# A Brief Introduction to Reverse Mathematics

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# Reverse Mathematics

**Goal:** Determine what set existence axioms are needed to prove familiar theorems.

**Method:** Prove results of the form

$$\mathbf{RCA_0} \vdash \mathbf{AX} \leftrightarrow \mathbf{THM}$$

where:

- $\mathbf{RCA_0}$ is a weak axiom system,
- **AX** is a set existence axiom selected from a small hierarchy of axioms, and
- **THM** is a familiar theorem.

# $\mathbf{RCA_0}$

Recursive Comprehension

**Language:**

$x, y, z$ variables representing integers

$X, Y, Z$ variables representing sets of integers

0, 1, +, $\times$, =, <, and $\in$

**Axioms:**

basic arithmetic axioms

(0, 1, +, $\times$, =, and < behave as usual.)

Restricted induction

$(\psi(0) \wedge \forall n(\psi(n) \rightarrow \psi(n+1))) \rightarrow \forall n \psi(n)$

where $\psi(n)$ has (at most) one $x$ quantifier.

Recursive set comprehension

If $\theta \in \Sigma^0_1$ and $\psi \in \Pi^0_1$, and

$\forall n(\theta(n) \leftrightarrow \psi(n))$, then there is

a set $X$ such that $\forall n(n \in X \leftrightarrow \theta(n))$

## What can $\mathbf{RCA_0}$ prove?

Arithmetic needed for coding.

Lots of finite graph theory, e.g.

**Thm** ($\mathbf{RCA_0}$) Every finite graph with no odd cycles is bipartite.

A little analysis, e.g.

**Thm** ($\mathbf{RCA_0}$) If $\langle I_n \rangle_{n \in \mathbb{N}}$ is a sequence of nested real intervals, then there is a real number in their intersection.

# Weak König's Lemma

**Statement:** Big 0-1 trees are tall.

More formally: If $T$ is an infinite tree in which each node is labeled 0 or 1, then $T$ contains an infinite path.

$\mathbf{WKL_0}$ is $\mathbf{RCA_0}$ plus Weak König's Lemma.

Note: $\mathbf{RCA_0} \nvdash \mathbf{WKL_0}$

Some reverse mathematics!

**Thm** ($\mathbf{RCA_0}$) The following are equivalent:

1) $\mathbf{WKL_0}$.

2) Every graph with no cycles of odd length is bipartite.

Proof: To prove that 1) $\to$ 2), we should 2-color the nodes of an arbitrary graph with no odd cycles by using a tree.

The reversal

Proof that "bipartite thm" implies $\mathbf{WKL_0}$

A reversal tool:

**Thm** ($\mathbf{RCA_0}$) T.F.A.E.:

1) $\mathbf{WKL_0}$

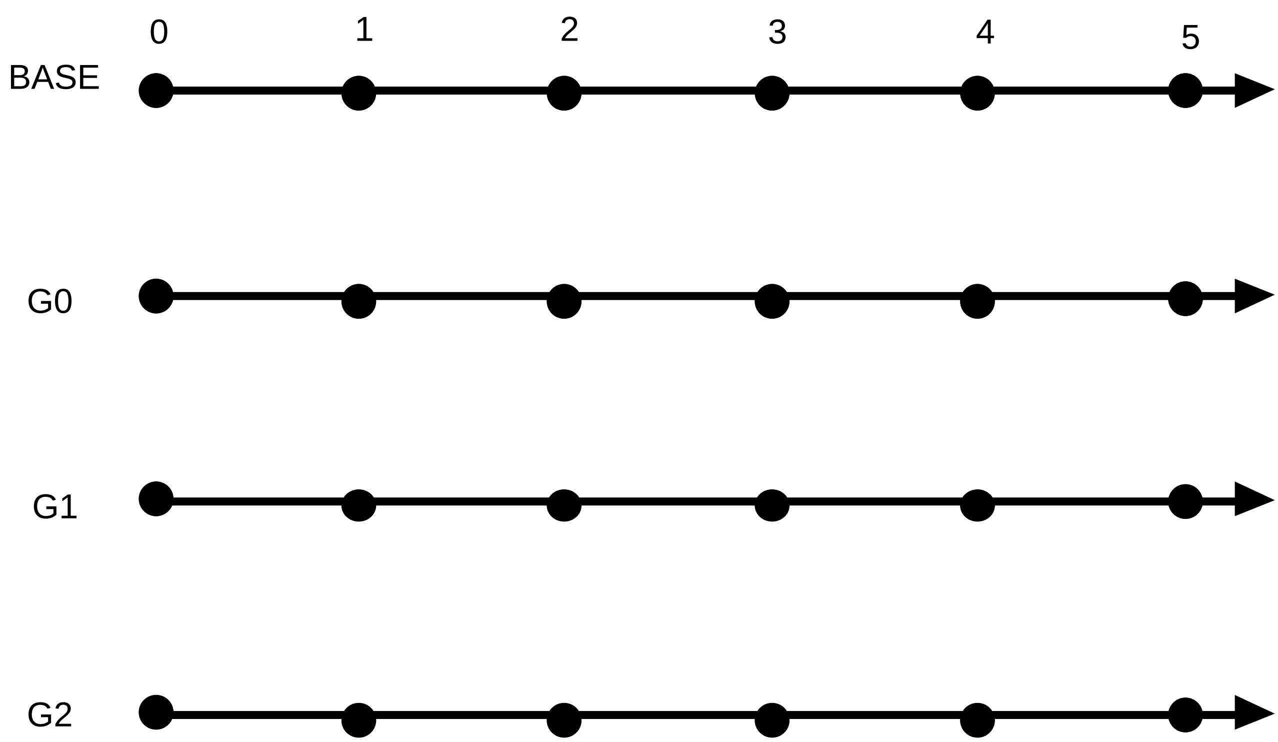
2) If $f$ and $g$ are 1-1 functions from $\mathbb{N}$ into $\mathbb{N}$ and $Ran(f) \cap Ran(g) = \emptyset$, then there is a set $X$ such that $Ran(f) \subset X$ and $X \cap Ran(g) = \emptyset$.

