Sets and Sequences

Jeff Hirst
Appalachian State University
Boone, NC

April 23, 2015

Dartmouth Mathematics Colloquium

Introduction: Turning sets into sequences

$\{3, 1, 0\}$ into $\langle 3, 1, 0 \rangle$ or ...

Introduction: Turning sets into sequences

$\{3, 1, 0\}$ into $\langle 3, 1, 0\rangle$ or ...

We can identify a set with its characteristic function:

n	3	2	1	0
$f(n)$	1	1	0	1

Introduction: Turning sets into sequences

$\{3, 1, 0\}$ into $\langle 3, 1, 0 \rangle$ or ...

We can identify a set with its characteristic function:

n	3	2	1	0
$f(n)$	1	1	0	1

and for finite sets, identify the binary sequence with a number.

$$1101_2 = 13_{10}$$

This is the canonical coding for finite sets from Soare's text [5].

Introduction: Turning sequences into sets

We can think of a sequence as a function:

$\langle 3, 1, 0 \rangle$ is

n	0	1	2
$f(n)$	3	1	0

and view a function as a set of ordered pairs

$$\{(0, 3), (1, 1), (2, 0)\}$$

Introduction: Turning sequences into sets

We can think of a sequence as a function:

$\langle 3, 1, 0 \rangle$ is

n	0	1	2
$f(n)$	3	1	0

and view a function as a set of ordered pairs

$$\{(0, 3), (1, 1), (2, 0)\}$$

Or we could let the set be the range of the function so

$$\langle 3, 1, 0, 3 \rangle \text{ is translated into } \{0, 1, 3\}$$

Infinite sequences into sets

Viewpoint: The Zermelo-Fraenkel axioms for set theory.

- In ZF sequences *are* sets.

Infinite sequences into sets

Viewpoint: The Zermelo-Fraenkel axioms for set theory.

- In ZF sequences *are* sets.
- Can we convert sequences (viewed as functions) into the associated range sets?

Yes, we could use the Axiom of Replacement:

Informal version: If $f(x)$ is a class function and D is a set then the range set $R = \{f(x) \mid x \in D\}$ exists.

Infinite sequences into sets

Viewpoint: The Zermelo-Fraenkel axioms for set theory.

- In ZF sequences *are* sets.
- Can we convert sequences (viewed as functions) into the associated range sets?

Yes, we could use the Axiom of Replacement:

Informal version: If $f(x)$ is a class function and D is a set then the range set $R = \{f(x) \mid x \in D\}$ exists.

Skolem-esque version: If $\psi(x, y)$ is a formula satisfying

$$\forall x \forall y \forall z ((\psi(x, y) \wedge \psi(x, z)) \rightarrow y = z)$$

then for every set D there is a set R such that

$$\forall y (y \in R \leftrightarrow \exists x (x \in D \wedge \psi(x, y)))$$

Infinite sets into sequences

Working in ZF, can we turn sets into sequences?

- The answer depends on our concept of *sequence*.
- We like to have “next elements” in sequences.

Infinite sets into sequences

Working in ZF, can we turn sets into sequences?

- The answer depends on our concept of *sequence*.
- We like to have "next elements" in sequences.
- A function from a (possibly transfinite) ordinal into a set S is certainly a sequence of elements from S.

Theorem

(ZF) Given $f : \alpha \to S$ *define* $\min(s) = \min\{\beta < \alpha \mid f(\beta) = s\}$ *and define* $s <_f t$ *if and only if* $\min(s) < \min(t)$*. Then* $<_f$ *is a well-ordering of* S*.*

Infinite sets into sequences

Working in ZF, can we turn sets into sequences?

- The answer depends on our concept of *sequence*.
- We like to have "next elements" in sequences.
- A function from a (possibly transfinite) ordinal into a set S is certainly a sequence of elements from S.

Theorem

(ZF) Given $f : \alpha \to S$ define $\min(s) = \min\{\beta < \alpha \mid f(\beta) = s\}$ and define $s <_f t$ if and only if $\min(s) < \min(t)$. Then $<_f$ is a well-ordering of S.

Theorem

ZF proves the following are equivalent:

1. *Every set is the range of a sequence.*
2. *Every set can be well-ordered.*
3. *The Axiom of Choice* [3]

Sets and sequences in ZF

The Axiom of Choice is not included in ZF [3].

