematical concepts can be encoded in terms of such sequences. Second order arithmetic is remarkably expressive.

Examples

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WKL_0

Weak König's Lemma

Statement: Big very skinny trees are tall.

More formally: If T is an infinite tree in which each node is labeled 0 or 1, then T contains an infinite path.

The subsystem WKL_0 is RCA_0 plus Weak König's Lemma.

There is an infinite computable $0 - 1$ tree with no infinite computable path, so $\langle \omega, \mathsf{REC} \rangle$ is not a model of WKL_0.

Conclusion: $\mathsf{RCA}_0 \nvdash \mathsf{WKL}_0$

Finally! Some reverse mathematics!

Theorem

(RCA_0) *The following are equivalent:*

1. WKL_0.
2. *Every graph with no cycles of odd length can be 2-colored.*

WKL_0 implies the 2-coloring theorem

Suppose G is a graph with vertices $v_0, v_1, v_2, \ldots$ and no odd cycles.

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Let T be the tree consisting of sequences of the form $\langle i_0, i_1, \ldots, i_n \rangle$ where the sequence is a correct 2-coloring of the subgraph of G on the vertices $v_0, v_1, \ldots, v_n$.

Since G has no odd cycles, RCA_0 proves T contains infinitely many nodes.

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Since G has no odd cycles, RCA_0 proves T contains infinitely many nodes.

Any path through T is the desired 2-coloring.

A tool for reversals

Theorem

(RCA_0) *The following are equivalent:*

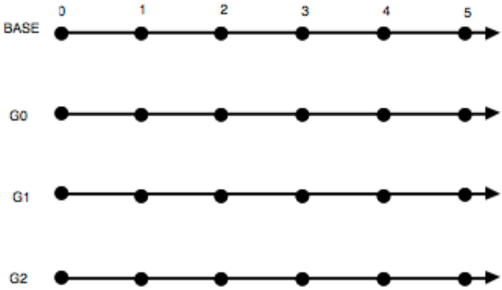
1. WKL_0.
2. *If f and g are injective functions from $\mathbb{N}$ into $\mathbb{N}$ and $Ran(f) \cap Ran(g) = \emptyset$, then there is a set X such that $Ran(f) \subset X$ and $X \cap Ran(g) = \emptyset$.*

Comment: X in (2) is like a separating set for disjoint computably enumerable sets.

The 2-coloring theorem implies WKL_0. A reversal!

Suppose we are given f and g with $Ran(f) \cap Ran(g) = \emptyset$.

If, for example, $f(3) = 0$ and $g(2) = 2$, we will construct the graph G as follows:



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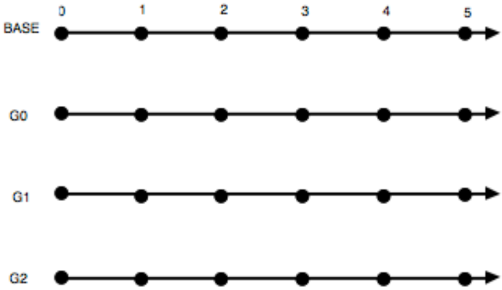

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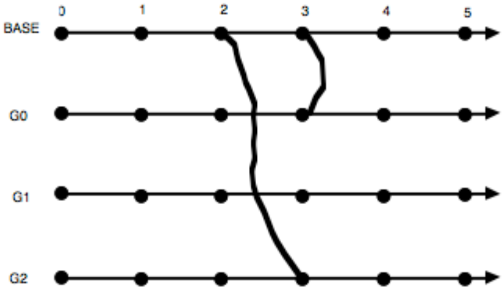

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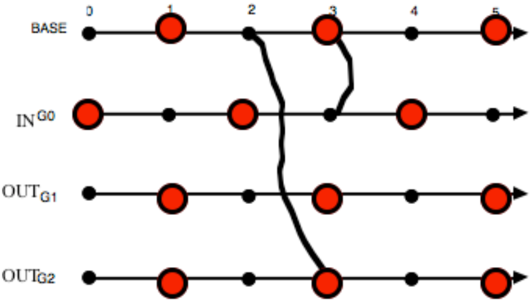

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A few other theorems equivalent to WKL_0.