Sketch of the reversal: Use a 2-coloring of a graph with no odd cycles to separate the ranges of some arbitrary functions.

Sample construction: Suppose we are given $f$ and $g$ such that $\mathbb{N}$ and $Ran(f) \cap Ran(g) = \emptyset$.

If, for example, $f(3) = 0$ and $g(4) = 2$, we will construct the graph $G$ as follows:

Associate straight links with $f$
Associate shifted links with $g$



## Related Computability Results

(Bean) There is a computable graph with no cycles of odd length that has no computable 2-coloring.

(Bean) Every computable graph with no cycles of odd length has a low 2-coloring.

# Other theorems equivalent to $\mathbf{WKL_0}$

**Thm** ($\mathbf{RCA_0}$) T.F.A.E.:

1) Every continuous function on [0,1] is bounded. (Simpson)

2) The closed interval [0,1] is compact. (Friedman)

3) Existence theorem for solutions to ODEs. (Simpson)

4) The line graph of a bipartite graph is bipartite. (Hirst)

5) Every countable partial order with no chains of length $k+1$ can be decomposed into $k$ antichains. (Hirst)

## Schmerl's new result

Theorem: If $2 \leq k \leq m < \omega$, then the following are equivalent over $\mathbf{RCA_0}$:

1) $\mathbf{WKL_0}$.

2) Every locally $k$-colorable graph is $m$-colorable.

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## Questions...

Does "every countable partial order with no chains of length 3 can be decomposed into 6 antichains" imply $\mathbf{WKL_0}$?

Over $\mathbf{RCA_0}$, does Ramsey's Theorem for pairs imply $\mathbf{WKL_0}$?

# Arithmetical Comprehension

$\mathbf{ACA_0}$ consists of $\mathbf{RCA_0}$ plus the following arithmetical comprehension scheme:

For any formula $\theta(n)$ with only number quantifiers, the set $\{n \in \mathbb{N} \mid \theta(n)\}$ exists.

Note: $\mathbf{WKL_0} \nvdash \mathbf{ACA_0}$, but $\mathbf{ACA_0} \vdash \mathbf{WKL_0}$

A reversal tool:

**Thm** ($\mathbf{RCA_0}$) T.F.A.E.:

1) $\mathbf{ACA_0}$

2) If $f : \mathbb{N} \to \mathbb{N}$ is 1-1, then $Ran(f)$ exists.

# $\mathbf{ACA_0}$ and Graph Theory

**Thm** ($\mathbf{RCA_0}$) T.F.A.E.:

1) $\mathbf{ACA_0}$

2) Every graph can be decomposed into its connected components.

