More reverse mathematics motivated by finite complexity theory (Preliminary results)

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Motivating example

Problems of finite complexity theory:

P: Determine whether a finite graph has an Euler path (one that uses all edges once).

NP complete: Determine whether a finite graph has a Hamilton path (one that uses all vertices once).

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Reverse math results

Thm: ACA is equivalent to: If $\langle G_i \rangle_{i\in\mathbb{N}}$ is a sequence of graphs, then the set of indices of graphs with Euler paths exists. [1]

Thm: Π^1_1-CA_0 is equivalent to: If $\langle G_i \rangle_{i\in\mathbb{N}}$ is a sequence of graphs, then the set of indices of graphs with Hamilton paths exists. [1]

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Answer: Nope.

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See *Infinite versions of some problems from finite complexity theory* (1996), by Jeff Hirst and Steffen Lempp [3]

A result from *Infinite versions...*

Thm: RCA_0 The following are equivalent:

1. Π^1_1-CA_0
2. (Isomorphic subgraph): If $\langle H_i, G_i \rangle_{i \in \mathbb{N}}$ is a sequence of ordered pairs of graphs then there is a function $s : \mathbb{N} \to 2$ such that $s(n) = 1$ if and only if H_n is isomorphic to a subgraph of G_i.
3. (Fixed isomorphic subgraph): For any graph H and any sequence of graphs $\langle G_i \rangle_{i \in \mathbb{N}}$, there is a function $s : \mathbb{N} \to 2$ such that $s(n) = 1$ if and only if H is isomorphic to a subgraph of G_n.
4. (Isomorphic subgraphs of a fixed graph): For any graph G an any sequence of graphs $\langle H_i \rangle_{i \in \mathbb{N}}$, there is a function $s : \mathbb{N} \to 2$ such that $s(n) = 1$ if and only if H_n is isomorphic to a subgraph of G.

A result from *Infinite versions...*

Thm: RCA_0 The following are equivalent:

1. Π^1_1-CA_0
2. (Isomorphic subgraph): Input pairs $\langle H_i, G_i \rangle$. Is H_i a subgraph of G_i? (Finite version is NP-complete.)
3. (Fixed isomorphic subgraph): Fix H. Input G_i. Is H a subgraph of G_i? (Finite version is polynomial in the size of G_i.)
4. (Isomorphic subgraphs of a fixed graph): Fix G. Input H_i. Is H_i a subgraph of G? (Finite version is constant in the size of G.)

View from the next millenium

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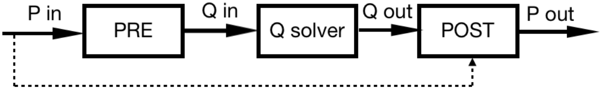
A Weihrauch problem accepts an input set and outputs a set or number.

We say that the problem P is Weihrauch reducible to the problem Q (and write P $\leqslant_W$ Q) if there are computable pre-processing and post-processing functionals such that P can be solved by:



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Question: Does examination of the Weihrauch reduciblity of infinite versions of results from finite complexity theory yield new insights into the nature of the classes P and NP?

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Question: Does examination of the Weihrauch reduciblity of infinite versions of results from finite complexity theory yield new insights into the nature of the classes P and NP?

Answer: Not so far.

Weihrauch results

Work with Asuka Wallace:

Consider the Weihrauch problems:

WF: Input a tree. Is it well-founded?

HG: Input graphs H and G. Is H a subgraph of G?

G_H: For a fixed graph H, input G. Is H a subgraph of G?

H_G: For a fixed graph G, input H. Is H a subgraph of G?

Add hats for sequential versions:
$\widehat{\mathrm{WF}}$: Input a sequence of trees. Which are well-founded?

Claim: $\widehat{\mathrm{WF}} \equiv_W \widehat{\mathrm{HG}} \equiv_W \widehat{\mathrm{G}_H} \equiv_W \widehat{\mathrm{H}_G}$.

Claim: There is a graph H_0 such that $\mathrm{WF} \equiv_W \mathrm{HG} \equiv_W \mathrm{G}_{H_0}$.
For every choice of H_0, $\mathrm{G}_{H_0} \leqslant_W \mathrm{WF}$. For every choice of G_0, $\mathrm{H}_{G_0} \leqslant_W \mathrm{WF}$.

Methodology

We prove the Weihrauch results formally, using the following restriction of a result of Hirst and Mummert [2].

Lemma: Suppose $P : \forall x \exists y p(x, y)$ and $Q : \forall u \exists v q(u, v)$ are total Weihrauch problems and $q(u, v) \to p(x, y)$ is in Γ_1. Then

$$i\mathrm{RCA}_0^\omega \vdash \forall x \exists u \forall v \exists y (q(u, v) \to p(x, y))$$

if and only if $i\mathrm{RCA}_0^\omega \vdash P \leqslant_W Q$.

$i\mathrm{RCA}_0^\omega$ is an intuitionistic version Kohlenbach's extension of RCA_0 to all finite types [4].

Γ_1 is Troelstra's class of formulas that avoid certain uses of existential quantifiers in the hypotheses of implications.

Demonstration

Steps for proving that $\widehat{HG} \leqslant_W \widehat{WF}$:

1. Working in $i\mathrm{RCA}_0^\omega$, prove that given any pair of graphs $\langle H, G\rangle$ there is a tree T such that T is well founded if and only if H is a subgraph of G. This amounts to verifying Hirst and Lempp's construction in a constructive analysis setting.
2. Apply the Lemma and conclude that:

$$i\mathrm{RCA}_0^\omega \vdash HG \leqslant_W WF$$

3. For any problems P and Q, $i\mathrm{RCA}_0^\omega \vdash P \leqslant_W Q \to \widehat{P} \leqslant_W \widehat{Q}$. Apply this fact and conclude that $i\mathrm{RCA}_0^\omega \vdash \widehat{HG} \leqslant_W \widehat{WF}$.
4. Apply the Lemma and some intuitionistic predicate calculus and conclude $i\mathrm{RCA}_0^\omega \vdash \widehat{WF} \to \widehat{HG}$.

References

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