Questions about Hindman's Theorem

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IHT: Let $\langle f_i \rangle_{i \in \mathbb{N}}$ be a sequence of finite colorings of $\mathbb{N}$. There is an infinite set $H = \{h_0 < h_1 < \dots\}$ such that for each i, $\{h_j \mid j \geqslant i\}$ is monochromatic for f_i in the sense of Hindman's theorem. That is, for each i, f_i is constant on the collection of finite sums of distinct elements in $\{h_j \mid j \geqslant i\}$.

Blass, Hirst, and Simpson [1] proved IHT in ACA_0^+, which appends the existence of the ω-jump to ACA_0. In the same paper, it was shown that Hindman's Theorem implies ACA_0 over RCA_0.

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Montalbán and Shore's [3] conservation results over IHT heighten the significance of finding the exact strength of this theorem.

For example, if IHT is provable in ACA_0, then the first order consequences of their statements that are conservative over IHT are also conservative over Peano Arithmetic.

IHT and ultrafilters

IHT is related to the existence of some special ultrafilters on countable fields of sets. [2]

A countable field of sets is a countable collection of sets that is closed under intersection, union, and relative complementation. In ACA_0 we may assume that the sets are represented by a non-repeating sequence of characteristic functions, and the set operations are represented by functions from indices of sets to indices of sets.

Given a set $X \subset \mathbb{N}$ and an number n, let $X - n = \{x - n \mid x \in X \wedge x \geqslant n\}$. A translation algebra is a countable field of sets which is closed under translation.

For a sequence of colorings $\langle f_i \rangle$, ACA_0 suffices to prove the existence of the translation algebra containing all the monochromatic sets for the colorings and all the finite sets.

IHT and ultrafilters

Given a countable field of sets, an ultrafilter u is a subset that:

- $\emptyset$ is not in u,
- u is closed under intersection,
- if $X \supset Y \in u$, then $X \in U$,and
- for all X either $X \in u$ or $X^c \in u$.

For the countable field of sets $\{A_0, A_1, A_2, \ldots\}$, we can identify an ultrafilter with a string of binary digits. For example, $\langle 1, 1, 0, 1 \ldots \rangle$ would indicate that $A_0 \in u$, $A_1 \in u$, $A_2 \notin u$, $A_3 \in u$, and so on. Indeed, the ultrafilters on a countable field of sets can be viewed as a closed subset of Cantor space.

IHT and ultrafilters

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a.d.t.i. ultrafilters are closely related to Hindman's theorem. The article [2] includes:

Thm: RCA_0 proves the following are equivalent:

(1) IHT.

(2) Every countable translation algebra has an a.d.t.i. ultrafilter.

Addition on ultrafilters

There are well-known connections between Galvin-Glazer addition on ultrafilters and Hindman's Theorem. If u and v are ultrafilters, we can define the ultrafilter $u \dot{+} v$ by:

$$A \in u \dot{+} v \text{ iff } \{x \mid A - x \in u\} \in v$$

In the countable translation algebra setting, this definition is problematic. For some choices of A and u, the set $\{x \mid A - x \in u\}$ may not be an element of the algebra. When working with ultrafilters over the full power set of $\mathbb{N}$, this is not an issue.

Addition on ultrafilters

However, in the countable translation algebra setting, we can add 1 to an ultrafilter. Define:

$$A \in u + 1 \text{ iff } A - 1 \in u$$

If we write $[1]$ for the principal ultrafilter consisting of the sets containing 1, then $u + 1$ is the same as $u \dot{+} [1]$.

Adding 1 is well behaved on the principle ultrafilters. For example,

$$\begin{aligned} A \in [3] + 1 &\leftrightarrow A - 1 \in [3] \\ &\leftrightarrow 3 \in A - 1 \\ &\leftrightarrow 4 \in A \\ &\leftrightarrow A \in [4] \end{aligned}$$

More importantly, the map $S : u \to u + 1$ is continuous.