Proof: To prove that 1) implies 2), let $G$ be a graph with vertices $v_0, v_1, ...$

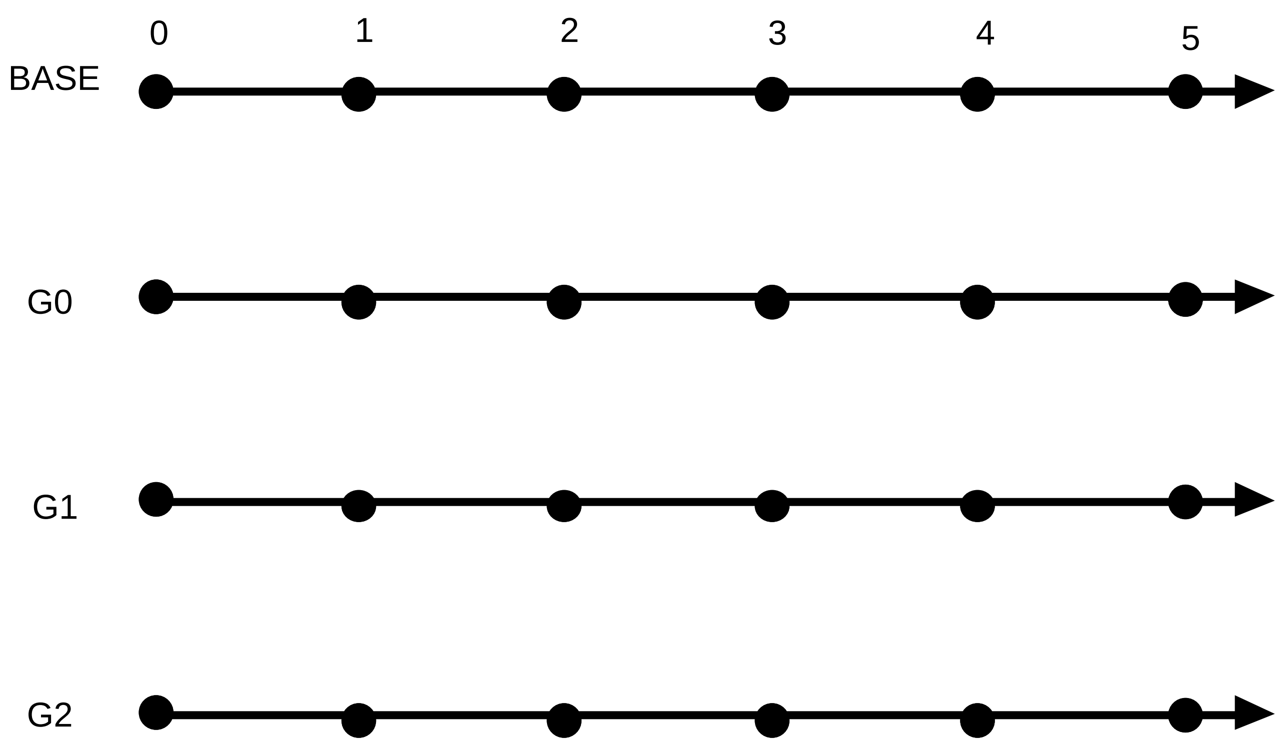
Define $f$ by letting $f(n)$ be the minimum of $n$ and the least $j$ such that there is a path from $v_n$ to $v_j$.

By $\mathbf{ACA_0}$, $f$ exists.
$f$ is the desired decomposition.

The reversal

Sketch: We will use a graph decomposition to define the range of an arbitrary function. Suppose we want to find the range of the function $f$.

Sample construction: Suppose that $f(4) = 2$ and $f(3) = 0$.



## Related computability results

Every arithmetically definable graph has an arithmetically definable decomposition into connected components.

There is a computable graph such that $0'$ is computable from any decomposition of the graph into connected components.

Other theorems equivalent to $\mathbf{ACA_0}$

**Thm** ($\mathbf{RCA_0}$) T.F.A.E.:

1) Ramsey's Theorem for triples. (Friedman)

2) Cauchy sequences converge. (Simpson)

3) Every connected graph with at most one vertex of odd degree which has at least one vertex of odd or infinite degree, and which cannot be disconnected by the removal of any finite subgraph has an Euler path. (Gasarch and Hirst analysis of Erdős, Grünwald, and Vàzsonyi)

4) Arithmetical transfinite induction. (Hirst)

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Questions...

Does the thin set theorem (or free set theorem) for triples imply $\mathbf{ACA_0}$?

Is Hindman's theorem provable in $\mathbf{ACA_0}$?