In set theory:

turning sequences into sets uses Axiom of Replacement (easy in ZF)

turning sets into sequences requires Axiom of Choice (hard – requires adding an axiom)

In set theory:

sequences contain more information than sets.

A different viewpoint: Reverse mathematics

Subsystems of second order arithmetic [4]

Basic axiom system: RCA_0

- Variables for natural numbers and sets of natural numbers.
- Axioms describing 0, $+$, $\cdot$, etc.
- Induction for formulas with at most one number quantifier.
- Recursive comprehension axiom:

  If a set is computable, then it exists.

A different viewpoint: Reverse mathematics

Subsystems of second order arithmetic [4]

Basic axiom system: RCA_0

- Variables for natural numbers and sets of natural numbers.
- Axioms describing 0, $+$, $\cdot$, etc.
- Induction for formulas with at most one number quantifier.
- Recursive comprehension axiom:

  If a set is computable, then it exists.

  If there is a program that can answer every question of the form "Is *n* in the set?" then the set exists.

Reverse mathematics: sets into sequences

Theorem

(RCA_0) Every nonempty set is the range of some function. That is, if S is a nonempty set, then there is a function f such that for all $s \in \mathbb{N}$

$$s \in S \leftrightarrow \exists n(f(n) = s).$$

Reverse mathematics: sets into sequences

Theorem

(RCA_0) *Every nonempty set is the range of some function. That is, if S is a nonempty set, then there is a function f such that for all $s \in \mathbb{N}$*

$$s \in S \leftrightarrow \exists n(f(n) = s).$$

Proof: Let s_0 be the least element of S. Define f by

$$f(n) = \begin{cases} s_0 & \text{if } n = 0 \\ n & \text{if } n > 0 \text{ and } n \in S \\ f(n-1) & \text{if } n > 0 \text{ and } n \notin S \end{cases}$$

Example: $S = \{2, 4, 5, \dots\}$

n	0	1	2	3	4	5	6
$f(n)$	2	2	2	2	4	5	?

Reverse mathematics: sequences into range sets

Theorem

RCA_0 *proves the following are equivalent:*

1. ACA_0 *(Sets definable by arithmetical formulas exist.)*
2. *If f is a function then there is a set R such that for all y*

$$y \in R \leftrightarrow \exists x(f(x) = y).$$

Reverse mathematics: sequences into range sets

Theorem

RCA_0 *proves the following are equivalent:*

1. ACA_0 *(Sets definable by arithmetical formulas exist.)*
2. *If f is a function then there is a set R such that for all y*

$$y \in R \leftrightarrow \exists x (f(x) = y).$$

Note: ACA_0 cannot be proved in RCA_0. Computable functions do not necessarily have computable ranges. For example, if

$$f(n,m) = \begin{cases} n & \textit{if } \{n\}(n) \downarrow_m \\ \# & \textit{otherwise} \end{cases}$$

then f is computable, but $\mathrm{range}(f) \cap \mathbb{N}$ is the set of indices of self-halting Turing machines, which is not computable.

Reverse math and set theory

In reverse mathematics:

turning sets into sequences can be done in RCA_0

turning sequences into sets requires ACA_0

sets contain more information than sequences.

In set theory:

turning sequences into sets can be done in ZF

turning sets into sequences requires Axiom of Choice

sequences contain more information than sets.

Which is correct? Set theory or second order arithmetic?

Reverse math and set theory

Question: Which is correct: ZF or RCA_0?

Answer: Both and neither.

- When we add axioms (ACA_0 or Choice), each theory can translate freely between sequences and sets.
- ZF and RCA_0 talk about different aspects of the mathematical cosmos.

Reverse math and set theory

Question: Which is correct: ZF or RCA_0?

Answer: Both and neither.

- When we add axioms (ACA_0 or Choice), each theory can translate freely between sequences and sets.
- ZF and RCA_0 talk about different aspects of the mathematical cosmos. ZF tells us about uncountable sets and RCA_0 gives us information about computability.
- Many tractable and interesting axiom systems are incomplete. They are neither oracles nor the creations of oracles.

More sets into sequences in RCA_0

For the set $S = \{2, 4, 5\}$ our old algorithm gives the sequence

n	0	1	2	3	4	5	6...
$f(n)$	2	2	2	2	4	5	5...

