Introduction to Reverse Mathematics with Applications

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Reverse Mathematics

Goal: Determine exactly which set existence axioms are needed to prove familiar theorems.

Method: Prove results of the form

$$\mathsf{RCA}_0 \vdash \mathbf{AX} \leftrightarrow \mathbf{THM}$$

where:

- RCA_0 is a weak axiom system,
- $\mathbf{AX}$ is a set existence axiom selected from a small hierarchy of axioms, and
- $\mathbf{THM}$ is a familiar theorem.

Why bother?

Work in reverse mathematics can:

- precisely categorize the logical strength of theorems.
- differentiate between different proofs of theorems.
- provide insight into the foundations of mathematics.
- utilize and contribute to work in many subdisciplines of mathematical logic – including proof theory, computability theory, models of arithmetic, etc.

RCA_0

Language:

Integer variables: x, y, z Set variables: X, Y, Z

Axioms:

basic arithmetic axioms

(0, 1, +, $\times$, =, and $<$ behave as usual.)

Restricted induction

$(\psi(0) \wedge \forall n(\psi(n) \to \psi(n+1))) \to \forall n \psi(n)$

where $\psi(n)$ has (at most) one number quantifier.

Recursive set comprehension

If $\theta \in \Sigma^0_1$ and $\psi \in \Pi^0_1$, and $\forall n(\theta(n) \leftrightarrow \psi(n))$,

then there is a set X such that $\forall n(n \in X \leftrightarrow \theta(n))$

A model of RCA_0

The smallest ω-model of RCA_0 consists of the usual natural numbers and the computable sets of natural numbers.
We write $\mathfrak{M} = \langle \omega, \mathsf{REC} \rangle$.

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The intuition gained from the minimal model is useful, but sometimes misleading.

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- RCA_0 can prove the arithmetic associated with pairing functions.
- Sets of pairs correspond to functions and/or countable sequences.
- Many mathematical concepts can be encoded in terms of such sequences. Second order arithmetic is remarkably expressive.
- Some coding can be averted: See Friedman's work on *Strict Reverse Mathematics* or Kohlenbach's *Higher Order Reverse Mathematics* in *Reverse Mathematics 2001*.

An example

Theorem

(RCA_0) *If* $\langle x_n \rangle_{n \in \mathbb{N}}$ *is a sequence of real numbers, then there is a sequence* $\langle y_n \rangle_{n \in \mathbb{N}}$ *such that for every* j, $y_j = \min\{x_i \mid i \leq j\}$.

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The idea behind the proof:

Here's a sequence of three real numbers, each represented as a rapidly converging Cauchy sequence of rationals.

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\begin{array}{lcccccc}
x_0 = \langle & 0 & .1 & .12 & .121 & .1212 & \ldots \rangle \\
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Build the minimum y_2 by choosing the least entry in each component. So $y_2 = \langle 0\ .09\ .101\ .1001\ .999 \ldots \rangle$.

WKL_0

Weak König's Lemma

Statement: Big very skinny trees are tall.

More formally: If T is an infinite tree in which each node is labeled 0 or 1, then T contains an infinite path.

The subsystem WKL_0 is RCA_0 plus Weak König's Lemma.

There is an infinite computable $0 - 1$ tree with no infinite computable path, so $\langle \omega, \mathsf{REC} \rangle$ is not a model of WKL_0.

Conclusion: $\mathsf{RCA}_0 \nvdash \mathsf{WKL}_0$

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- There is a model of WKL_0 in which every set is low. (Apply the Jockusch-Soare low basis theorem.)

For more details, see Chapter VIII of Simpson's *Subsystems of Second Order Arithmetic*.

Finally! Some reverse mathematics!

Theorem

(RCA_0) *The following are equivalent:*

1. WKL_0.
2. *Every graph with no cycles of odd length is bipartite.*

Note: RCA_0 proves that a graph is bipartite if and only if there is a 2-coloring of its nodes.

Also, RCA_0 proves (2) for finite graphs.

WKL_0 implies bipartite graph theorem

Suppose G is a graph with vertices $v_0, v_1, v_2, \ldots$ and no odd cycles.

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Let T be the tree consisting of sequences of the form $\langle i_0, i_1, \ldots, i_n\rangle$ where the sequence is a correct 2-coloring of the subgraph of G on the vertices $v_0, v_1, \ldots, v_n$.

Since G has no odd cycles, RCA_0 proves T contains infinitely many nodes.

