Formalized reducibility

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Motivation

Goal: Explore the relationship between Weihrauch (and other) reducibilities and results in reverse mathematics.

Observation: Some reducibility results and reverse mathematics results have proofs with significant common content.

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Observation: Some reducibility results and reverse mathematics results have proofs with significant common content.
For example, in [1], Gura, Hirst, and Mummert prove:

$$\mathsf{RCA}_0 \vdash \mathsf{FC1} \leftrightarrow \mathsf{FC3} \quad \text{and} \quad \mathsf{FC1} \equiv_{\mathrm{sW}} \mathsf{FC3}$$

where

> FC1 says: every infinite graph in which every connected component is finite has a sequence of canonical indices of different components.
>
> FC3 says: every infinite graph in which every connected component is finite has an infinite totally disconnect set.

Motivation

Goal: Explore the relationship between Weihrauch (and other) reducibilities and results in reverse mathematics.

Observation: Some reducibility results and reverse mathematics results have proofs with significant common content.

We can reduce duplication in our arguments if we can prove single results that have both desired consequences as immediate corollaries.

Formalizing *sW* reduction

One characterization of *sW* reduction is to consider *problems*:

> The problem P is a sentence $\forall X \exists Y\ p(X, Y)$, where $p(X, Y)$ is a formula of second order arithmetic.
>
> If $p(X_P, Y_P)$, we say X_P is an instance of the problem P and Y_P is a solution of X_P.

In this setting $Q \leqslant_{sW} P$ means there are computable functionals ψ and ϕ such that

$$\begin{array}{ccc} & \psi & \\ X_Q & \longrightarrow & X_P \\ \downarrow & & \downarrow \\ Y_Q & \longleftarrow & Y_P \\ & \phi & \end{array}$$

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In this setting $Q \leqslant_{sW} P$ means there are computable functionals ψ and ϕ of type $1 \to 1$ such that

$$\begin{array}{ccc} & \psi & \\ X_Q & \longrightarrow & X_P \\ \downarrow & & \downarrow \\ Y_Q & \longleftarrow & Y_P \\ & \phi & \end{array}$$

Kohlenbach's axioms

Kohlenbach [3] presents axioms for reverse mathematics in higher types.

- RCA_0^ω consists of $\widehat{\text{E-HA}}^\omega_\restriction$ plus law of the exclude middle plus $\text{QF-AC}^{1,0}$:

  $$\forall X \exists y\, A(X, y) \ \rightarrow\ \exists Y \forall X\, A(X, Y(X))$$

  for A quantifier free.

- $\widehat{\text{E-HA}}^\omega_\restriction$ is intuitionistic arithmetic in all finite types. (See §3.4 of Kohlenbach [4]).

- $\widehat{\text{E-HA}}^\omega_\restriction$ includes combinators allowing λ-abstraction.

Formalizing sW reduction

Since the functionals defining *sW* reduction are of finite type, statements about their existence can be formulated in higher order reverse mathematics.

Since $i\mathrm{RCA}_0^\omega$ admits term extraction, for many formulas if $i\mathrm{RCA}_0^\omega \vdash Q \leqslant_{sW} P$, then $Q \leqslant_{sW} P$. (Using the intuitionistic system is necessary here. Corrected after the talk; for details see “Using Ramsey’s Theorem Once” by Hirst and Mummert.)

By composition of functionals,

$$\mathrm{RCA}_0^\omega \vdash Q \leqslant_{sW} P \ \rightarrow (P \rightarrow Q \ \wedge \widehat{P} \rightarrow \widehat{Q})$$

where $\widehat{P}$ is the infinite parallelization of P.

By Proposition 3.1 of Kohlenbach [3]:

If $\mathrm{RCA}_0^\omega \vdash \theta$ then $\mathrm{RCA}_0 \vdash \theta$.

A sample problem

Goal: Prove $\mathrm{RCA}_0^\omega \vdash \widehat{\mathrm{LPO}} \equiv_{sW} \mathrm{RAN}$.

$\widehat{\mathrm{LPO}}$ is $\forall \langle p_n \rangle \, \exists g \, (g(i) = 1 \leftrightarrow \exists t \; p_i(t) = 0)$

So g selects those i such that 0 is in the range of p_i.
Infinite parallelization of the limited principle of omniscience.

RAN is "Every injective function has a range."

$$\forall f \, \exists \chi_f \, \forall y \, (\chi_f(y) = 1 \leftrightarrow \exists t \; f(t) = y)$$

$\widehat{\mathrm{LPO}} \leqslant_{sW}$ RAN: Construction of ϕ in RCA_0^ω

Given $\langle p_n \rangle$ for $\widehat{\mathrm{LPO}}$, define an injection f by $f((i,j)) = k$ if and only if the following formula (denoted $\theta(\langle p_n \rangle, (i,j), k)$) holds:

$$(k = 2i + 1 \wedge p_i(j) = 0 \wedge \forall t < j\ p_i(t) \neq 0) \vee \\ (k = 2(i,j) \wedge (p_i(j) \neq 0 \vee \exists t < j\ p_i(t) = 0))$$

Note that $2i + 1 \in \mathrm{RAN}(f)$ if and only if $\exists t\ p_i(t) = 0$, so

$$\chi_{\mathrm{RAN}(f)}(2i+1) = \begin{cases} 0 & \text{if } \forall t\ p_i(t) \neq 0 \\ 1 & \text{if } \exists t\ p_i(t) = 0 \end{cases}$$

which is the solution to the instance $\langle p_n \rangle$ of $\widehat{\mathrm{LPO}}$.