We might like the sequence to look like this:

n	0	1	2	3	4	5	6...
$f(n)$	2	4	5	5	5	5	5...

Theorem

(RCA_0) *If S is a nonempty set then there is an increasing sequence f with at most one repeated value such that for all $s \in \mathbb{N}$*

$$s \in S \leftrightarrow \exists n(f(n) = s).$$

At most one repeater for one set

Theorem

(RCA_0) *If S is a nonempty set then there is an increasing sequence f with at most one repeated value such that the range of f is exactly S.*

At most one repeater for one set

Theorem

(RCA_0) *If S is a nonempty set then there is an increasing sequence f with at most one repeated value such that the range of f is exactly S.*

Proof: Suppose $S \neq \emptyset$ and let $s_0 = \min S$.
Define s_1: If S is finite let $s_1 = \max S$, otherwise let $s_1 = \#$.
Define f: $f(0) = s_0$ and

$$f(n+1) = \begin{cases} s_1 & \text{if } f(n) = s_1 \\ \text{least } y \in S \text{ with } y > f(n) & \text{otherwise.} \end{cases}$$

At most one repeater for one set

Theorem

(RCA_0) *If S is a nonempty set then there is an increasing sequence f with at most one repeated value such that the range of f is exactly S.*

Proof: Suppose $S \neq \emptyset$ and let $s_0 = \min S$.
Define s_1: If S is finite let $s_1 = \max S$, otherwise let $s_1 = \#$.
Define f: $f(0) = s_0$ and

$$f(n+1) = \begin{cases} s_1 & \text{if } f(n) = s_1 \\ \text{least } y \in S \text{ with } y > f(n) & \text{otherwise.} \end{cases}$$

Example: $S = \{2, 4, 5\}$, so $s_0 = 2$ and $s_1 = 5$.

n	0	1	2	3	4	5	6…
$f(n)$	2						

At most one repeater for one set

Theorem

(RCA_0) *If S is a nonempty set then there is an increasing sequence f with at most one repeated value such that the range of f is exactly S.*

Proof: Suppose $S \neq \emptyset$ and let $s_0 = \min S$.
Define s_1: If S is finite let $s_1 = \max S$, otherwise let $s_1 = \#$.
Define f: $f(0) = s_0$ and

$$f(n+1) = \begin{cases} s_1 & \text{if } f(n) = s_1 \\ \text{least } y \in S \text{ with } y > f(n) & \text{otherwise.} \end{cases}$$

Example: $S = \{2, 4, 5\}$, so $s_0 = 2$ and $s_1 = 5$.

n	0	1	2	3	4	5	6...
$f(n)$	2	4					

At most one repeater for one set

Theorem

(RCA$_0$) *If S is a nonempty set then there is an increasing sequence f with at most one repeated value such that the range of f is exactly S.*

Proof: Suppose $S \neq \emptyset$ and let $s_0 = \min S$.
Define s_1: If S is finite let $s_1 = \max S$, otherwise let $s_1 = \#$.
Define f: $f(0) = s_0$ and

$$f(n+1) = \begin{cases} s_1 & \text{if } f(n) = s_1 \\ \text{least } y \in S \text{ with } y > f(n) & \text{otherwise.} \end{cases}$$

Example: $S = \{2, 4, 5\}$, so $s_0 = 2$ and $s_1 = 5$.

n	0	1	2	3	4	5	6…
$f(n)$	2	4	5				

At most one repeater for one set

Theorem

(RCA$_0$) *If S is a nonempty set then there is an increasing sequence f with at most one repeated value such that the range of f is exactly S.*

Proof: Suppose $S \neq \emptyset$ and let $s_0 = \min S$.
Define s_1: If S is finite let $s_1 = \max S$, otherwise let $s_1 = \#$.
Define f: $f(0) = s_0$ and

$$f(n+1) = \begin{cases} s_1 & \text{if } f(n) = s_1 \\ \text{least } y \in S \text{ with } y > f(n) & \text{otherwise.} \end{cases}$$

Example: $S = \{2, 4, 5\}$, so $s_0 = 2$ and $s_1 = 5$.

n	0	1	2	3	4	5	6...
$f(n)$	2	4	5	5			

At most one repeater for one set

Theorem

(RCA_0) *If S is a nonempty set then there is an increasing sequence f with at most one repeated value such that the range of f is exactly S.*

Proof: Suppose $S \neq \emptyset$ and let $s_0 = \min S$.
Define s_1: If S is finite let $s_1 = \max S$, otherwise let $s_1 = \#$.
Define f: $f(0) = s_0$ and

$$f(n+1) = \begin{cases} s_1 & \text{if } f(n) = s_1 \\ \text{least } y \in S \text{ with } y > f(n) & \text{otherwise.} \end{cases}$$

Example: $S = \{2, 4, 5\}$, so $s_0 = 2$ and $s_1 = 5$.

n	0	1	2	3	4	5	6...
$f(n)$	2	4	5	5	5	5	5...