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Any path through T is the desired 2-coloring.

A tool for reversals

Theorem

(RCA_0) *The following are equivalent:*

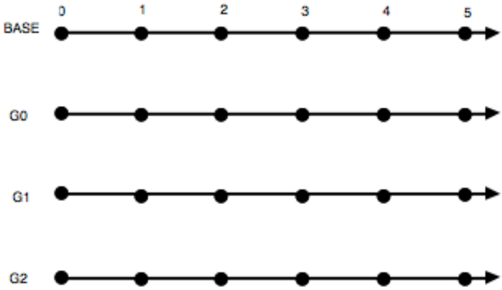
1. WKL_0.
2. *If* f *and* g *are injective functions from* $\mathbb{N}$ *into* $\mathbb{N}$ *and* $Ran(f) \cap Ran(g) = \emptyset$*, then there is a set* X *such that* $Ran(f) \subset X$ *and* $X \cap Ran(g) = \emptyset$*.*

Comment: X in (2) is like a separating set for disjoint computably enumerable sets.

The bipartite graph theorem implies WKL_0. A reversal!

Suppose we are given f and g with $Ran(f) \cap Ran(g) = \emptyset$.

If, for example, $f(3) = 0$ and $g(2) = 2$, we will construct the graph G as follows:



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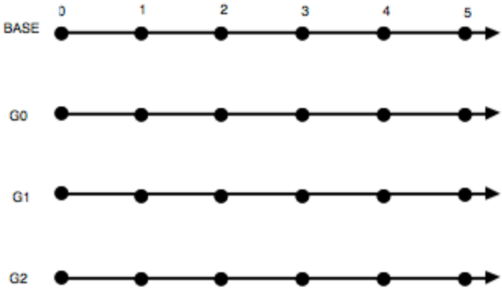

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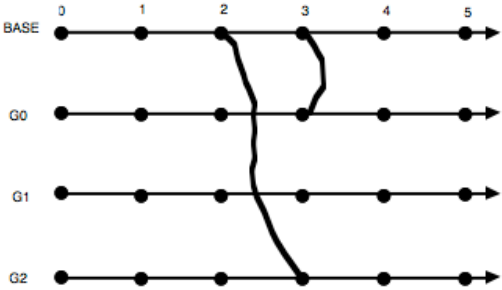

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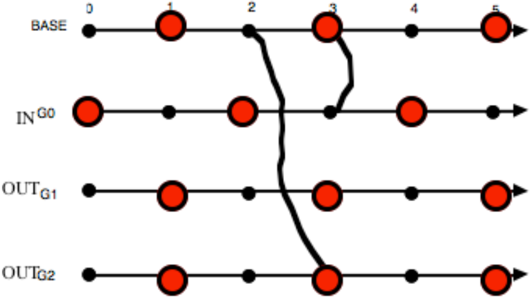

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A few other theorems equivalent to WKL_0.

Theorem

(RCA_0) *The following are equivalent:*

1. WKL_0.
2. *Every ctn. function on $[0,1]$ is bounded. (Simpson)*
3. *The closed interval $[0,1]$ is compact. (Friedman)*
4. *Every closed subset of $\mathbb{Q} \cap [0,1]$ is compact. (Hirst)*
5. *Existence theorem for solutions to ODEs. (Simpson)*
6. *The line graph of a bipartite graph is bipartite. (Hirst)*
7. *If $\langle x_n \rangle_{n \in \mathbb{N}}$ is a sequence of real numbers then there is a sequence of natural numbers $\langle i_n \rangle_{n \in \mathbb{N}}$ such that for each j, $x_{i_j} = \min\{x_n \mid n \leq j\}$. (Hirst)*

Arithmetical Comprehension

ACA_0 is RCA_0 plus the following comprehension scheme:

> For any formula $\theta(n)$ with only number quantifiers, the set $\{n \in \mathbb{N} \mid \theta(n)\}$ exists.

The minimum ω model of ACA_0 contains all the arithmetically definable sets.

Note: $\mathsf{WKL}_0 \nvdash \mathsf{ACA}_0$, but $\mathsf{ACA}_0 \vdash \mathsf{WKL}_0$.

ACA_0 and Graph Theory

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(RCA_0) *The following are equivalent:*

1. ACA_0
2. *Every graph can be decomposed into its connected components.*

Half of the proof: To prove that 1) implies 2), let G be a graph with vertices $v_0, v_1, \ldots$

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By ACA_0, f exists. f is the desired decomposition.

