Graphs and Models

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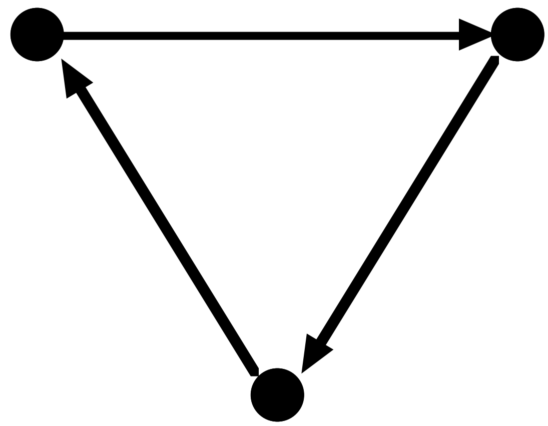
Material in blue was added at the talk.

Given a directed graph, we can determine the truth of certain statements.

Example:



Every vertex starts some arrow.

True in example $\forall x \exists y A(x, y)$

There is one vertex that is pointed to by every vertex.

False in example $\exists y \forall x A(x, y)$

We could formalize these statements.

$\forall x$ – **for every vertex** x

$\exists y$ – **there is a vertex** y

$A(x, y)$ – **there's an arrow from** x **to** y

Some formulas are true in every graph. (Logically valid formulas)

$$\forall x (A(x,x) \vee \neg A(x,x))$$

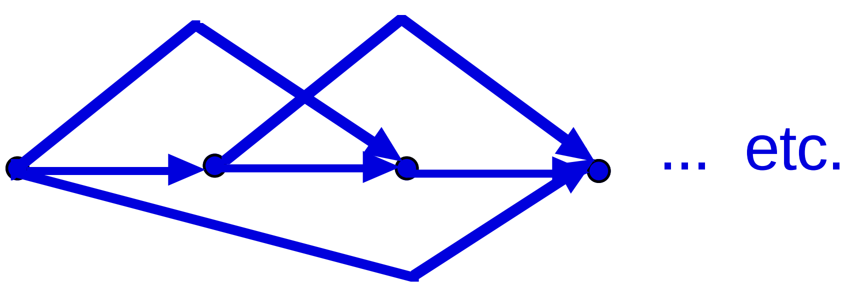
Some formulas aren't true in any graph. (Contradictions)

$$\exists x (A(x,x) \wedge \neg A(x,x))$$

Many formulas are true in some graphs but not true in others.

$$\forall x \neg A(x,x) \wedge \forall x \exists y (A(x,y) \wedge \forall z (A(z,x) \rightarrow A(z,y)))$$

No vertex points to itself.
Every vertex (and all its predecessors) point to some vertex.

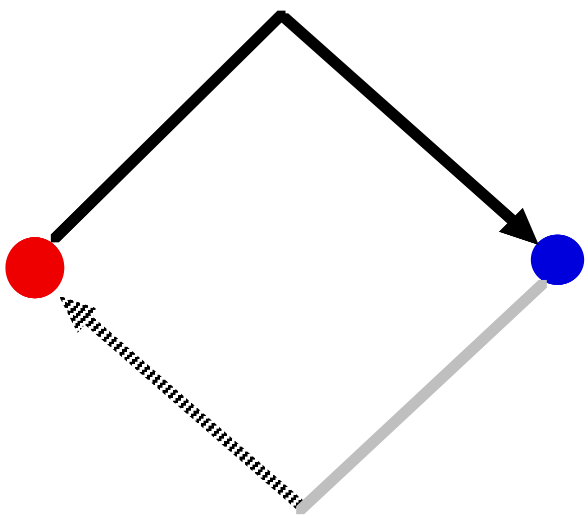


This formula is not true in any finite graph. Every model of the formula is infinite.

Some formulas require colorful graphs.

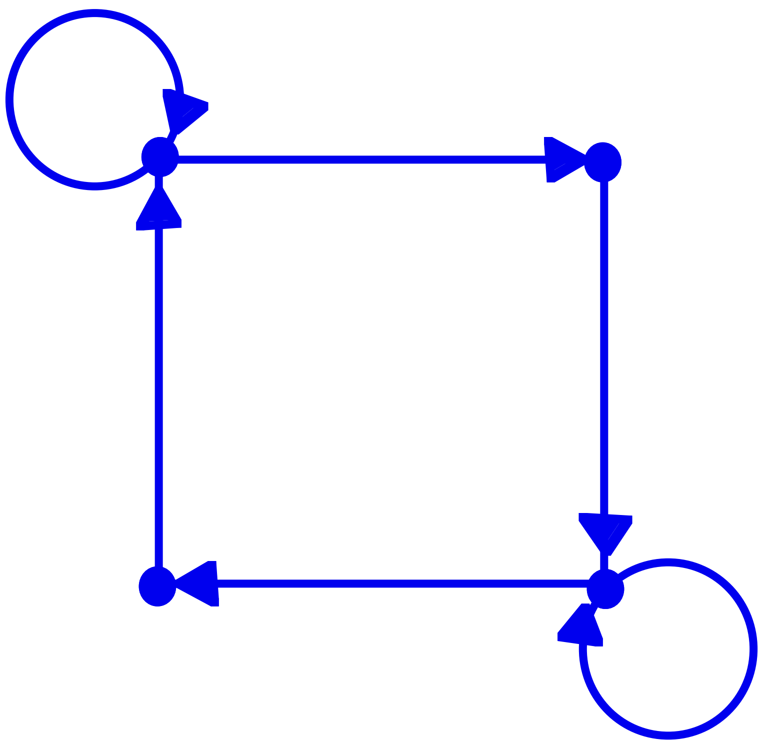
$$\forall x(R(x) \rightarrow \exists y(B(y) \wedge S(x, y) \wedge D(y, x)))$$

If x is a red vertex, then there is a blue vertex y with a solid arrow from x to y and a dotted arrow from y to x.



Find a graph that is a model of:

$$\forall x \exists y((A(x,y) \wedge \neg A(y,x)) \wedge \neg(A(x,x) \leftrightarrow A(y,y)))$$



Is this formula true in a graph with one vertex? How about 2 or 3?

Not true in graphs of size 1, 2, or 3

Is this formula true in graphs of size 5? How about bigger numbers?

True in (some) graphs of size n for each $n \geq 4$.

Suppose S is a formula. How hard are these questions?

1. **Does S have a model of size n?**

   ANS: Computable, at least NP complete.

2. **Does S have a finite model?**

   ANS: Not computable! Trahtenbrot's Theorem

3. **Does S have a model?**

   ANS: Not computable! Undecidability of dyadic predicate calculus.

More questions:

4. **Can we find formulas S_1, S_2, ... so that the only graphs satisfying S_n have size n?**

5. **Can we find a formula S such that for each n there is exactly one graph (up to isomorphism) where S is true?**

6. **Can we find a formula S such that for each $n \geq 2$ there are exactly three graphs (up to isomorphism) where S is true?**

7. **Can we find a formula S which is true in infinitely many finite graphs but not in any infinite graph?**

5 is yes, 7 is no, 4 and 6 are not immediately obvious.