At most one repeater for many sets

Theorem

RCA_0 *proves that the following are equivalent:*

1. ACA_0.
2. *If* $\langle S_i \mid i \in \mathbb{N} \rangle$ *is a sequence of nonempty sets then there is a sequence* $\langle f_i \mid i \in \mathbb{N} \rangle$ *of increasing sequences with at most one repeated value such that for each* i*, the range of* f_i *is exactly* S_i.

Sketch of $(2) \to (1)$: Suppose $h : \mathbb{N} \to \mathbb{N}$. We want to use (2) to compute the range of h. For each i, put $m + 1 \in S_i$ iff $h(m) = i$.

At most one repeater for many sets

Theorem

RCA_0 *proves that the following are equivalent:*

1. ACA_0*.*
2. *If $\langle S_i \mid i \in \mathbb{N} \rangle$ is a sequence of nonempty sets then there is a sequence $\langle f_i \mid i \in \mathbb{N} \rangle$ of increasing sequences with at most one repeated value such that for each i, the range of f_i is exactly S_i.*

Sketch of $(2) \to (1)$: Suppose $h : \mathbb{N} \to \mathbb{N}$. We want to use (2) to compute the range of h. For each i, put $m + 1 \in S_i$ iff $h(m) = i$. For example:

if h(3)=2 then $S_2 \supset \{0, 4\}$ and if $5 \notin \mathrm{Range}(h)$ then $S_5 = \{0\}$.

At most one repeater for many sets

Theorem

RCA_0 *proves that the following are equivalent:*

1. ACA_0*.*
2. *If* $\langle S_i \mid i \in \mathbb{N} \rangle$ *is a sequence of nonempty sets then there is a sequence* $\langle f_i \mid i \in \mathbb{N} \rangle$ *of increasing sequences with at most one repeated value such that for each* i*, the range of* f_i *is exactly* S_i*.*

Sketch of $(2) \to (1)$: Suppose $h : \mathbb{N} \to \mathbb{N}$. We want to use (2) to compute the range of h. For each i, put $m + 1 \in S_i$ iff $h(m) = i$. For example:

if h(3)=2 then $S_2 \supset \{0, 4\}$ and if $5 \notin \mathrm{Range}(h)$ then $S_5 = \{0\}$.

The sequence $\langle S_i \mid i \in \mathbb{N} \rangle$ is computable from h. Apply (2). $i \in \mathrm{Range}(h) \leftrightarrow f_i(1) \neq 0$, so $\langle f_i \mid i \in \mathbb{N} \rangle$ computes $\mathrm{Range}(h)$.

Recap:

Given a non-empty set S we can compute an increasing sequence f with at most one repeater such that the range of f is exactly S. (RCA_0 proves the existence of the sequence for each set.)

However, the choice of the computing algorithm depends on S, since there is no single algorithm that works for every set. (RCA_0 can't prove the existence of a sequence of sequences for a sequence of sets.)

The computation of the sequence (of this type) for the set is **not uniform**.

If we allow more repeaters, we can make the computation uniform.

References

[1] François Dorais et al., *On uniform relationships between combinatorial properties*. to appear in TAMS.

[2] Jean van Heijenoort, *From Frege to Gödel. A source book in mathematical logic, 1879–1931*, Harvard University Press, Cambridge, Mass., 1967.

[3] Thomas J. Jech, *The axiom of choice*, North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1973. Studies in Logic and the Foundations of Mathematics, Vol. 75.

[4] Stephen G. Simpson, *Subsystems of second order arithmetic*, 2nd ed., Perspectives in Logic, Cambridge University Press, Cambridge, 2009.

[5] Robert I. Soare, *Recursively enumerable sets and degrees*, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1987.