Introduction to Reverse Mathematics

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Reverse Mathematics

Goal: Determine exactly which set existence axioms are needed to prove familiar theorems.

Method: Prove results of the form

$$\mathrm{RCA}_0 \vdash \mathbf{AX} \leftrightarrow \mathbf{THM}$$

where:

- RCA_0 is a weak axiom system,
- **AX** is a set existence axiom selected from a small hierarchy of axioms, and
- **THM** is a familiar theorem.

Why bother?

Work in reverse mathematics can:

- precisely categorize the logical strength of theorems.
- differentiate between different proofs of theorems.
- provide insight into the foundations of mathematics.
- utilize and contribute to work in many subdisciplines of mathematical logic – including proof theory, computability theory, models of arithmetic, etc.

RCA_0

Language:

Integer variables: x, y, z Set variables: X, Y, Z

Axioms:

basic arithmetic axioms

(0, 1, +, $\times$, =, and $<$ behave as usual.)

Restricted induction

$(\psi(0) \wedge \forall n(\psi(n) \rightarrow \psi(n+1))) \rightarrow \forall n \psi(n)$

where $\psi(n)$ has (at most) one number quantifier.

Recursive set comprehension

If $\theta \in \Sigma^0_1$ and $\psi \in \Pi^0_1$, and $\forall n(\theta(n) \leftrightarrow \psi(n))$,

then there is a set X such that $\forall n(n \in X \leftrightarrow \theta(n))$

A model of RCA_0

The smallest ω-model of RCA_0 consists of the usual natural numbers and the computable sets of natural numbers.
We write $\mathfrak{M} = \langle \omega, \mathsf{REC} \rangle$.

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RCA_0 proves that if $f : \mathbb{N} \to 2$, then there is an infinite set X such that f is constant on X.

The intuition gained from the minimal model is useful, but sometimes misleading.

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- Many mathematical concepts can be encoded in terms of such sequences. Second order arithmetic is remarkably expressive.

- Some coding can be averted: See Friedman's *Strict Reverse Mathematics* (2pm Sat) or Kohlenbach's *Higher Order Reverse Mathematics* in *Reverse Mathematics 2001*.

An example

Theorem

(RCA_0) *If $\langle x_n \rangle_{n\in\mathbb{N}}$ is a sequence of real numbers, then there is a sequence $\langle y_n \rangle_{n\in\mathbb{N}}$ such that for every j, $y_j = \min\{x_i \mid i \le j\}$.*

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The idea behind the proof:

Here's a sequence of three real numbers, each represented as a rapidly converging Cauchy sequence of rationals.

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\begin{array}{lllllll}
x_0 = \langle & 0 & .1 & .12 & .121 & .1212 & \ldots \rangle \\
x_1 = \langle & .1 & .11 & .101 & .1001 & .100 & \ldots \rangle \\
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Build the minimum y_2 by choosing the least entry in each component. So $y_2 = \langle 0\ .09\ .101\ .1001\ .999 \ldots \rangle$.

WKL_0

Weak König's Lemma

Statement: Big very skinny trees are tall.

More formally: If T is an infinite tree in which each node is labeled 0 or 1, then T contains an infinite path.

The subsystem WKL_0 is RCA_0 plus Weak König's Lemma.

There is an infinite computable $0-1$ tree with no infinite computable path, so $\langle \omega, \mathrm{REC} \rangle$ is not a model of WKL_0.

Conclusion: $\mathsf{RCA}_0 \nvdash \mathsf{WKL}_0$

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- There is a model of WKL_0 in which every set is low. (Apply the Jockusch-Soare low basis theorem.)

For more details, see Chapter VIII of Simpson's *Subsystems of Second Order Arithmetic*.

Finally! Some reverse mathematics!

Theorem

(RCA_0) *The following are equivalent:*

1. WKL_0.
2. *Every graph with no cycles of odd length is bipartite.*

Note: RCA_0 proves that a graph is bipartite if and only if there is a 2-coloring of its nodes.

Also, RCA_0 proves (2) for finite graphs.