To see this, fix u and consider a basic open neighborhood of $u + 1$. The neighborhood consists of the ultrafilters that agree with $u + 1$ on some finite initial segment of the list of sets in the translation algebra. Suppose this list is $A_0, A_1 \ldots A_n$. If v agrees with u on $A_0 - 1, A_1 - 1 \ldots A_n - 1$, then $u + 1$ and $v + 1$ will agree on $A_0, A_1 \ldots A_n$. Thus for every ultrafilter v in the basic open neighborhood that agrees with u on an initial segment of the list of sets including $A_0 - 1, A_1 - 1 \ldots A_n - 1$, $v + 1$ will be in the desired open neighborhood of $u + 1$.

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S is continuous on a closed compact set. We can do topological dynamics!

Topological dynamics

Here is a lemma from Blass, Hirst, and Simpson [1]:

Lemma 5.10: The following is provable in ACA_0. Let X be a compact metric space and let $S : X \to X$ be a continuous function. For all $x \in X$ there exists $y \in \bar{x}$ such that every $z \in \bar{y}$ is uniformly recurrent.

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What does it mean?

$\bar{x}$ denotes the orbit closure of x, that is, (roughly) the collection of all points that are accumulation points of the set $\{x, S(x), S(S(x)), \dots\}$.

z is uniformly recurrent means that for every $\varepsilon > 0$ there is an $m \in \mathbb{N}$ such that for any n, there is a $j \leqslant m$ such that $S^{n+j}(z)$ is ε close to z. Informally, z is closely revisited remarkably regularly.

The question

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We can prove that p exists in ACA_0. If we can answer the question affirmatively in ACA_0, then we have a proof of Hindman's theorem in ACA_0.

Note that such a proof would rely on the representation of the ultrafilter space (on a countable Boolean algebra) as a complete separable metric space. We're not working in the Stone-Cech compactification.

Ultrafilters that are not a.d.t.i.

An ultrafilter p is a.d.t.i. (almost downward translation invariant) if for every $X \in p$, there is an $x \in X$ such that $X - x \in p$.

If p is not a.d.t.i., then there must be a witness, namely, a set $X \in p$ such that for every $x \in X$, $X - x \notin p$.

If $x \in X$, then for every ultrafilter p such that X witnesses that p is not a.d.t.i., p is an element of the open set of all ultrafilters for which $X \cap (X - x)^c$ is an element.

Conclusion: There are natural countable open covers of the ultrafilters that are not a.d.t.i.

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Strategy 1: Assume that every ultrafilter in $\bar{p}$ is not a.d.t.i. Enumerate potential witnesses and build a countable cover of $\bar{p}$. Apply Heine-Borel and find a finite subcover. Use it to contradict the construction of p.

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Strategy 2: Build a tree of initial segments of ultrafilters in $\bar{p}$ so that every length n sequence can be extended to an ultrafilter u such that none of $A_0, \dots, A_{n-1}$ witness that u is not a.d.t.i. Any path through the tree codes an element of $\bar{p}$ that is a.d.t.i..

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Both of these strategies boil down to finding an a.d.t.i. ultrafilter by avoiding the ultrafilters that are not a.d.t.i.

Another strategy

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Observations:

- The principle ultrafilters are a dense subset of the space of ultrafilters (on a countable Boolean algebra).
- Every ultrafilter can be approximated by a sequence of principle ultrafilters.
- Using Ramsey's theorem, we can find an approximating sequence for p such that $p \dotplus p$ makes sense.
- For p uniformly recurrent, we can find an ultrafilter r such $p \dotplus p \dotplus r = p$.
- If r could be chosen so that $p \dotplus r \dotplus p = p$, then $p \dotplus r$ would be an idempotent for Glazer addition, and an a.d.t.i.

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- If r could be chosen so that $p \dot{+} r \dot{+} p = p$, then $p \dot{+} r$ would be an idempotent for Glazer addition, and an a.d.t.i.
- I tried this. I think the strategies on the previous slide hold more promise.

References

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