Reverse Analysis

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George Washington University, 11/19/2004

These slides appear at: `www.mathsci.appstate.edu/~jlh`

For much more reverse mathematics, come to the AMS-ASL Special Session on Reverse Mathematics at the Joint Math Meetings in Atlanta on 1/5 and 1/6.
`http://www.ams.org/amsmtgs/2091_program_ss6.html`

Reverse Mathematics

Goal: Determine what set existence axioms are needed to prove familiar theorems.

Method: Prove results of the form

$$\mathsf{RCA}_0 \vdash \mathsf{AX} \leftrightarrow \mathsf{THM}$$

where:

- RCA_0 is a weak axiom system,
- AX is a set existence axiom selected from a small hierarchy of axioms, and
- THM is a familiar theorem.

RCA_0: Recursive Comprehension

Language:

Integer variables (x, y, z) and set variables (X, Y, Z)

Axioms:

basic arithmetic axioms

(0, 1, +, ×, =, and < behave as usual.)

Restricted induction

$(\psi(0) \wedge \forall n(\psi(n) \rightarrow \psi(n+1))) \rightarrow \forall n \psi(n)$

where $\psi(n)$ has (at most) one number quantifier.

Recursive set comprehension

If $\theta \in \Sigma^0_1$ and $\psi \in \Pi^0_1$, and $\forall n(\theta(n) \leftrightarrow \psi(n))$, then there is a set X such that $\forall n(n \in X \leftrightarrow \theta(n))$

RCA_0 suffices to prove the existence of pairing functions.

Encoding the reals

A *real number* is a function $x : \mathbb{N} \to \mathbb{Q}$ such that

$$\forall k \forall i \; |x(k) - x(k+i)| \leq 2^{-k}$$

$\langle x(i) \rangle_{i \in \mathbb{N}}$ is a rapidly converging Cauchy seq. of rationals.

Examples of reals

$\sqrt{2}$: $1, 1.4, 1.41, 1.414, 1.4142, \ldots$

π : $3, 3.1, 3.14, 3.141, 3.1415, \ldots$

0 : $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

0 : $0, 0, 0, 0, 0, \ldots$

Definition. ($\mathsf{RCA_0}$) If x is a real number, then a decimal expansion for x is a sequence y such that

$$y(0) \in \mathbb{Z},$$

$$y(i) \in \{0, \ldots, 9\} \text{ for all } i > 0, \text{ and}$$

$$\text{for all } k,\ |x(k) - y(0).y(1)y(2)\ldots y(k)| \leq 2^{-k+1}.$$

Theorem 1. ($\mathsf{RCA_0}$) *If x is a real number, then it has a decimal expansion.*

Proof. If x is rational, do the long division. If x is irrational, list enough of x to find $y(0) \in \mathbb{Z}$ so that $y(0) < x < y(0)+1$. Divide $[y(0), y(0)+1]$ into ten subintervals and list enough of x to find a $y(1)$ such that $y(0).y(1) < x < y(0).y(1) + .1$ Iterate. □

Weak König's Lemma

Statement: Big 0-1 trees are tall.

More formally: If T is an infinite tree in which each node is labeled 0 or 1, then T contains an infinite path.

WKL_0 is RCA_0 plus Weak König's Lemma.

Some reverse mathematics!

Theorem 2. (RCA_0) *The following are equivalent:*

1. WKL_0.
2. WKL_0 *with* $\{0,1\}$ *replaced by* $\{0,1,2,\ldots,9\}$.
3. *If* f *and* g *are injective functions with disjoint ranges, then there is a set* X *such that for all* j, $f(j) \in X$ *and* $g(j) \notin X$.

A consequence: $\mathsf{RCA}_0 \not\vdash \mathsf{WKL}_0$.

Some reverse analysis!

Theorem 3. (RCA_0) *The following are equivalent:*

1. WKL_0.
2. *If* $\langle x_n \rangle_{n \in \mathbb{N}}$ *is a sequence of real numbers, then there is a sequence* $\langle y_n \rangle_{n \in \mathbb{N}}$ *such that for each* n, y_n *is a decimal expansion for* x_n.
3. *If* $\langle x_n \rangle_{n \in \mathbb{N}}$ *is a sequence of rationals in* $[0,1]$, *then there is a sequence* $\langle d_n \rangle_{n \in \mathbb{N}}$ *such that for each* n, d_n *is the first digit of the decimal expansion of* x_n.

Proof. For (1)→(2), build a tree. (2)→(3) is trivial. For (3)→(1), use (3) to separate ranges of disjoint functions. □

Irrationals are different

If we know that a real x is irrational, we can always calculate x to a sufficient degree of accuracy to show that it is strictly greater or strictly less than any rational. Consequently...

Theorem 4. (RCA_0) *If* $\langle x_n \rangle_{n \in \mathbb{N}}$ *is a sequence of* **irrational** *reals, then there is a sequence* $\langle y_n \rangle_{n \in \mathbb{N}}$ *such that for each* n*,* y_n *is a decimal expansion for* x_n*.*

Computability theoretic consquences

Theorem 5. *Every computable real has a computable decimal expansion.*

Theorem 6. *Every computable sequence of irrationals has a computable sequence of decimal expansions.*

Theorem 7. *There is a computable sequence of computable reals (each of which is equal to a rational number) for which there is no computable sequence of decimal expansions.*

Theorem 8. *Every computable sequence of computable reals has a sequence of decimal expansions of low degree, that is of degree* $\mathbf{a}$ *where* $\mathbf{a}' = \mathbf{0}'$.

Constructive analysis vs. Computable analysis

Computable analyst:

> We can find the decimal expansion of any single real.
>
> We can find the decimal expansions for all the elements of a sequence of irrationals.
>
> We can't always find the decimal expansions for all the elements of sequences of rationals.

Constructive analyst:

> We can find the decimal expansions for all the elements of a sequence of irrationals.
>
> We can't always find the decimal expansion for a real.

Other results equivalent to WKL_0

Theorem 9. (RCA_0) *The following are equivalent:*

1. WKL_0.
2. *Every ctn function on* $[0,1]$ *is bounded. (Simpson)*
3. *If* f *is ctn on* $[0,1]$*, then* $\int_0^1 f\ dx$ *exists and is finite. (Simpson)*
4. $[0,1]$ *is Heine-Borel compact. (Friedman)*
5. *If* $\langle x_k \rangle_{k \in \mathbb{N}}$ *is a sequence of reals, then there are integers* $\langle \mu_k \rangle_{k \in \mathbb{N}}$ *such that* $x_{\mu_k} = \min\{x_j \mid j \leq k\}$ *for all* k*. (H)*
6. *Graphs with no cycles of odd length are bipartite. (H)*

Stronger subsystems and associated results

Theorem 10. RCA_0 *proves the following equivalences:*

1. ACA_0 *iff Bolzano/Weierstraß Theorem: Every bounded sequence of reals has a convergent subsequence. (Friedman)*

2. ATR_0 *iff Every countable closed subset of a complete separable metric space has a derived sequence. (Hirst)*

3. Π^1_1-CA_0 *iff* ACA_0+ *Cantor/Bendixson Theorem for Cantor space: Every closed subset of Cantor space is the union of a perfect closed set and a countable set. (Simpson)*

Reverse Mathematics

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