# Arithmetical Transfinite Recursion

$\mathbf{ATR_0}$ consists of $\mathbf{RCA_0}$ plus axioms that allow iteration of arithmetical comprehension along any well ordering. This allows transfinite constructions.

A reversal tool:

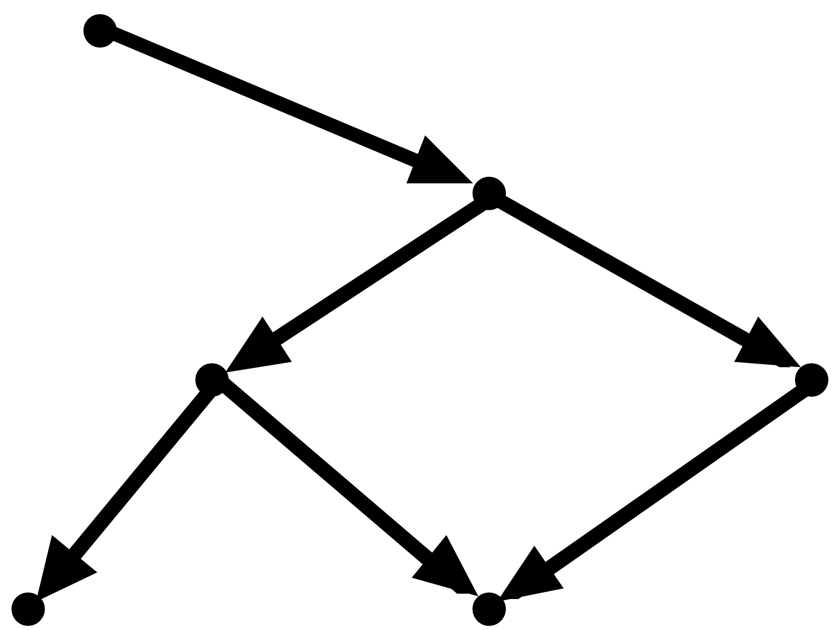
**Thm** ($\mathbf{RCA_0}$) T.F.A.E.:

1) $\mathbf{ATR_0}$

2) If $\alpha$ and $\beta$ are well orderings, then $\alpha \leq \beta$ or $\beta \leq \alpha$.

## $\mathbf{ATR_0}$ and Graph Theory

A rank function for a directed acyclic graph is a function that maps the vertices onto a well ordering, preserving the ordering induced by the edges in a nice way.



**Thm** ($\mathbf{RCA_0}$) T.F.A.E.:

1) $\mathbf{ATR_0}$

2) Every well-founded directed acyclic graph has a rank function.

# Related computability result

There is a hyperarithmetical directed acyclic graph with no hyperarithmetical infinite descending sequences that has no hyperarithmetical rank function.

## Other theorems equivalent to $\mathbf{ATR_0}$

**Thm** ($\mathbf{RCA_0}$) T.F.A.E.:

1) $\mathbf{ATR_0}$

2) Lusin's Separation Theorem: Any two disjoint analytic sets can be separated by a Borel set. (Simpson)

3) Mahlo's Theorem: Given any two countable closed compact subsets of the reals, one can be homeomorphically embedded in the other. (Friedman and Hirst)

4) Sherman's Inequality: If $\alpha$, $\beta$, and $\gamma$ are countable well orderings, then

$$(\alpha + \beta)\gamma \leq \alpha\gamma + \beta\gamma$$

(Hirst)

# $\Pi^1_1$ comprehension

The system $\mathbf{\Pi^1_1-CA_0}$ consists of $\mathbf{RCA_0}$ and the axioms asserting the existence of the set $\{n \in \mathbb{N} \mid \theta(n)\}$ for $\theta \in \Pi^1_1$. (That is, $\theta$ has one universal set quantifier and no other set quantifiers.)

A reversal tool followed by graph theory:

**Thm** ($\mathbf{RCA_0}$) T.F.A.E.:

1) $\mathbf{\Pi^1_1-CA_0}$

2) If $\langle T_i \rangle_{n \in \mathbb{N}}$ is a sequence of trees then there is a function $f : \mathbb{N} \to \{0, 1\}$ such that $f(n) = 1$ iff $T_n$ is well founded.

3) For any graph $H$, and any sequence of graphs $\langle G_i \rangle_{i \in \mathbb{N}}$, there is a function $f : \mathbb{N} \to \{0, 1\}$ such that $f(n) = 1$ iff $H$ is isomorphic to a subgraph of $G$. (Hirst and Lempp)

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