Counterintuitive Aspects of Computable Analysis

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(Copies of these slides are available at:
`www.mathsci.appstate.edu/~jlh/snp/slides.html`)

A *computable real number* is a computable function $x : \mathbb{N} \to \mathbb{Q}$ such that $\forall k \forall i\ |x(k) - x(k+i)| \leq 2^{-k}$ (that is, $\langle x(i) \rangle_{i \in \mathbb{N}}$ is a rapidly converging Cauchy sequence of rational numbers.)

Examples of some computable reals:

$$\sqrt{2} : \quad 1,\ 1.4,\ 1.41,\ 1.414,\ 1.4142,\ \ldots$$

$$\pi : \quad 3,\ 3.1,\ 3.14,\ 3.141,\ 3.1415,\ \ldots$$

$$0 : \quad 1,\ \tfrac{1}{2},\ \tfrac{1}{4},\ \tfrac{1}{8},\ \tfrac{1}{16},\ \ldots$$

$$0 : \quad 0,\ 0,\ 0,\ 0,\ 0,\ \ldots$$

Relationships between computable reals

$x = y$ means: $\forall k \; |x(k) - y(k)| \leq 2^{-k+1}$

$x \leq y$ means: $\forall k \; (x(k) \leq y(k) + 2^{-k+1})$

$y < x$ means $x \not\leq y$,
which is $\exists k \; (y(k) + 2^{-k+1} < x(k))$

Finding minima

Theorem 1. *Let* $\langle x_k \rangle_{k\in\mathbb{N}}$ *be a computable sequence of computable real numbers. Then there is a computable sequence of computable reals* $\langle u_k \rangle_{k\in\mathbb{N}}$ *such that for each* k, $u_k = \min\{x_j \mid j \leq k\}$. *That is, for each* k *we have:*

(1) $\forall j \leq k \ (u_k \leq x_j)$, *and*

(2) $\exists j \leq k \ (u_k = x_j)$.

Picking the minima

Let $u_k(j) = \min\{x_n(j) \mid n \leq k\}$.

Example:

x_0:	1	1.4	1.41	1.414	1.4142	...	$(\sqrt{2})$
x_1:	3	3.1	3.14	3.141	3.1415	...	(π)
x_2:	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	...	(0)
x_3:	0	$\frac{1}{8}$	$\frac{3}{16}$	$\frac{7}{32}$	$\frac{15}{64}$	...	$(\frac{1}{4})$

Questions:

- Is $\langle u_k \rangle_{k\in\mathbb{N}}$ a computable sequence?
- Is each u_k a computable real?
- Is $u_k = \min\{x_j \mid j \leq k\}$?
- Do we know which x_j is equal to u_k?

Question: Do we know which x_j is equal to u_k?

Answer: Not necessarily.

Theorem 2. *There is a computable sequence of computable reals* $\langle x_k \rangle_{k \in \mathbb{N}}$ *such that if* $\langle \mu_k \rangle_{k \in \mathbb{N}}$ *is any sequence of integers satisfying*

$$\forall k \big(x_{\mu_k} = \min\{x_j \mid j \leq k\} \big)$$

then $\langle \mu_k \rangle_{k \in \mathbb{N}}$ *is not computable.*

Question: Do we know which x_j is equal to u_k?

Answer: Sometimes.

Theorem 3. *If* $\langle x_k \rangle_{k\in\mathbb{N}}$ *is a* **nonrepeating** *computable sequence of computable reals, then there is a computable sequence of integers* $\langle \mu_k \rangle_{k\in\mathbb{N}}$ *such that*

$$\forall k\big(x_{\mu_k} = \min\{x_j \mid j \le k\}\big).$$

Picking μ_k

Example (of a nonrepeating sequence):

x_0:	1	1.4	1.41	1.414	1.4142	$\ldots$	$(\sqrt{2})$
x_1:	3	3.1	3.14	3.141	3.1415	$\ldots$	(π)
x_2:	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\ldots$	(0)
x_3:	0	$\frac{1}{8}$	$\frac{3}{16}$	$\frac{7}{32}$	$\frac{15}{64}$	$\ldots$	$(\frac{1}{4})$

Constructive Analysis vs. Computable Analysis

Computable Analyst:

> We can select minima of initial segments.
>
> We can't select the indices of the minima.

Constructive Analyst:

> We can't select the indices, so we can't select the minima.
>
> We can select the minima of nonrepeating sequences.

Theorem 4. (RCA_0) *If $\langle x_k \rangle_{k \in \mathbb{N}}$ is a countable sequence of reals, then there is a sequence of reals $\langle u_k \rangle_{k \in \mathbb{N}}$ such that for each k, $u_k = \min\{x_j \mid j \leq k\}$.*

Theorem 5. (RCA_0) *The following are equivalent:*

1. WKL_0
2. *If $\langle x_k \rangle_{k \in \mathbb{N}}$ is a sequence of reals, then there is a sequence of integers $\langle \mu_k \rangle_{k \in \mathbb{N}}$ such that $\forall k (x_{\mu_k} = \min\{x_j \mid j \leq k\})$.*

Reverse Mathematics

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