Coloring Problems and Reverse Mathematics

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These slides are available at:
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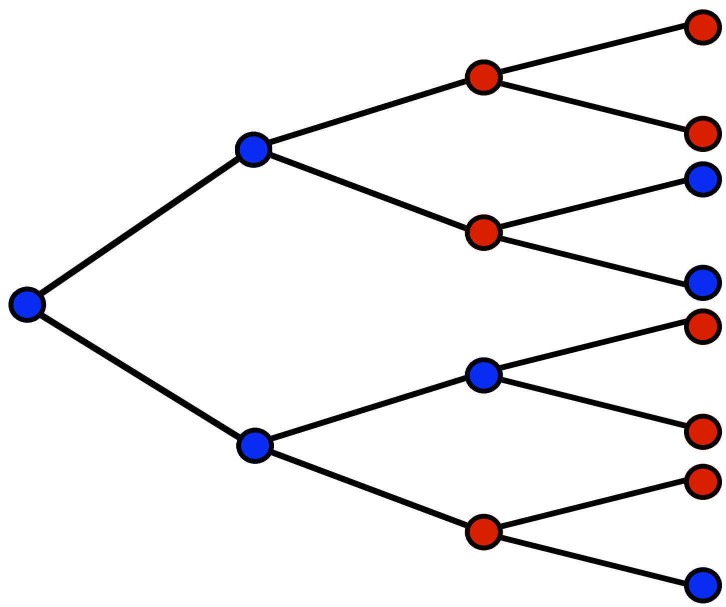
Pigeonhole principles

RT^1: If $f : \mathbb{N} \to k$ then there is a $c \leq k$ and an infinite set H such that $\forall n \in H \ f(n) = c$.

TT^1: For any finite coloring of $2^{<\mathbb{N}}$, there is a monochromatic subtree order-isomorphic to $2^{<\mathbb{N}}$.



A proof of TT^1

Lef FIN denote the set of finite subsets of $\mathbb{N}$.

A version of Hindman's theorem:

Finite Union Theorem (FUT): If $f : \mathsf{FIN} \to \mathsf{k}$ then there is a $c \leq k$ and an infinite increasing sequence $\langle H_i \rangle_{i \in \mathbb{N}}$ of elements of FIN such that for every $F \in \mathsf{FIN}$

$$f(\cup_{i \in F} H_i) = c.$$

Claim: TT^1 is an easy consequence of FUT.
Sketch: Identify finite sets with sequences.

Question: Do we need FUT to prove TT^1?
Answer: No.
Reverse mathematics is often useful for answering this sort of question.

Brief overview of reverse mathematics

Reverse mathematics uses a hierarchy of axiom systems for second order arithmetic to analyze the relative strength of mathematical theorems.

RCA_0 : basic arithmetic axioms, induction for Σ^0_1 formulas, comprehension for computable sets

WKL_0 : RCA_0 plus a weak form of König's lemma

ACA_0 : RCA_0 plus comprehension for sets defined by arithmetical formulas

Theorem [BHS] (RCA_0) FUT implies ACA_0.

Theorem [CHM] (RCA_0) The least element principle for Σ^0_2 formulas ($\Sigma^0_2 - \mathsf{IND}$) implies TT^1.

Sketch: Find a smallest set of colors such that for some node, every extension has a color in the set.

Corollary: The natural numbers together with the computable sets form a model of RCA_0 and TT^1 that is not a model FUT.

Related computability theoretic result: Every computable coloring of $2^{<\mathbb{N}}$ has a computable monochromatic subtree order isomorphic to $2^{<\mathbb{N}}$.

In reverse mathematics, equivalence results are optimal. The preceding results could be improved.

Question: Do we need $\Sigma^0_2 - \mathsf{IND}$ to prove TT^1?

Answer: Maybe. RCA_0 plus RT^1 does not prove TT^1 [CGM].

Question: Does ACA_0 prove FUT?

Answer: Maybe. The best known result is that the stronger system ACA_0^+ proves FUT [BHS].

Some more results on Ramsey's theorem

RT^n_k: If $f : [\mathbb{N}]^n \to k$ then there is a c and an infinite $H \subset \mathbb{N}$ such that $f([H]^n) = c$.

RT^n: $\forall k \mathsf{RT}^n_k$

RT: $\forall n \mathsf{RT}^n$

Sample reverse mathematics

- $\mathsf{RCA}_0 \vdash \mathsf{RT}^1 \leftrightarrow \mathsf{B}\Pi^0_1$
- $\mathsf{RCA}_0 \nvdash \mathsf{RT}^2_2$ (Specker) $\quad \mathsf{WKL}_0 \nvdash \mathsf{RT}^2_2$ (Jockusch)
- For $n \geq 3$ and $k \geq 2$, $\mathsf{RCA}_0 \vdash \mathsf{RT}^n_k \leftrightarrow \mathsf{ACA}_0$ (Simpson)
- $\mathsf{RCA}_0 \vdash \mathsf{RT} \leftrightarrow \mathsf{ACA}'_0$ (Mileti)

TT^n_k parallels RT^n_k

TT^n_k: For any k coloring of the n-tuples of comparable nodes in $2^{<\mathbb{N}}$, there is a color and a subtree order-isomorphic to $2^{<\mathbb{N}}$ in which all n-tuples of comparable nodes have the specified color.

Note: RT^n_k is an easy consequence of TT^n_k

- For $n \geq 3$ and $k \geq 2$, $\mathsf{RCA}_0 \vdash \mathsf{TT}^n_k \leftrightarrow \mathsf{ACA}_0$ [CHM].
- $\mathsf{RCA}_0 \vdash \mathsf{TT} \leftrightarrow \mathsf{ACA}_0'$. [AH plus Mileti]

Cholak, Jockusch, and Slaman showed $\mathsf{RCA}_0 + \mathsf{RT}^2_2 \nvdash \mathsf{RT}^2$.

