Reverse Mathematics and Ramsey's Theorem

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These slides are available at:
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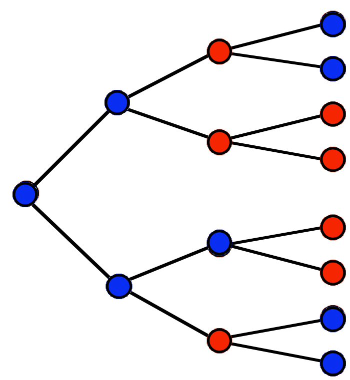
A pigeonhole principle and Hindman's Theorem

TT^1: For any finite coloring of $2^{<\mathbb{N}}$, there is a monochromatic subtree order-isomorphic to $2^{<\mathbb{N}}$.



A version of Hindman's theorem:

Finite Union Theorem (FUT): If $f : \mathsf{FIN} \to \mathsf{k}$ then there is a $c \leq k$ and an infinite increasing sequence $\langle H_i \rangle_{i \in \mathbb{N}}$ of elements of FIN such that for every $F \in \mathsf{FIN}$

$$f(\cup_{i \in F} H_i) = c.$$

Question (McNicholl): Do we need FUT to prove TT^1?

Answer: No.

(CHM[3]): $\mathsf{RCA}_0 + \Sigma^0_2 - \mathsf{IND} \vdash \mathsf{TT}^1$

(BHS[2]): $\mathsf{RCA}_0 \vdash \mathsf{FUT} \rightarrow \mathsf{ACA}_0$

ω together with the computable sets forms a model of RCA_0 plus $\Sigma^0_2 - \mathsf{IND}$ which is not a model of ACA_0.

Brief overview of reverse mathematics

Reverse mathematics uses a hierarchy of axiom systems for second order arithmetic to analyze the relative strength of mathematical theorems.

RCA_0 : basic arithmetic axioms, induction for Σ^0_1 formulas, comprehension for computable sets

ACA_0 : RCA_0 plus comprehension for sets defined by arithmetical formulas

Friedman presented the axiom systems used here (with restricted induction) in a talk at the meeting of the ASL in Chicago in April 1975. (See [7] for these abstracts and [6] for the related paper from the ICM in Vancouver in 1974.)

How strong is TT^1?

RT^1: Usual infinite pigeonhole principle. If $f : \mathbb{N} \longrightarrow k$ then for some c and some infinite X, $f(x) = c$ for all $x \in X$.

Theorem (CGM[4]): $\mathsf{RCA}_0 + \mathsf{RT}^1 \nvdash \mathsf{TT}^1$

Question: Does TT^1 imply $\Sigma^0_2 - \mathsf{IND}$?

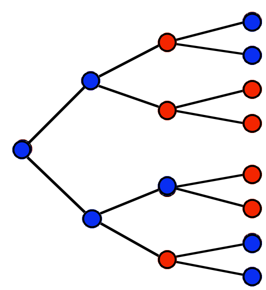
Partial answer: The known proofs of TT^1 use $\Sigma^0_2 - \mathsf{IND}$, but the use may not be necessary.

ECT: Eventually constant tails

ECT($\mathbb{N}$): If $f : \mathbb{N} \to k$, then for some b, the range of f on $[x, \infty)$ is the same for every $x \geq b$.

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$f(x)$	⚫	🟢	🟡	⚫	🟢	⚫	⚫	🔴	🟢	🔵	🔴	🟢	🔵	🔴	🔴	🟢	🟢	🔵	🔴	🔵	🟢

ECT($2^{<\mathbb{N}}$): If $f : 2^{<\mathbb{N}} \to k$, then for some node τ, the range of f on the tree of nodes extending σ is the same for every $\sigma \supset \tau$.



ECT, induction, and TT^1

Note: $\mathsf{RCA}_0 \vdash \mathsf{ECT}(2^{<\mathbb{N}}) \to \mathsf{TT}^1$

Theorem: RCA_0 proves the following are equivalent:

(1) $\Sigma^0_2 - \mathsf{IND}$

(2) $\mathsf{ECT}(2^{<\mathbb{N}})$

(3) $\mathsf{ECT}(\mathbb{N})$

Hints: Prove $1 \to 2 \to 3 \to 1$. For $3 \to 1$ use Simpson's exercise II.3.13 [9]:

$$\Sigma^0_2 - \mathsf{IND} \leftrightarrow \text{bounded } \Sigma^0_2 \text{ comprehension.}$$

More Ramsey's theorem

RT^n: If $f : [\mathbb{N}]^n \to k$ then there is a c and an infinite $H \subset \mathbb{N}$ such that $f([H]^n) = c$.

TT^n: For any k coloring of the n-tuples of comparable nodes in $2^{<\mathbb{N}}$, there is a color and a subtree order-isomorphic to $2^{<\mathbb{N}}$ in which all n-tuples of comparable nodes have the specified color.

IPT^n: If $f : [\mathbb{N}]^n \to k$ then there is a c and a sequence of infinite sets $H_1 \ldots H_n$ so that for any $x_1 < \cdots < x_n$ (with $x_i \in H_i$ for all i) we have $f(x_1 \ldots x_n) = c$.

Theorem: For $n \geq 3$, RCA_0 proves these equivalences:

$$\mathsf{ACA}_0 \leftrightarrow \mathsf{RT}^n \leftrightarrow \mathsf{TT}^n \leftrightarrow \mathsf{IPT}^n$$

References: RT: Simpson[9] TT: CHM[3] IPT: DH[5]

If we let RT denote $\forall n \mathsf{RT}^n$, we can prove:

Theorem: For $n \geq 3$, RCA_0 proves these equivalences:

$$\mathsf{ACA}_0' \leftrightarrow \mathsf{RT} \leftrightarrow \mathsf{TT} \leftrightarrow \mathsf{IPT}$$

References: RT: Mileti[8] TT: AH[1] IPT: DH[5]

RCA_0 proves that $\mathsf{RT}^2 \rightarrow \mathsf{IPT}^2 \rightarrow \mathsf{SRT}^2$. How strong are the converses?

References

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