A tool for reversals to ACA_0

Theorem

(RCA_0) *The following are equivalent:*

1. ACA_0
2. *If $f : \mathbb{N} \to \mathbb{N}$ is 1-1, then $Ran(f)$ exists.*

Item (2) is analogous to asserting the existence of the Turing jump.

To prove that the graph decomposition theorem implies ACA_0, we want to use a graph decomposition to calculate the range of a function.

The graph decomposition theorem implies ACA_0

Suppose we are given an injection f.

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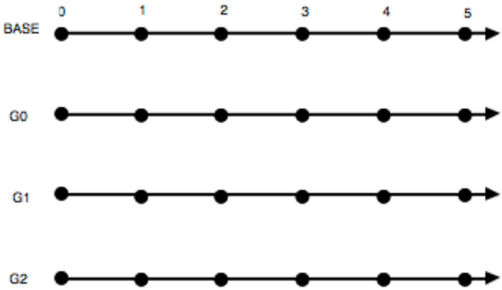

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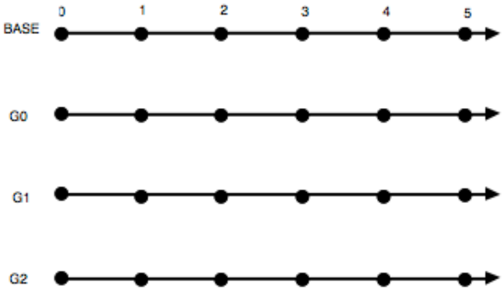

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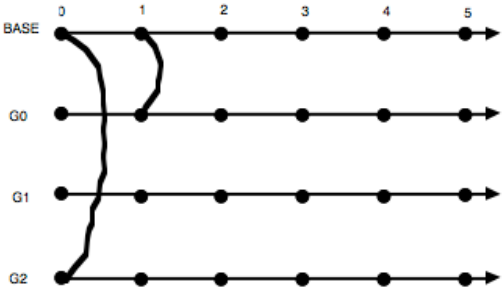

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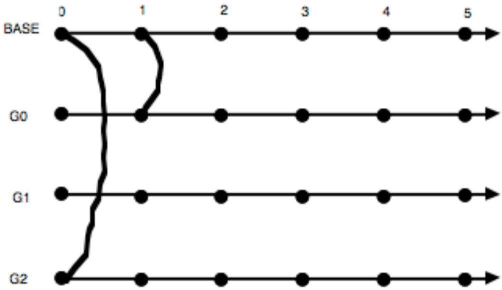

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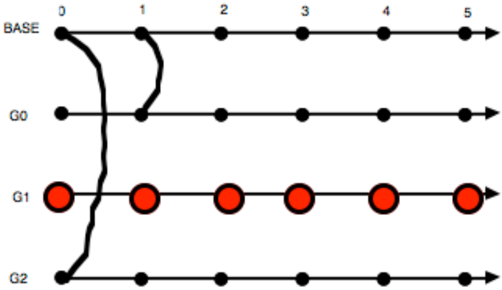

The range of f is computable from the decomposition.

Other theorems equivalent to ACA_0

Theorem

(RCA_0) *The following are equivalent:*

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2. *Bolzano-Weierstraß theorem. (Friedman)*
3. *Cauchy sequences converge. (Simpson)*
4. *Ramsey's theorem for triples. (Simpson)*

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Conclusion: All undergraduate math can be done with transfinite induction arguments.

Arithmetical Transfinite Recursion

ATR_0 consists of RCA_0 plus axioms that allow iteration of arithmetical comprehension along any well ordering. This allows transfinite constructions.

A tool for proofs:

Theorem

(ATR_0) *If $\psi(X)$ is a Σ^1_1 formula that is only satisfied by well ordered sets, then there is a well ordering β such that $\psi(X)$ implies $X < \beta$.*

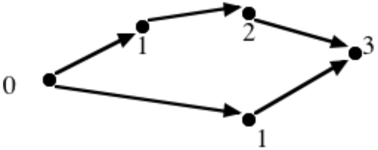
A tool for reversals:

Theorem

(RCA_0) ATR_0 *is equivalent to "If α and β are well orderings, then $\alpha \leq \beta$ or $\beta \leq \alpha$."*

ATR_0 and graph theory

A rank function for a directed acyclic graph is a function that maps the vertices onto a well ordering, preserving the ordering induced by the edges in a nice way.



