A *Real* Tour of Reverse Mathematics

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These slides appear at: `www.mathsci.appstate.edu/~jlh`

For much more reverse mathematics, come to the AMS-ASL Special Session on Reverse Mathematics at the Joint Math Meetings in Atlanta on 1/5 and 1/6.
`http://www.ams.org/amsmtgs/2091_program_ss6.html`

Reverse Mathematics

Goal: Determine what set existence axioms are needed to prove familiar theorems.

Method: Prove results of the form $\mathsf{RCA}_0 \vdash \mathsf{AX} \leftrightarrow \mathsf{THM}$, where:

- RCA_0 is a weak axiom system,
- AX is a set existence axiom selected from a small hierarchy of axioms, and
- THM is a familiar theorem.

Hierarchy: $\mathsf{RCA}_0 < \mathsf{WKL}_0 < \mathsf{ACA}_0 < \mathsf{ATR}_0 < \Pi^1_1\text{-}\mathsf{CA}_0$

RCA_0: Recursive Comprehension

Language:

Integer variables (x, y, z) and set variables (X, Y, Z)

Axioms:

basic arithmetic axioms

(0, 1, +, ×, =, and < behave as usual.)

Restricted induction

$(\psi(0) \wedge \forall n(\psi(n) \rightarrow \psi(n+1))) \rightarrow \forall n \psi(n)$

where $\psi(n)$ has (at most) one number quantifier.

Recursive set comprehension

If $\theta \in \Sigma^0_1$ and $\psi \in \Pi^0_1$, and $\forall n(\theta(n) \leftrightarrow \psi(n))$, then there is a set X such that $\forall n(n \in X \leftrightarrow \theta(n))$

RCA_0 suffices to prove the existence of pairing functions.

Encoding the reals

A *real number* is a function $x : \mathbb{N} \to \mathbb{Q}$ such that

$$\forall k \forall i \; |x(k) - x(k+i)| \leq 2^{-k}$$

$\langle x(i) \rangle_{i \in \mathbb{N}}$ is a rapidly converging Cauchy seq. of rationals.

Examples of reals

$\sqrt{2}$: $1, 1.4, 1.41, 1.414, 1.4142, \ldots$

π : $3, 3.1, 3.14, 3.141, 3.1415, \ldots$

0 : $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

0 : $0, 0, 0, 0, 0, \ldots$

Weak König's Lemma

Statement: WKL_0 consists of RCA_0 together with "every infinite 0-1 tree contains an infinite path."

Theorem 1. (RCA_0) *The following are equivalent:*

1. WKL_0.
2. *If f and g are injective functions with disjoint ranges, then there is a set X such that for all j, $f(j) \in X$ and $g(j) \notin X$.*
3. $[0,1]$ *is Heine-Borel compact. (Friedman)*

Compactness vs. Compactness

Definition. A complete separable metric space X is *compact* if there exists an infinite sequence of finite sequences of points in X, $\langle\langle x_{ij} \mid i \leq n_j\rangle \mid j \in \mathbb{N}\rangle$ such that for every $z \in X$ and $j \in \mathbb{N}$ there exists an $i \leq n_j$ such that $d(x_{ij}, z) < 2^{-j}$.

Theorem 2. (RCA_0) $[0,1]$ *is compact.*

Definition. A set is *Heine-Borel compact* if every open cover contains a finite subcover.

Theorem 3. (RCA_0) *The following are equivalent:*

1. WKL_0.
2. $[0,1]$ *is Heine-Borel compact. (Friedman)*

Minima vs. Minima

Theorem 4. (RCA_0) *If $\langle x_i \rangle_{i \in \mathbb{N}}$ is a sequence of real numbers, then there is a sequence of real numbers $\langle y_i \rangle_{i \in \mathbb{N}}$ such that $y_i = \min\{x_j \mid j \leq i\}$ for each $i \in \mathbb{N}$.*

Theorem 5. (RCA_0) *The following are equivalent:*

1. WKL_0.

2. *If $\langle x_i \rangle_{i \in \mathbb{N}}$ is a sequence of real numbers, then there is a sequence of indices $\langle \mu_i \rangle_{i \in \mathbb{N}}$ such that for each $i \in \mathbb{N}$,*

$$x_{\mu_i} = \min\{x_j \mid j \leq i\}.$$

Arithmetical Comprehension

Statement: ACA_0 consists of RCA_0 together with "if $\theta(x)$ is a formula with no set quantifiers, i.e. an arithmetical formula, then the set $X = \{x \in \mathbb{N} \mid \theta(x)\}$ exists."

Theorem 6. (RCA_0) *The following are equivalent:*

1. ACA_0.
2. *If f injects $\mathbb{N}$ into $\mathbb{N}$, then the range of f exists.*
3. *(Bolzano-Weierstraß) Every bounded sequence of numbers contains a convergent subsequence. (Friedman)*
4. *Every increasing sequence of rationals in $(0,1)$ converges. (Friedman)*

Closed vs. Closed

Definition. An open set in $\mathbb{R}$ is a countable sequence of balls with real centers and rational radii. A closed set is the complement of an open set.

Definition. A separably closed set is the collection of limit points of a countable sequence of points.

Theorem 7. (RCA_0) *The following are equivalent:*

1. ACA_0.
2. *If X is a separably closed set then X is closed. (Brown)*
3. *If X is a closed subset of a compact set, then X is separably closed. (Brown and Hirst)*

Induction and Set Comprehension

Theorem 8. ($\mathsf{RCA_0}$) *The following are equivalent:*

1. $\mathsf{ACA_0}$.

2. *The arithmetical transfinite induction scheme:*
If X is a well-ordered set with minimum element 0 and $\theta(x)$ is an arithmetical formula, then if

$$\theta(0) \ \textit{and} \ \forall y \in X(\forall x < y \ \theta(x) \rightarrow \theta(y)),$$

then $\forall y \in X \ \theta(y)$.

Corollary 9. *All undergraduate analysis theorems can be proved by transfinite induction.*

Arithmetical Transfinite Recursion

Statement: ATR_0 consists of ACA_0 together with a scheme for iterating arithmetical comprehension along countable well orderings.

Definition. A derived sequence for a set of reals is constructed by repeatedly ejecting the isolated points, and taking intersections at limit stages.

Theorem 10. (RCA_0) *The following are equivalent:*

1. ATR_0.
2. *If X and Y are well-ordered then $X \leq Y$ or $Y \leq X$. (Friedman et. al.)*
3. *Every countable closed subset of $[0, 1]$ has a derived sequence.*

Π^1_1 comprehension

Statement: Π^1_1-CA_0 consists of RCA_0 together with "if $\theta(x)$ is a formula with exactly one leading universal set quantifier, i.e. a Π^1_1 formula, then the set $X = \{x \in \mathbb{N} \mid \theta(x)\}$ exists."

Theorem 11. (RCA_0) *The following are equivalent:*

1. Π^1_1-CA_0.

2. ACA_0+*Cantor/Bendixson Theorem for Cantor space: Every closed subset of Cantor space is the union of a perfect closed set and a countable set. (Simpson)*

3. *Every closed set is separably closed. (Brown)*

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