WKL_0 implies bipartite graph theorem

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Let T be the tree consisting of sequences of the form $\langle i_0, i_1, \ldots, i_n \rangle$ where the sequence is a correct 2-coloring of the subgraph of G on the vertices $v_0, v_1, \ldots, v_n$.

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Any path through T is the desired 2-coloring.

A tool for reversals

Theorem

(RCA_0) *The following are equivalent:*

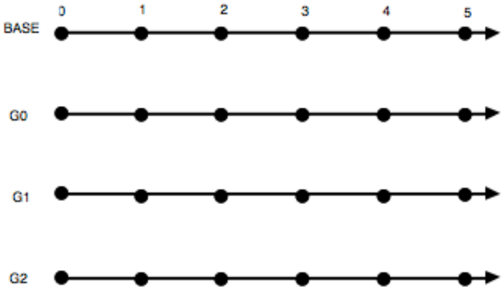
1. WKL_0.
2. *If f and g are injective functions from $\mathbb{N}$ into $\mathbb{N}$ and $Ran(f) \cap Ran(g) = \emptyset$, then there is a set X such that $Ran(f) \subset X$ and $X \cap Ran(g) = \emptyset$.*

Comment: X in (2) is like a separating set for disjoint computably enumerable sets.

The bipartite graph theorem implies WKL_0. A reversal!

Suppose we are given f and g with $Ran(f) \cap Ran(g) = \emptyset$.

If, for example, $f(3) = 0$ and $g(2) = 2$, we will construct the graph G as follows:



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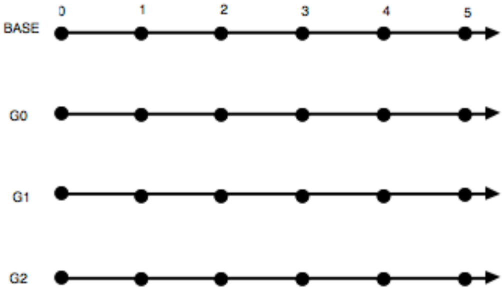

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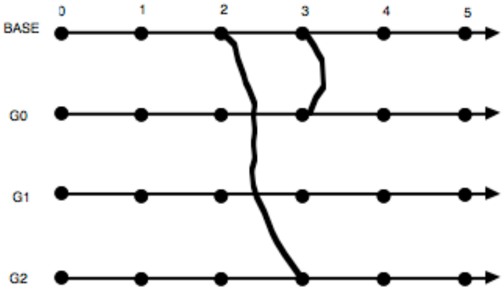

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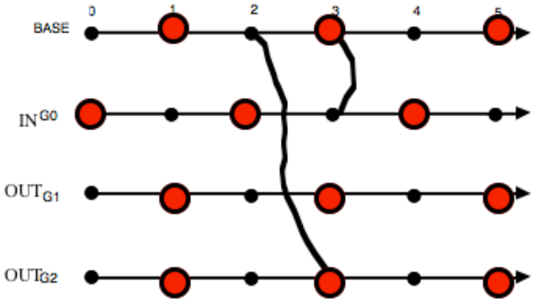

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A few other theorems equivalent to WKL_0.

Theorem

(RCA_0) *The following are equivalent:*

1. WKL_0.
2. *Every ctn. function on* $[0, 1]$ *is bounded. (Simpson)*
3. *The closed interval* $[0, 1]$ *is compact. (Friedman)*
4. *Every closed subset of* $\mathbb{Q} \cap [0, 1]$ *is compact. (Hirst)*
5. *Existence theorem for solutions to ODEs. (Simpson)*
6. *The line graph of a bipartite graph is bipartite. (Hirst)*
7. *If* $\langle x_n \rangle_{n \in \mathbb{N}}$ *is a sequence of real numbers then there is a sequence of natural numbers* $\langle i_n \rangle_{n \in \mathbb{N}}$ *such that for each* j, $x_{i_j} = \min\{x_n \mid n \leq j\}$. *(Hirst)*

Arithmetical Comprehension

ACA_0 is RCA_0 plus the following comprehension scheme:

> For any formula $\theta(n)$ with only number quantifiers, the set $\{n \in \mathbb{N} \mid \theta(n)\}$ exists.

