Reverse Mathematics of Matroids

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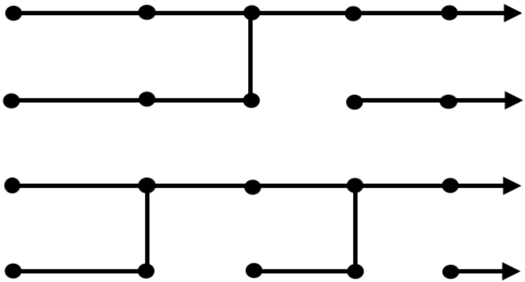
Motivation

We are interested in decomposing graphs into their connected components. It is helpful to find an “antichain” of vertices.



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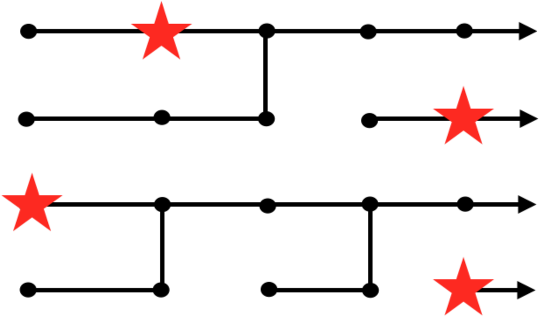

Observation: The antichain vertices are like a basis.

Matroids

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Defn: (RCA_0) An e-matroid is a pair (M, e) consisting of a set M and a function $e : \mathbb{N} \to M^{<\mathbb{N}}$ that lists all the finite dependent sets. That is,

- For all n, $e(n) \neq \emptyset$.
- If $e(n) = X$ and Y is a finite superset of X, then $\exists m(e(m) = Y)$.
- (Exchange property) If $\forall m(e(m) \neq X \wedge e(m) \neq Y)$, and X is smaller than Y, then for some $y \in Y$, $\forall n(e(n) \neq X \cup \{y\})$.

Examples of e-matroids

Thm: (RCA_0) Let G be a graph with vertex set V and at least one edge. Then there is a function e that enumerates every finite subset of V that contain a path connected pair of vertices. (V, e) is an e-matroid.

Thm: (RCA_0) Let V be a countable vector space. Then there is a function e that enumerates every finite subset of V that is linearly dependent. (V, e) is an e-matroid.

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There are many ways to formulate countable vector spaces in second order arithmetic. For example, the formulation in Simpson's book consists of a countable Abelian group of vectors together with scalar multiplication over a countable field.

Notions from linear algebra

Many concepts from linear algebra have natural analogs for matroids.

- B *spans* (M, e) if adjoining any additional element to B results in a dependent set.
- If B is independent and spans (M, e) then B is a *basis*.
- If every subset of size n is dependent, we say (M, e) has *rank* no more than n.

Matroids and graphs

Thm: (RCA_0) The following are equivalent:

1. For every *n*, every e-matroid of rank no more than *n* has a basis.

2. For every *n*, if *G* is a countable graph and every collection of *n* vertices contains at least one path connected pair, then *G* can be decomposed into its connected components. (Equivalently, *G* has a maximal antichain of pairwise disconnected vertices.)

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3. The induction scheme for Σ^0_2 formulas.

What happens if the rank is not bounded?

A not unexpected result

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Thm: (RCA_0) The following are equivalent:

1. ACA_0.
2. Every e-matroid has a basis.
3. Every countable graph has a maximal antichain of pairwise disconnected vertices.
4. Every vector space has a basis. (Friedman, Simpson, Smith)

There is a substantial early literature on recursion theory of matroids and vector spaces by authors including Crossley, Downey, Metakides, Nerode, Remmel, and many others.

sW reduction

Strong Weihrauch (or uniform) reduction can be described in terms of *problems*:

The problem P is a sentence $\forall X \exists Y \; p(X, Y)$, where $p(X, Y)$ is a formula of second order arithmetic.

If $p(X_P, Y_P)$, we say X_P is an instance of the problem P and Y_P is a solution of X_P.

In this setting $Q \leqslant_{sW} P$ means there are computable functionals ψ and ϕ such that

$$\begin{array}{ccc} & \psi & \\ X_Q & \longrightarrow & X_P \\ \downarrow & & \downarrow \\ Y_Q & \longleftarrow & Y_P \\ & \phi & \end{array}$$

Uniform reduction: unbounded dimension

Thm: EMB$\equiv_{sW}$GAC$\equiv_{sW}$VSB$\equiv_{sW}$ $\widehat{\mathsf{C}}_{\mathbb{N}}$

Problem	Input	Output
EMB	e-matroid	basis
GAC	graph	antichain
VSB	vector space	basis

$\widehat{\mathsf{C}}_{\mathbb{N}}$ – the parallelization of choice for $\mathbb{N}$
Input: A sequence of nonsurjective functions from $\mathbb{N}$ to $\mathbb{N}$
Output: A sequence $y_0, y_1, \ldots$
such that y_n is not in the range of f_n

Uniform reduction: fixed dimension

Thm: For each $n \geqslant 2$, $\mathrm{EMB}_n \equiv_{sW} \mathrm{GAC}_n \equiv_{sW} \mathrm{VSB}_n \equiv_{sW} \mathrm{C}_{\mathbb{N}}$

Problem	Input	Output
EMB_n	n dimensional e-matroid	basis
GAC_n	graph with n components	antichain
VSB_n	n dimensional vector space	basis
$\mathrm{C}_{\mathbb{N}}$	nonsurjective function	non-range number

Uniform reduction: bounded dimension

Thm: $\mathsf{EMB}_{<\omega} \equiv_{sW} \mathsf{GAC}_{<\omega} \equiv_{sW} \mathsf{VSB}_{<\omega} \equiv_{sW} \mathsf{C}^{\subseteq}_{\max}$

Problem	Input	Output
$\mathsf{EMB}_{<\omega}$	n and an e-matroid of dimension $\leqslant n$	basis
$\mathsf{GAC}_{<\omega}$	n and a graph with $\leqslant n$ components	antichain
$\mathsf{VSB}_{<\omega}$	n and a vector space of dimension $\leqslant n$	basis

$\mathsf{C}^{\subseteq}_{\max}$ – Choosing a maximal subset
Input: A number n and a nonsurjective function $f : \mathbb{N} \to \mathbb{N}^{<\mathbb{N}}$
whose range includes all sets of size $\geqslant n$.
Output: A maximal set not in the range of f.

Back to reverse mathematics

Formalizing aspects of the proof of the previous result yields:

Thm: (RCA_0) The following are equivalent:

1. The induction scheme for Σ^0_2 formulas.
2. Let V be a vector space such that for some n, every subset of n vectors is linearly dependent. Then V has a basis.
3. $C^{\subseteq}_{max}$

Some references

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