Theorem

(RCA_0) *The following are equivalent:*

1. WKL_0.
2. *Every ctn. function on* $[0,1]$ *is bounded. (Simpson)*
3. *The closed interval* $[0,1]$ *is compact. (Friedman)*
4. *Every closed subset of* $\mathbb{Q} \cap [0,1]$ *is compact. (Hirst)*
5. *Existence theorem for solutions to ODEs. (Simpson)*
6. *The line graph of a bipartite graph is bipartite. (Hirst)*
7. *If* $\langle x_n \rangle_{n \in \mathbb{N}}$ *is a sequence of real numbers then there is a sequence of natural numbers* $\langle i_n \rangle_{n \in \mathbb{N}}$ *such that for each* j*,* $x_{i_j} = \min\{x_n \mid n \leq j\}$*. (Hirst)*

Arithmetical Comprehension

ACA_0 is RCA_0 plus the following comprehension scheme:

> For any formula $\theta(n)$ with only number quantifiers, the set $\{n \in \mathbb{N} \mid \theta(n)\}$ exists.

The minimum ω model of ACA_0 contains all the arithmetically definable sets.

Note: $\mathsf{WKL}_0 \nvdash \mathsf{ACA}_0$, but $\mathsf{ACA}_0 \vdash \mathsf{WKL}_0$.

ACA_0 and Graph Theory

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2. *Every graph can be decomposed into its connected components.*

Observation: The proof of "every graph with no odd cycles can be two colored" that starts by decomposing the graph into its connected components makes use of the strong axiom ACA_0. That proof is provably distinct from our proof in WKL_0.

Other theorems equivalent to ACA_0

Theorem

(RCA_0) *The following are equivalent:*

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2. *Bolzano-Weierstraß theorem. (Friedman)*
3. *Cauchy sequences converge. (Simpson)*
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General rule of thumb: ACA_0 suffices for undergraduate math.

RCA_0 proves transfinite induction for arithmetical formulas implies ACA_0. (Hirst)

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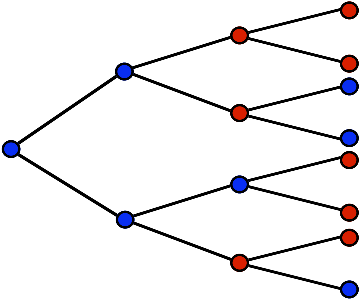
Conclusion: All undergraduate math can be done via transfinite induction arguments.

Ramsey's theorem on trees

RT^1: If $f : \mathbb{N} \to k$ then there is a $c \leq k$ and an infinite set H such that $\forall n \in H\ f(n) = c$.

TT^1: For any finite coloring of $2^{<\mathbb{N}}$, there is a monochromatic subtree order-isomorphic to $2^{<\mathbb{N}}$.



These results extend to colorings of n-tuples.

TT^n_k parallels RT^n_k

TT^n_k: For any k coloring of the n-tuples of comparable nodes in $2^{<\mathbb{N}}$, there is a color and a subtree order-isomorphic to $2^{<\mathbb{N}}$ in which all n-tuples of comparable nodes have the specified color.

Note: RT^n_k is an easy consequence of TT^n_k

Results in Chubb, Hirst, and McNichol:

- There is a computable coloring with no Σ^0_n monochromatic subtree. (Free.)
- Every computable coloring has a Π^0_n monochromatic subtree. (Not free.)
- For $n \geq 3$ and $k \geq 2$, $\mathsf{RCA}_0 \vdash \mathsf{TT}^n_k \leftrightarrow \mathsf{ACA}_0$.

TT^1 and TT^2 are problematic

$\mathsf{RCA}_0 + \Sigma^0_2 - \mathsf{IND}$ can prove TT^1.

$\mathsf{RCA}_0 + \mathsf{RT}^1$ does not suffice to prove TT^1.
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Question: Does TT^1 imply $\Sigma^0_2 - \mathsf{IND}$?

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Question: Does TT^1 imply $\Sigma^0_2 - \mathrm{IND}$?

$\mathrm{RCA}_0 + \mathrm{RT}^2$ does not imply ACA_0. (Seetapun)

Does $\mathrm{RCA}_0 + \mathrm{TT}^2$ imply ACA_0?
Does $\mathrm{RCA}_0 + \mathrm{TT}^2$ imply WKL_0?

References

[1] Harvey Friedman, *Abstracts: Systems of second order arithmetic with restricted induction, I and II*, J. Symbolic Logic **41** (1976), 557–559.

[2] Stephen G. Simpson, *Subsystems of second order arithmetic*, 2nd ed., Perspectives in Logic, Cambridge University Press, Cambridge, 2009.

[3] Jennifer Chubb, Jeffry L. Hirst, and Timothy H. McNicholl, *Reverse mathematics, computability, and partitions of trees*, J. Symbolic Logic **74** (2009), no. 1, 201–215.

[4] Jared Corduan, Marsha Groszek, and Joseph Mileti, *Reverse mathematics and Ramsey's property for trees*, J. Symbolic Logic. To appear.