The minimum ω model of ACA_0 contains all the arithmetically definable sets.

Note: $\mathsf{WKL}_0 \nvdash \mathsf{ACA}_0$, but $\mathsf{ACA}_0 \vdash \mathsf{WKL}_0$.

ACA_0 and Graph Theory

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Half of the proof: To prove that 1) implies 2), let G be a graph with vertices $v_0, v_1, \ldots$

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By ACA_0, f exists. f is the desired decomposition.

A tool for reversals to ACA_0

Theorem

(RCA_0) *The following are equivalent:*

1. ACA_0
2. *If* $f : \mathbb{N} \to \mathbb{N}$ *is 1-1, then* $Ran(f)$ *exists.*

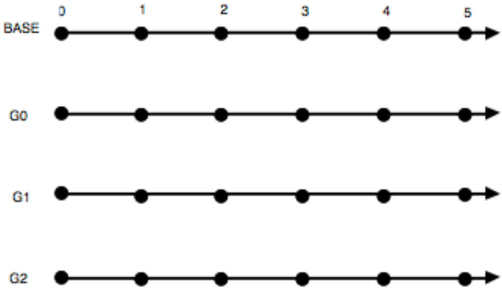
Item (2) is analogous to asserting the existence of the Turing jump.

To prove that the graph decomposition theorem implies ACA_0, we want to use a graph decomposition to calculate the range of a function.

The graph decomposition theorem implies ACA_0

Suppose we are given an injection f.

If, for example, $f(0) = 2$ and $f(1) = 0$, we will construct the graph G as follows:



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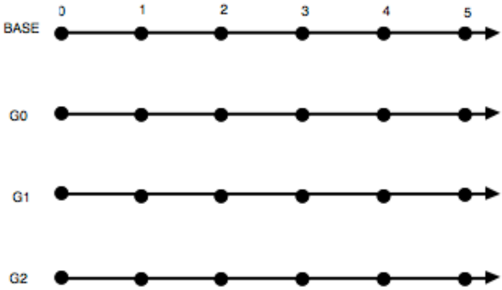

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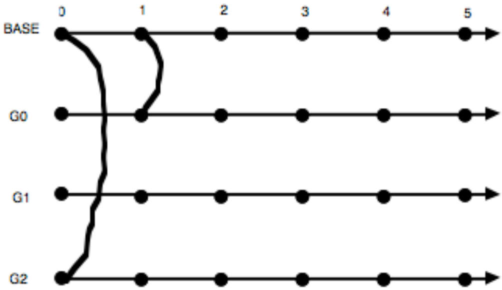

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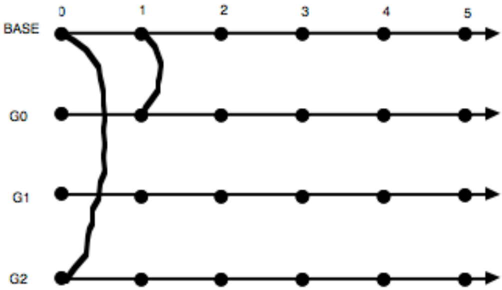

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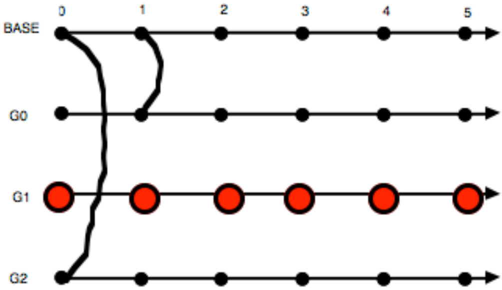

The range of f is computable from the decomposition.

Other theorems equivalent to ACA_0

Theorem

(RCA_0) *The following are equivalent:*

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2. *Bolzano-Weierstraß theorem. (Friedman)*
3. *Cauchy sequences converge. (Simpson)*
4. *Ramsey's theorem for triples. (Simpson)*

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Conclusion: All undergraduate math can be done with transfinite induction arguments.