Theorem

(RCA_0) *The following are equivalent:*

1. ATR_0
2. *Every well founded directed acyclic graph with a source node has a rank function*

Other theorems equivalent to ATR_0

Theorem

(RCA_0) *The following are equivalent:*

1. ATR_0.
2. *Lusin's Separation Theorem: Any two disjoint analytic sets can be separated by a Borel set. (Simpson)*
3. *Mahlo's Theorem: Given any two countable closed compact subsets of the reals, one can be homeomorphically embedded in the other. (Friedman and Hirst)*
4. *Every countable reduced Abelian p-group has an Ulm resolution. (Friedman, Simpson, and Smith)*
5. *Sherman's Inequality: If α, β, and γ are countable well orderings, then $(\alpha+\beta)\gamma \leq \alpha\gamma + \beta\gamma$. (Hirst)*

Π^1_1 comprehension

The system $\Pi^1_1 - \mathsf{CA}_0$ is RCA_0 plus the axioms asserting the existence of the set $\{n \in \mathbb{N} \mid \theta(n)\}$ for $\theta \in \Pi^1_1$. (That is, θ has one universal set quantifier and no other set quantifiers.)

A tool for reversals and some graph theory:

Theorem

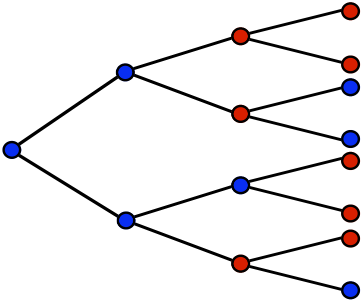
(RCA_0) *The following are equivalent:*

1. $\Pi^1_1 - \mathsf{CA}_0$.
2. *If $\langle T_i \rangle_{n \in \mathbb{N}}$ is a sequence of trees then there is a function $f : \mathbb{N} \to 2$ such that $f(n) = 1$ iff T_n is well founded.*
3. *For any graph H, and any sequence of graphs $\langle G_i \rangle_{i \in \mathbb{N}}$, there is a function $f : \mathbb{N} \to 2$ such that $f(n) = 1$ iff H is isomorphic to a subgraph of G. (Hirst and Lempp)*

Ramsey's theorem on trees

RT[1]: If $f : \mathbb{N} \to k$ then there is a $c \leq k$ and an infinite set H such that $\forall n \in H\, f(n) = c$.

TT[1]: For any finite coloring of $2^{<\mathbb{N}}$, there is a monochromatic subtree order-isomorphic to $2^{<\mathbb{N}}$.



TT^n_k parallels RT^n_k

TT^n_k: For any k coloring of the n-tuples of comparable nodes in $2^{<\mathbb{N}}$, there is a color and a subtree order-isomorphic to $2^{<\mathbb{N}}$ in which all n-tuples of comparable nodes have the specified color.

Note: RT^n_k is an easy consequence of TT^n_k

Results in Chubb, Hirst, and McNichol:

- There is a computable coloring with no Σ^0_n monochromatic subtree. (Free.)
- Every computable coloring has a Π^0_n monochromatic subtree. (Not free.)
- For $n \geq 3$ and $k \geq 2$, $\mathsf{RCA}_0 \vdash \mathsf{TT}^n_k \leftrightarrow \mathsf{ACA}_0$.

TT^1 is problematic

$\mathsf{RCA}_0 + \Sigma^0_2 - \mathsf{IND}$ can prove TT^1.

$\mathsf{RCA}_0 + \mathsf{RT}^1$ does not suffice to prove TT^1.

Corduan, Groszek, and Mileti

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$f : [2^{<\mathbb{N}}]^2 \to k$ is 3-stable if

$$\forall \sigma \in 2^{<\mathbb{N}} \exists c < k \forall \sigma' \supseteq \sigma \exists \tau \supset \sigma' \forall \rho \supseteq \tau \, f(\sigma, \rho) = c$$

$\mathsf{S}^3\mathsf{TT}^2_2$ asserts that every 3-stable coloring of pairs in a tree has a monochromatic subtree order isomorphic to $2^{<\mathbb{N}}$.

Theorem: $\mathsf{RCA}_0 + \mathsf{S}^3\mathsf{TT}^2_2 \vdash \mathsf{TT}^1$.

Question: Does this theorem hold with 1-stable colorings?

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