Does $\mathsf{RCA}_0 + \mathsf{TT}^2_2 \vdash \mathsf{TT}^2$?

Does $\mathsf{RCA}_0 + \mathsf{TT}^2_2 \vdash \mathsf{RT}^2$?

Polarized partitions

Work with Damir Dzhafarov [DH]:

[IPT^n_k:] If $f : [\mathbb{N}]^n \to k$ then there is a c and a sequence of infinite sets $H_1 \dots H_n$ such that for any $x_1 < \cdots < x_n$ (with $x_i \in H_i$ for all i) we have $f(x_1 \dots x_n) = c$.

Note: IPT^n_k is an easy consequence of RT^n_k.

Theorem: If $n \geq 3$ and $k \geq 2$, $\mathsf{RCA}_0 \vdash \mathsf{IPT}^n_k \leftrightarrow \mathsf{ACA}_0$.

Theorem: $\mathsf{RCA}_0 \vdash \mathsf{IPT} \leftrightarrow \mathsf{ACA}'_0$.

IPT^2

$f : [\mathbb{N}]^2 \to k$ is *stable* if $\lim_m f(n,m)$ exists for every n.

SRT^2 is RT^2 for stable partitions.

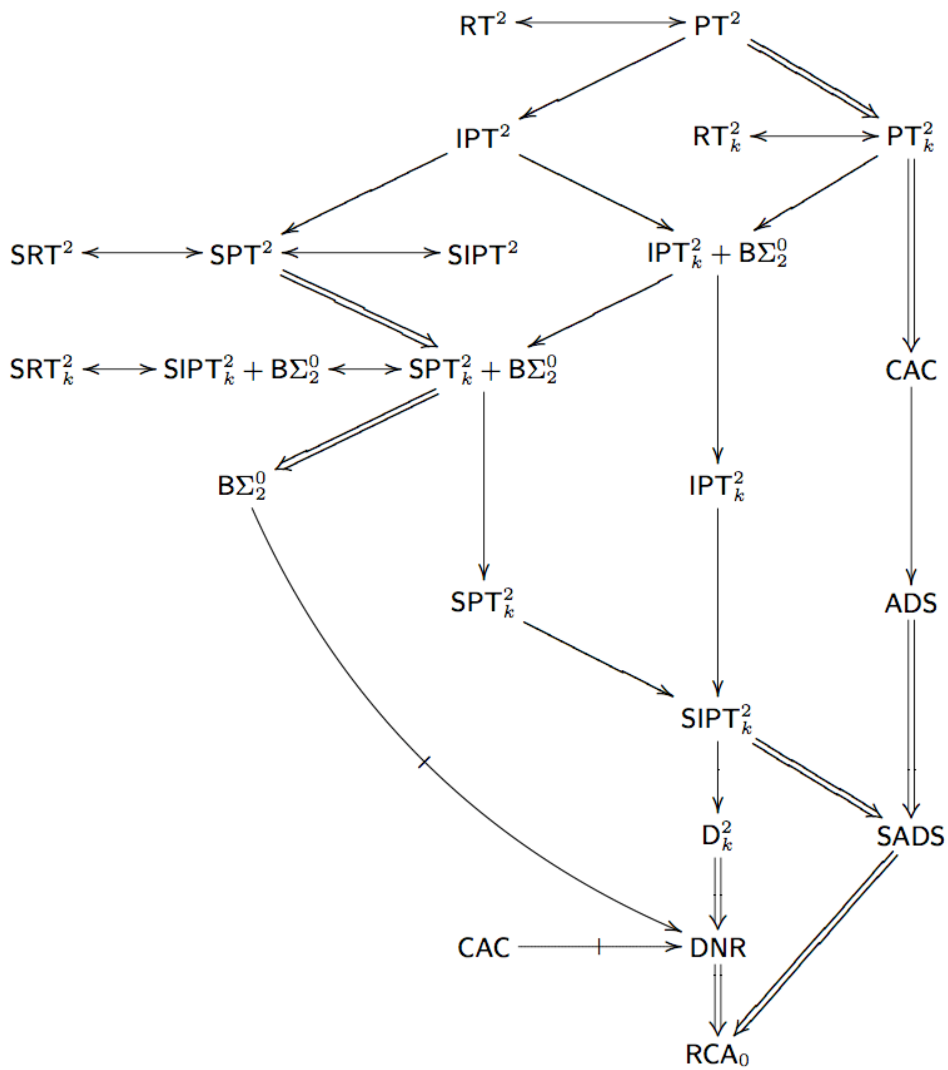
SIPT^2 is IPT^2 for stable partitions.

Theorem: $\mathsf{RCA}_0 \vdash \mathsf{SIPT}^2 \to \mathsf{RT}^1$

Theorem: $\mathsf{RCA}_0 \vdash \mathsf{SIPT}^2 \leftrightarrow \mathsf{SRT}^2$

Consequence: $\mathsf{RCA}_0 \vdash \mathsf{RT}^2 \to \mathsf{IPT}^2 \to \mathsf{SRT}^2$

Question: Which of the converses hold?



Results contributed by: Cholak, Dzhafarov, Hirschfeldt, Hirst, Jockusch, Kjos-Hanssen, Lempp, Slaman, and Shore

A coloring problem equivalent to WKL_0

Theorem(RCA_0) The following are equivalent:

(1) Weak König's Lemma.

(2) Suppose G is a graph with no cycles of odd length. Then the vertices of G can be colored with two colors so that no neighboring vertices match.

(3) Every graph with no cycles of odd length is bipartite.

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