Arithmetical Transfinite Recursion

ATR_0 consists of RCA_0 plus axioms that allow iteration of arithmetical comprehension along any well ordering. This allows transfinite constructions.

A tool for proofs:

Theorem

(ATR_0) *If $\psi(X)$ is a Σ^1_1 formula that is only satisfied by well ordered sets, then there is a well ordering β such that $\psi(X)$ implies $X < \beta$.*

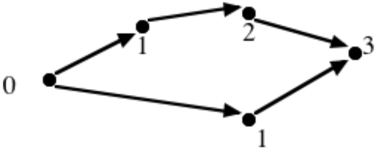
A tool for reversals:

Theorem

(RCA_0) ATR_0 *is equivalent to "If α and β are well orderings, then $\alpha \leq \beta$ or $\beta \leq \alpha$."*

ATR_0 and graph theory

A rank function for a directed acyclic graph is a function that maps the vertices onto a well ordering, preserving the ordering induced by the edges in a nice way.



Theorem

(RCA_0) *The following are equivalent:*

1. ATR_0
2. *Every well founded directed acyclic graph with a source node has a rank function*

Other theorems equivalent to ATR_0

Theorem

(RCA_0) *The following are equivalent:*

1. ATR_0.
2. *Lusin's Separation Theorem: Any two disjoint analytic sets can be separated by a Borel set. (Simpson)*
3. *Mahlo's Theorem: Given any two countable closed compact subsets of the reals, one can be homeomorphically embedded in the other. (Friedman and Hirst)*
4. *Every countable reduced Abelian p-group has an Ulm resolution. (Friedman, Simpson, and Smith)*
5. *Sherman's Inequality: If α, β, and γ are countable well orderings, then $(\alpha+\beta)\gamma \leq \alpha\gamma + \beta\gamma$. (Hirst)*

Π^1_1 comprehension

The system $\Pi^1_1 - CA_0$ is RCA$_0$ plus the axioms asserting the existence of the set $\{n \in \mathbb{N} \mid \theta(n)\}$ for $\theta \in \Pi^1_1$. (That is, θ has one universal set quantifier and no other set quantifiers.)

A tool for reversals and some graph theory:

Theorem

(RCA$_0$) *The following are equivalent:*

1. $\Pi^1_1 - CA_0$.
2. *If $\langle T_i \rangle_{n\in\mathbb{N}}$ is a sequence of trees then there is a function $f : \mathbb{N} \to 2$ such that $f(n) = 1$ iff T_n is well founded.*
3. *For any graph H, and any sequence of graphs $\langle G_i \rangle_{i\in\mathbb{N}}$, there is a function $f : \mathbb{N} \to 2$ such that $f(n) = 1$ iff H is isomorphic to a subgraph of G. (Hirst and Lempp)*

An abbreviated list of references

[1] Harvey Friedman, *Some systems of second order arithmetic and their use*, Proceedings of the International Congress of Mathematicians (Vancouver, B. C., 1974), Vol. 1, 1975, pp. 235–242.

[2] Harvey Friedman, *Abstracts: Systems of second order arithmetic with restricted induction, I and II*, J. Symbolic Logic **41** (1976), 557–559.

[3] Stephen G. Simpson, *Subsystems of second order arithmetic*, 2nd ed., Perspectives in Logic, Cambridge University Press, Cambridge, 2009.

Things that don't fit

Theorems are interesting when they are equivalent to one of the "big five," and also when they aren't.

- The infinite pigeon-hole principle, RT(1), is not provable in WKL_0. RT(1) is equivalent to the Σ^0_2 bounding principle.
- The infinite pigeon-hole principle on trees, TT(1), is not provable from RT(1) (Corduan, Mileti, and Groszek). Does RT^2_2 prove TT(1)?
- RT(2) $\nvdash$ ACA_0 (Seetapun) and $WKL_0 \nvdash$ RT(2). Does RT(2) prove WKL_0?
- Full Ramsey's theorem is equivalent to ACA_0^+ (Mileti).