Define ϕ by $\phi(\chi_{\mathrm{RAN}(f)}) = \chi_{\mathrm{RAN}(f)}(2i+1)$.

$\widehat{\mathrm{LPO}} \leqslant_{sW}$ RAN: Construction of ψ in RCA_0^ω

Working in RCA_0^ω, we need to prove the existence of the functional ψ mapping $\langle p_n \rangle$ to f (as defined on the previous slide).

Our main tool is $\mathrm{QF\text{-}AC}^{1,0}$: $\forall X \exists y\, A(X, y) \rightarrow \exists Y \forall X\, A(X, Y(X))$

$\theta(\langle p_n \rangle, (i, j), k)$ is Σ_0^0 and $\forall(\langle p_n \rangle, (i, j)) \exists k\, \theta(\langle p_n \rangle, (i, j), k)$, so $\mathrm{QF\text{-}AC}^{1,0}$ proves the existence of a functional F such that $\theta(\langle p_n \rangle, (i, j), F(\langle p_n \rangle, (i, j)))$.

F maps $(\langle p_n \rangle, (i, j))$ to $f((i, j))$.

Thus f is $\lambda(i, j).F((\langle p_n \rangle, (i, j)))$

and $\psi = \lambda \langle p_n \rangle.[\lambda(i, j).F((\langle p_n \rangle, (i, j)))]$.

RAN $\leqslant_{sW}$ $\widehat{\text{LPO}}$: Construction of ϕ in RCA_0^ω

Given an injection f, define an instance $\langle p_n \rangle$ of $\widehat{\text{LPO}}$ by:

$$p_n(t) = \begin{cases} 0 & \text{if } f(t) = n \\ 1 & \text{if } f(t) \neq n \end{cases}$$

Note that $n \in \text{RAN}(f)$ if and only if $\exists t\ p_n(t) = 0$.

So the solution of $\langle p_n \rangle$ is $\chi_{\text{RAN}(f)}$ and ϕ is the identity functional.

RAN $\leqslant_{sW}$ $\widehat{\text{LPO}}$: Construction of ψ in RCA_0^ω

Working in RCA_0^ω, we need to prove the existence of a functional ψ mapping f to $\langle p_n \rangle$ (as defined on the previous page).

$$\forall (f, n, t) \exists k \, ((k = 0 \wedge f(t) = n) \vee (k = 1 \wedge f(t) \neq n))$$

so QF-$\text{AC}^{1,0}$ proves the existence of S such that S maps (f, n, t) to $p_n(t)$.

$p_n = \lambda t.S((f, n, t))$ and $\langle p_n \rangle = \lambda n.[\lambda t.S((f, n, t))]$, so

$$\psi = \lambda f.(\lambda n.[\lambda t.S((f, n, t))])$$

Summarizing the demonstration problem:

We showed $\mathrm{RCA}_0^\omega \vdash \widehat{\mathrm{LPO}} \equiv_{sW} \mathrm{RAN}$

Consequently,

By Kohlenbach's conservation result,

$$\mathrm{RCA}_0 \vdash \widehat{\mathrm{LPO}} \leftrightarrow \mathrm{RAN} \quad \text{and} \quad \mathrm{RCA}_0 \vdash \widehat{\widehat{\mathrm{LPO}}} \leftrightarrow \widehat{\mathrm{RAN}}$$

Because $\mathrm{RCA}_0^\omega \vdash \widehat{P} \equiv_{sW} \widehat{\widehat{P}}$,

$$\mathrm{RCA}_0 \vdash \mathrm{ACA}_0 \leftrightarrow \widehat{\mathrm{LPO}} \leftrightarrow \widehat{\widehat{\mathrm{LPO}}} \leftrightarrow \mathrm{RAN} \leftrightarrow \widehat{\mathrm{RAN}}$$

If we reprove an appropriate formalization of the reduction in $i\mathrm{RCA}_0^\omega$, we can conclude

$$\widehat{\mathrm{LPO}} \equiv_{sW} \mathrm{RAN}$$

(See Hirst and Mummert "Using Ramsey's theorem once" for more detail.)

Questions

- How unfaithful is this formalization of sW reduction? Find good examples where $P \leqslant_{sW} Q$ but $\mathrm{RCA}_0^\omega \nvdash P \leqslant_{sW} Q$.

- If $\mathrm{RCA}_0^\omega \nvdash P \leqslant_{sW} Q$, then we can view the formalization of $P \leqslant_{sW} Q$ as a "functional existence axiom" which is not provable in RCA_0^ω. What is the logical strength of these functional existence axioms? Are there large natural classes that are provably equivalent in RCA_0^ω? What is the analog of the big five?

- What about other reducibilities?

Some references

[1] Jeffry L. Hirst, Carl Mummert, and Kirill Gura, *On the existence of a connected component of a graph*, Computability **4** (2015), 103–117. DOI 10.3233/COM-150039.
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[2] Jeffry L. Hirst and Carl Mummert, *Reverse mathematics and uniformity in proofs without excluded middle*, Notre Dame J. Form. Log. **52** (2011), no. 2, 149–162, DOI 10.1215/00294527-1306163.
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[3] Ulrich Kohlenbach, *Higher order reverse mathematics*, Reverse mathematics 2001, Lect. Notes Log., vol. 21, Assoc. Symbol. Logic, La Jolla, CA, 2005, pp. 281–295.

[4] U. Kohlenbach, *Applied proof theory: proof interpretations and their use in mathematics*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2008.